A Micro-mechanical Damage and Fracture Model for Concrete under Tension

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ABSTRACT

Some particularities of a crack along the interface of circular inclusion (aggregate) embedded in an infinite solid (hardened cement paste) are discussed. Especial attentions are paid to the propagation of a circular arc cracks of different open angle and different position along the interface with increasing load. By computing the effective strain tensor components, which are combinations of the elastic strain of perfect material and the inelastic strain due to the crack open displacement and propagation, a constitutive theory for concrete containing many de-bonding cracks is presented. The material parameters used within the context of the proposed formulation are those identifiable and associated with the macroscopic response. In order to validate this model, the case of uniaxial tension is worked out in detail.

KEY WORDS: fracture, damage, concrete, interface crack.

INTRODUCTION

Concrete is a kind of composite material composed by aggregates in a brittle matrix: hydrated cement paste. The de-bonding cracks lying along the interface between matrix and grains can appear during the moulding of concrete and the volumetric evolutions of the cement at the time of hydration [1]. Under applied load, cracks will result in stress concentration, which will make in reverse cracks stably and even unstably extend. It has been a key problem in damage mechanics how to establish a rational damage model by which the macro-scale mechanical behaviors taking into consideration the meso-structure of concrete can be described [1, 2].

Some particularities of the microstructure of concrete were first presented by Mazars [1], and an isotropic continuum damage model by using the coupling of two damage variables, D_t (tensile effects) and D_c (compressive effects), was proposed sequentially. An approach [2] on the influence of the birth and growth of interface cracks in concrete described by using effective compliance demonstrated that it is possible to formulate a rational constitutive model for concrete without introducing a large number of unidentifiable material constants. However, in reference [2], the pristine de-bonding cracks were simplified to be straight. Recently, a damage model of domain of microcrack growth to describe damage evolution in brittle materials such as concrete was given in references [3-4], and a micro-plane model [5] for concrete was proposed to study mechanical properties of cracked solids.

The present study addresses the case of damage and fracture in concrete affected by the microcrack evolution, assuming the grains in concrete to be cylindrical and adopting the approximation of no-interacting cracks [6, 7]. For simplicity, the plane model of the circular arc interface cracks is discussed in detail. Thereafter a constitutive theory for concrete containing multiple de-bonding cracks is proposed by using effective compliance (stiffness) tensor, in which the propagation of interface cracks with increasing load is taken into consideration.

MESOSTRUCTURE OF CONCRETE

Concrete is composed of grains in hydrated cement paste. These two phases play different roles and the interface between them is relatively weakened area. The cement matrix is a microporous material where the size of porous (from $10^{-4}$ A) depends on the cement-water ratio and decreases with aging. For classical concrete, the grains are stronger than the matrix, and around the grains the crystallization of the hydrated cement is different and the porosity is greater than anywhere else. The interface between matrix and grains, the transition halo [8], is the weakest link of the composite.

From mentioned above and the results in reference [2], it seems reasonable to assume that:

- The matrix is composed of cylindrical grains in an homogeneous matrix.
- The matrix and grains are considered as perfectly linear elastic brittle in respect to the Griffith energy criterion.
- The strain tensor component attributable to the presence of cracks is adequately quantified by a volume average of the crack opening displacement (over the unit cell). The volume average of strain is defined in references [9-13]:

$\epsilon_{ij}^{v} = \frac{1}{V} \int_{V} \epsilon_{ij}^{c} dV$
\[
\varepsilon_{ij} = \frac{1}{V_s} \int_{V_s} \varepsilon_{ij} dV + \frac{1}{2V_s} \sum_{k=1}^{k=1} \int_{V_s} (b_i n_j + b_j n_i) \delta(s^{(i)}) dV
\]

where \(V_m\) and \(V_c\) are volumes occupied in the representative volume element (RVE) by the matrix (hydrated cement paste and inclusions) and interface cracks respectively, \(\varepsilon_{ij}^e\) are elastic strain components of the perfect solid. \(b = [u]\) the displacement discontinuity vector across the crack with normal \(n\), and \(\delta(s^{(i)})\) the delta function.

