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An Approximate Ultimate-Load Analysis for Brittle-Matrix Slabs and Pressurized Cylinders

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ABSTRACT

The paper concerns restrained bending analysis of elastic-plastic structures with no-tension or quasi-brittle matrix. Purely flexural response of such structures (in the absence of membrane forces) is rarely possible. When in-plane displacements at slab supports are restrained, important compressive membrane forces appear from the very beginning of the deformation process. They strengthen seriously the structure (the "dome effect") but make its response extremely unstable.

Analysis of such behavior using commercial FEM codes is, in principle, possible but the corresponding geometrically non-linear procedures become slow, capricious and frustrating potential users or simply fail. All that makes the direct FEM approach to restrained bending of limited utility for practical purposes now and simplified methods are needed. The ultimate goal of this research is furnishing a tool for analysis of restrained flexure, at the "strength-of-materials" level of complexity.

A hybrid method for elastic-plastic structures is proposed using an analytical approach with its parameters calibrated with the incremental FEM analysis performed in benchmark cases. It is based on an old authors' proposition and consists in applying the classical post-yield approach combined with elastic "spring model". The method permits for simple analytical formulae for the ultimate-peak loads and needs no more knowledge than the elementary limit analysis (plastic hinges, yield lines). Its details have been given elsewhere. Here, basic ideas are recalled and extended to cylindrical shells. The importance of the effect and the pertinence of the method is shown in the case of one-way bending, when compared with the FEM simulation and tests on reinforced concrete models. The same approach is applied also to the case of pressurized tubes and compared with FEM results.

Restrained bending response is very sensitive to "secondary effects" (e. g., shape and support imperfections). The approximate method permits easily to account for initial deflections, imperfect contact and compliant supports.

KEY WORDS: slabs, beams, pressurized tubes, ultimate load, post-yield analysis, limit analysis, elastic-plastic, brittle, no-tension matrix, non-linear analysis, concrete, arching action, dome effect, restrained bending.

RESTRAINED BENDING

Simple bending means here transversal flexure of flat structures in absence of membrane forces. Flexural response with in-plane displacements at supports prohibited or restricted is called restrained bending; the term has been introduced *per analogiam* to restrained torsion. For structures composed of classical "symmetric" materials (with the same strength/elasticity characteristics in compression and tension) the simple bending induces existence of a neutral plane (axis) free of deformation; therefore, even if in-plane restraints at support exist, they do not generate membrane forces. The latter will appear eventually and may be of some importance only at very advanced deformation. However, if elastic and/or strength characteristics of the material are different in tension and in compression, the restraints may change qualitatively the structure response from the very beginning of the deformation process. For example, in the case of brittle-matrix composite structures the end fixity generates important compressive membrane forces. This effect, known already long ago as the arching action in RC beams [1] and the dome effect in slabs [2], strengthens considerably the structure but makes its response strongly unstable. The character of the load-deflection response under conditions of restrained flexure is shown in Fig.1. The geometrical non-linearity inherent to the behavior of eccentrically compressed slender bars is enhanced here by the deformation-dependence of the membrane forces. The strengthening effect of the membrane compression attains its peak at small deformations of the order of 10% of the slab thickness and is followed with a dramatic decrease of structure strength; under non-decreasing load it will result in a dynamic snap- through to a new equilibrium configuration (if a tensile membrane action may appear) or in a complete collapse otherwise. The most drastic decrease appears in one-way bending; the minimum of the corresponding curve falls below the collapse load in simple bending and corresponds to the transition from the compressive arching action to membrane tension.

It is obvious that the strength reserve (q_u - q_Y in Fig.1) is the more important, the more strength properties in compression and tension are different, e. g., in weakly reinforced concrete. Of course, the extreme case concerns unreinforced structures made of no-tension material, which have no carrying capacity in conditions of simple bending, but reveal non-negligible strength due to the dome effect. Structural consultants are aware that it is frequently easier to show why the structure collapsed than to explain why its parts are still up.