**A CIRCULAR ARC CRACK EMBEDDED IN AN INFINITE SOLID**

**The Solution of a Circular arc Crack**

Considering a circular arc shaped crack embedded in an infinite solid shown in Fig. 1, the stress distribution near the crack tip (\(t_1\)) and the relative displacements of the faces of the crack are obtained in the polar coordinates system by Toya\(^{[14]}\). Thus, in the case of homogeneous cement paste-aggregate matrix \((\mu = \mu_2 = \mu, \nu = \nu_2 = \nu)\) and uniaxial tension \((T = T = \infty)\), the relative displacement is

\[
u_r + iu_\theta = A_1 \sigma_{rs} \delta \left[ \sin \frac{1}{2}(\alpha - \theta) \sin \frac{1}{2}(\alpha + \theta) \right]^{1/2} \left\{ G_0 + i H_0 + 2 + 2 \exp[i(2\theta - \theta)] \right\} \exp[-i(\frac{1}{2}\theta)]
\]

where \(G_0 = (-2 + 2 \cos \alpha - \sin^2 \alpha \cos 2\theta) / (3 - \cos \alpha)\), \(H_0 = (\cos \alpha - 1) \sin 2\theta\), \(A_1 = (1 - \nu) / \mu\) for plane strain, \(A_1 = 1 / [\mu(1 + \nu)]\) for plane stress, \(a\) is the radius of the inclusion, and \(\phi, \alpha\) and \(\theta\) are defined in Fig. 1.

The stress distribution near the crack tip is also obtained

\[
\sigma_r = \frac{\sqrt{2} \alpha}{4 \sqrt{r}} F_1
\]

where \(F_1\) is the coefficient of stress intensity factor (SIF)

\[
F_1 = \left( \sin \alpha \right)^{1/2} \cos \frac{\alpha}{3 - \cos \alpha} \left[ 2 - \frac{1}{2} \sin^2 \alpha \cos 2\theta + 3 \cos(2\theta - \alpha) - \cos(2\theta - \alpha) \cos \alpha \right]
\]

By means of Eq. (3), SIF can be defined as

\[
K_f = \lim_{r \to 0} \sqrt{2 \pi \sigma_r} = \frac{\sqrt{\mu \pi \sigma_{rs} r}}{2} F_1
\]

In order to apply the Griffith's criterion to the interface crack, it is necessary to analyze Eq. (4) further.

**Crack Symmetrically Extending on Both Ends**

When the crack locates symmetrically with respect to the tension axis, by substituting \(\phi = 0\) into Eq. (4), \(F_1\) can be rewritten as

\[
F_1 = (\sin \alpha)^{1/2} \left( \frac{1 + 3 \cos \alpha + \frac{1}{2} \sin^2 \alpha}{3 - \cos \alpha} \cos \frac{\alpha}{2} - \sin \alpha \sin \frac{1}{2} \alpha \right)
\]
From above Equation, we know that the half open angle $\alpha$ has dual influence upon $F_1$. With increase in the half open angle the length of crack increases so that the value of $F_1$ inclines to increase. On the other hand, the angle that the crack growth direction makes with the direction of the applied stress decreases from 90° so that $F_1$ inclines to decrease. If the SIF near the crack tip reaches critical value, there must be two critical half angles $\alpha'$ and $\alpha''$, at which $K_{\alpha_c}$ reaches a critical value $K_{\alpha_c}^\ell$. When $\alpha = \alpha'$, the crack will extend unstably until $\alpha = \alpha''$. Then the crack will extend slowly and stably as long as the applied load increases.