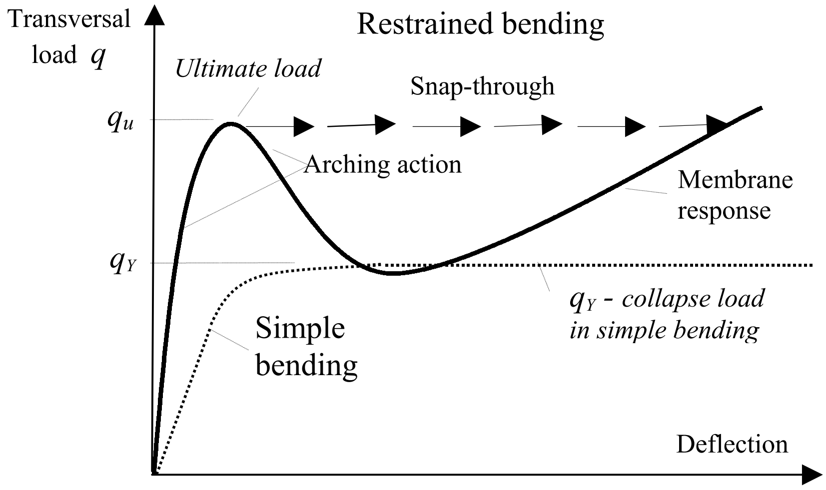


Fig.1. Quasi-static restrained bending response of reinforced brittle-matrix or no-tension slabs

Experimental evidence, concerning first of all, concrete slabs (e.g., [3], [4]) shows the importance of this effect. To give an idea on the quantitative importance of the effect, some results of test on reinforced concrete strips [4] are quoted in Fig. 2.

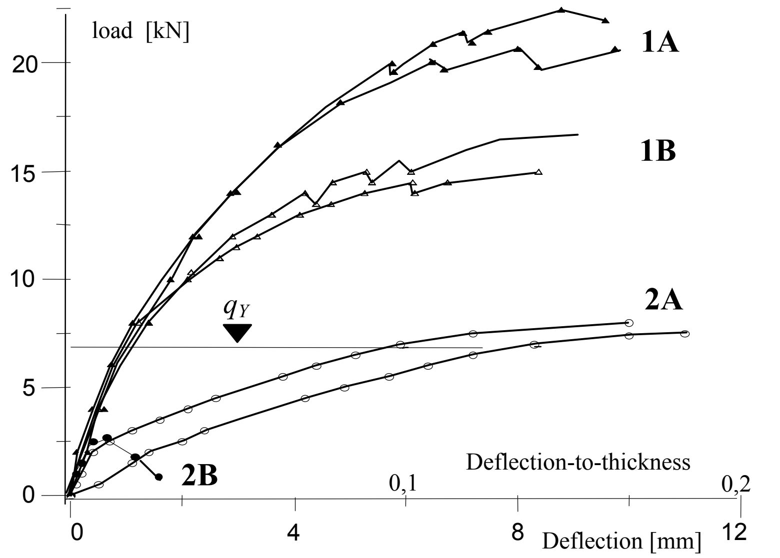


Fig. 2 Three-point loading tests on concrete strips [4]: **1** - restrained (**A** - strongly reinforced, **B** – unreinforced); **2** - unrestrained (**A** - strongly reinforced, **B** – unreinforced)

The value of the ultimate load in restrained bending q_u is very sensitive to such agents, as slight shape and support imperfections, temperature etc. that are commonly considered by engineers as irrelevant. That is why they are reluctant in accounting for this effect into the structural design. Taking into account this strength reserve may be necessary for safety assessment of structures that had undergone previously a heavy overloading. Tools for such diagnostics are needed, first of all, for deteriorated no-tension (segmented) historical structures, but may be also useful for heavy concrete protective structures in nuclear engineering.

A POST-YIELD APPROACH VS. F.E.M. ANALYSIS

Physical non-linearity (no-tension, plasticity), together with the geometrically non-linear character of the restrained bending response, require large-deflection analysis of the problem, even at an early deformation stage. The geometrically non-linear analysis is, in principle, feasible even using commercial FEM codes. However, the procedures are slow, laborious and frustrating or simply fail when concern brittle material behavior in presence of geometrical non-linearity. Topic-oriented FEM programs and laborious design charts (e.g., [5]) become necessary to deal with the problem. Last, but not least, the procedures and results are very sensitive to input data and to modeling of support conditions, whereas uncertainty level concerning these data is important. All that makes still the direct FEM approach to restrained bending of limited utility for practical purposes. Therefore, a qualitative analysis and simplified methods are still useful and need attention.