**SIF Near Crack Tip for Different Applied Load Directions**

For a fixed open angle $2\alpha$ of a crack, when $\phi$ varies from 0° to 90°, there must be a critical angle $\phi = \phi_{cr}$, for which $F_1$ reaches a maximum value. $\phi_{cr}$ can be defined by $\frac{dF_1}{d\phi}_{\phi=\phi_{cr}} = 0$. By application of Eq. (4), the critical angle can be evaluated as

$$
tg 2\phi_{cr} = \frac{\cos \frac{\pi}{6}(6\sin \alpha - \sin 2\alpha) + \sin \frac{\pi}{6}(1 + \cos \alpha)(3 - \cos \alpha)}{\cos \frac{\pi}{6}(6\cos \alpha - 1 - \cos^2 \alpha) - 2\sin \frac{\pi}{6}\sin \alpha(3 - \cos \alpha)}
$$

(7)

**Crack Non-Symmetrically (Unilaterally) Extending**

If an interface crack locates non-symmetrically with respect to the direction of applied load ($\phi \neq 0$) shown in Fig. 2(a), the extension always starts from the end $t_1$ ($\phi > 0$) according to reference [14]. Substituting $\phi_{0} - \alpha$ for $\phi$, Eq. (4) becomes

$$
F_1 = (\sin \alpha)^{\frac{\kappa}{2}} \cos \frac{\pi}{6} \left[ 2 - \frac{1}{2} \sin^2 \alpha \cos 2(\phi_{0} - \alpha) + 3 \cos(2\phi_{0} - 3\alpha) - \cos(2\phi_{0} - 3\alpha) \cos \alpha \right] 
$$

$$
+ (\sin \alpha)^{\frac{\kappa}{2}} \left[ \frac{1}{2} (1 - \cos \alpha) \sin(2\phi_{0} - 2\alpha) + \sin(2\phi_{0} - 3\alpha) \right]
$$

(8)

![Fig. 2](image)

**Fig. 2** (a) A crack extending non-symmetrically (b) Variation of the SIF with the crack half open angle

A set of $F_1(\alpha)$ curves with different initial angle $\phi_{0}$ is shown in Fig. 2(b). For the sake of contrast, the curve in Fig. 2(b) in the case of crack symmetrically extending ($\phi = 0$) is drawn in it too. From Fig. 2(b) some conclusion can be drawn as follows:

1. Any curve for $\phi > 0$ ends at the curve for $\phi = 0$, and the end ($\alpha = \phi_{0}$) represents that the crack is now locating symmetrically with respect to the tension axis.
When the initial half open angle $\alpha_0$ is relatively small, for example $\alpha_0 < 10^\circ$, the crack will be unstable until it becomes symmetrical with respect to tension axis: $\alpha = \phi_0$ as soon as it starts to extend. At that time, the SIF is still larger than the initial case, so that the crack will continue extend symmetrically. On the other hand, if the initial half open angle $\alpha_0$ is relative large, the crack can’t always extend to be symmetrical to tension axis.

(3). If $\phi_0$ is relatively large, for a same initial open angle, the SIF of the crack will decrease fast with the increase of $\phi_0$. Therefore, generally, the crack will not start to extend.

Crack Deviating into Matrix
The stress field near the crack tip shown in Fig. 1 can be expressed \[14\] as

$$
\sigma_{yy} = \frac{1}{2} (\sigma_x + \sigma_y) + \frac{1}{2} \left[ (\sigma_x - \sigma_y) \cos[2(\alpha - \beta)] + \sigma_{\phi\phi} \sin[2(\alpha - \beta)] \right]
$$

where $\sigma_x, \sigma_y$ and $\sigma_{\phi\phi}$ are defined by

$$
\frac{1}{\sigma_{\infty}} \left( \frac{\rho_1}{a} \right)^2 (\sigma_x + \sigma_y) \sigma_{\infty} = \text{Re} \left[ \frac{iM_a}{2(2\sin\alpha)^{3/2}} \exp\left\{ i \left[ \frac{\beta - \alpha}{2} \right] \right\} \right]
$$

(9)

$$
\frac{1}{\sigma_{\infty}} \left( \frac{\rho_1}{a} \right)^2 (\sigma_x - \sigma_y + 2i\sigma_{\phi\phi})
$$

(10)

$$
= \frac{1}{4(2\sin\alpha)^{3/2}} \left[ -M_a \sin \beta \exp\{i(-\frac{5}{2} - \frac{3}{2} \beta)\} + iM_b \exp\{i(-\frac{5}{2} + \frac{3}{2} \beta)\} + iM_b \exp\{i(-\frac{3}{2} + \frac{1}{2} \beta)\} \right]
$$

where $M_a = -i \sin \alpha \{G_a + iH_a + 2 + 2 \exp[i(2\theta - \alpha)]\}$, $G_b$ and $H_b$ are defined by Eq. (2). When the conditions

$$
\begin{align*}
\frac{\partial \sigma_{\phi\phi}}{\partial \beta} |_{\beta = \phi_0} &= 0 \\
\sqrt{2\pi \rho_1} \sigma_{\phi\phi} &= K_{IC}
\end{align*}
$$

are satisfied, the interface crack will deviate into the matrix, and complete brittle fracture may occur.

RELATIONSHIP BETWEEN STRESS AND STRAIN UNDER UNIAXIAL TENSION
For simplicity, consider the case where the radius $a$ of grains is the same, and there is only one initial crack (debonding) lying along the interface around each grain and the initial open angle 2$\alpha_0$ is taken to be the same as $10^\circ$. Suppose that the distribution of the initial crack direction is uniform, considering symmetry of the problem, the probability-density function is $p(\phi) = 2/\pi$ ( $\phi \in [0, \pi]$).

Consider a RVE containing $N$ aggregates, the volume of all aggregates in a RVE of concrete is $V = NV = f_N V$,

where $f_N$ is the volume fraction of the coarse aggregates (taken to be 0.33 as in reference [2]), and $V = \pi a^2$ the volume of a aggregate (for plane strain and unit thickness). So, the total volume of the RVE is $V = N \pi a^2 / f_N$.

The model is suitable for the case of monotonically increasing proportional load. The synopsis of the microcrack evolution includes the following phases:

(1) Elastic phase

For the load magnitude $\sigma_\infty$ below the threshold value $\sigma^e$ to be determined, all initial defects are stable and the concrete specimen behaves as a perfectly elastic, but slightly anisotropic, solid. From Eq. (1), the total strain includes two parts

$$
\varepsilon_{11} = \varepsilon_{11}^0 + \varepsilon_{11}^{ck}
$$

(12)

where $\varepsilon_{11}^0$ is the volume average of strain, the component of the strain tensor of the perfectly cement paste-aggregate matrix,

$$
\varepsilon_{11}^0 = \frac{1}{E_0} \sigma_{\infty} = \frac{1}{2(1+\nu)} \sigma_{\infty}
$$

and $\varepsilon_{11}^{ck}$ is the component of the strain tensor attributable to the existence of the cracks.
\[ \varepsilon_{11}^{ck} = \frac{1}{V} \sum_{k=1}^{N} \int_{\varphi=-\alpha}^{\alpha} (u_r^{(k)} (\varphi, \theta) \sin \theta + u_0^{(k)} (\varphi, \theta) \cos \theta) \sin \theta \cdot d\theta = \frac{a}{V} \int_{0}^{\alpha} \left[ \int_{-\varphi}^{\varphi} (u_r \sin \theta' + u_0 \cos \theta') \sin \theta' d\theta' \frac{2N}{\pi} \right] d\varphi \]

\[ = \frac{2f_c}{\pi^2} \int_{0}^{\alpha} \left[ \int_{-\varphi}^{\varphi} (u_r \sin \theta' + u_0 \cos \theta') \sin \theta' d\theta' d\varphi \right] \]

where \( u_r \) and \( u_0 \) are defined by Eq. (2), \( \theta' = \frac{\pi}{2} - \varphi + \theta \) and \( 0, \theta', \varphi, \alpha \) are defined in Fig. 3.

\( \varepsilon_{11}^{ck}, \varepsilon_{11}^{sck}, \varepsilon_{11}^{uck} \)

\[ \sigma = \varepsilon \frac{E}{\varepsilon} \]

\[ K_{IC} = \frac{\sigma}{\sqrt{a \pi}} \]

\[ K_I = K_{IC} \]

\[ K_{II} = K_{IC} \]

\( K_{III} \)

\[ F_1 (\varphi, \varphi', \alpha) = K_{IC} \]

\( K_{IC} \approx 0.5K_{IC} \) as suggested in reference [2]. \( K_{IC} \) is the critical stress intensity factor of the cement paste.

\( \varphi_{cr} \) is the critical angle for which SIF reaches a maximum value. \( \varphi_{cr} \) can be calculated by Eq. (7): for the case of \( \alpha = 5^\circ \), \( \varphi_{cr} = 3.76^\circ \).

According to the discussion in last Section, if a crack starts to extend, it will propagate unstably until it becomes symmetrical with respect to the tension axis. It will continue to extend symmetrical until \( \alpha = \alpha^* \) where \( \alpha^* \) is defined by Eq. (6). As long as the applied load increases, more cracks undergo an underlying unstable process to reach a new equilibrium with the applied load. At the same time, the extended cracks will propagate stably and slowly. Let \( \sigma_{cr} \) be a value larger than \( \sigma' \) but smaller than \( \sigma^* \), two angles \( \varphi_1, \varphi_2 \) can be calculated from Eq. (4). Considering symmetry, when \( \varphi_1 < 0 \) calculated from Eq. (4), let \( \varphi_1 = 0 \). Because of the uniform distribution of cracks along the perimeters of grains, the total number of cracks whose partial angle satisfy \( \varphi_1 < \varphi < \varphi_2 \), is \( \frac{2N}{\pi} (\varphi_2 - \varphi_1) \), and the other \( \frac{2N}{\pi} (\varphi_2 + \varphi_1) \) cracks will keep stable. At this time, the strain attributable to the existence of cracks includes two parts:

\[ \varepsilon_{11}^{ck} = \varepsilon_{11}^{sck} + \varepsilon_{11}^{uck} \]

where the superscripts ck, sck, uck stand for the components due to the displacement discontinuities associated with all
cracks, stable cracks and unstable cracks respectively. By means of Eq. (1), the expression of the strains can be obtained as

\[
\varepsilon_{11}^{\text{ick}} = \frac{2f_y}{\pi^2a} \left( \int_0^{\alpha_0} \int_0^{\pi/2} (u_r \sin \theta' + u_0 \cos \theta') \sin \theta' d\theta d\phi \right)
\]

\[
\varepsilon_{11}^{\text{ack}} = \frac{2f_y}{\pi^2a} \left( \phi_0 - \phi_1 \right) \int_0^{\pi/2} (u_r \sin \theta' + u_0 \cos \theta') \sin \theta' d\theta
\]

where \(u_r, u_0\) are defined by Eq. (2). In the second expression, \(\theta = \frac{\pi}{2} - \phi, \phi = 0\) for \(\phi = 0\).

(3) Fracture phase

When \(\sigma = \infty\) satisfies Eq. (11), some cracks will deviate into the cement paste matrix and then propagate in an unstable manner through the matrix. At last they will yield total failure of the material.

The analytical results for the total strain in function of the stress and the results in reference [2] are plotted in Fig. 4 from which close fit of the results[2] is obvious.

CONCLUSIONS

The presence of the circular arc interface cracks has serious influence on the meso- and macro mechanical properties of concrete structures. It is indispensable to analyze the heterogeneous mesostructure of concrete for formulating the kinetic equations of damage evolution. Some particularities of a circular arc crack along the interface of an inclusion (aggregate) embedded in an infinite solid (hardened cement paste) are discussed in this paper. Then a reasonable constitutive theory is proposed by computing the effective strain tensor components. The components include two parts, namely, the elastic strain of the perfect cement paste-aggregate matrix and the inelastic strain due to the crack open displacement and propagation. The material parameters needed in the proposed model are those identifiable and associated with the macroscopic and response. To validate this model, the case of uniaxial tension is worked out in detail. The result is in good agreement with the previous work. For simplicity, the same size of grains and the uniform distribution of the initial crack around the perimeter of grains are assumed in this approach, however, in fact, the model can deal with the cases where the distribution of the radius of grains and the direction of the initial cracks are arbitrary. Further more, the model can be applied to other particle-reinforced composites with slightly modification.

REFERENCES