In the pre-computer era, the only possible practical way of dealing with a geometrically non-linear response of elastic-perfectly plastic structures was the post-yield approach (PYA) - friendly and effective tool. It consists of determination of a sequence of instantaneous collapse loads for the structure configurations consecutively modified following the rigid-plastic flow mechanism (Fig.3). In this way the load-deflection relation corresponding to the quasi-static deformation path is established. This approach has been applied to restrained bending of RC slabs starting from early propositions [2, 6, 7] up to more recent applications (e.g., [8]). The PYA gives satisfactory results for rather advanced deformation and, therefore, may be reliable only in the absence of early geometrical softening. In the latter case the neglected early elastic deformation may be determinant for the value of the ultimate-peak load and, hence, for the structure safety. Unfortunately, the restrained bending response falls into the latter category. However, the PYA may provide a realistic description of the early yielding if a "spring model" is used to account for elastic in-plane deformations, whereas flexural deformation is considered rigid-plastic.

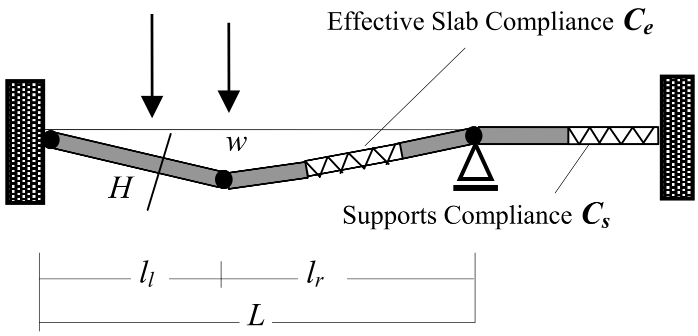


Fig.3. The three-hinge collapse mode for one-way slabs completed with the elastic spring model accounting for an effective slab in-plane compliance C_e, and resultant supports compliance C_s.

This method was proposed long ago by the first author [9] and was revisited when incremental FEM analysis became possible and proved the pertinence of the approach [4]. It consists of applying the elementary limit analysis (plastic hinges, yield lines) in the framework of the PYA methodology, with the elastic in-plane compliance modifying the kinematical compatibility relations inherent to the plastic flow. As details of the method are given in [9] and quoted in [10], we recall here only its principal assumptions leading to final results. Our attention is focused on one-way slabs and some results are quoted here. Identical methodology is applicable to simple-shaped two-way slabs. The load-deflection relation, describing the structure quasi-static response up to the beginning of tensile membrane action (see Fig.1) is given, in a non-dimensional form in Eq.1:

$$q = q_Y + (k-\alpha)^2 - [k - (1-e^{-\varepsilon\alpha})(k+\varepsilon^{-1})]^2 \tag{1}$$

$$\varepsilon = \frac{2LH}{l_l l_r \sigma_c C_{rs}} \tag{2}$$

Equation 1 contains a term representing the plastic collapse load in simple bending q_Y and a term corresponding to the decreasing arching action as follows from the rigid-perfectly plastic PYA. The third term due to the elastic in-plane compliance of the slab is controlled by the rigidity parameter ε (Eq.2) depending on the span-to-thickness ratio L/H, on the resulting elastic compliance of the system C_{rs} and yield point in compression σ_c, as well as on the geometry of the collapse mode (Fig.3). Depending upon the value of the resulting compliance C_{rs} equation (6) may describe a spectrum of the restrained flexure from the rigid-plastic arching action (C_{rs}=0) up to the simple bending response ($C_{rs}\rightarrow\infty$).

Notation used in the Eqs. 1 and 2: $\alpha = w/H$ is characteristic deflection-to-thickness ratio; the cross-sectional parameter is $k = 1 - \eta_b - \eta_t$ for near-to-face tensile reinforcement of intensity $\eta_i = t_i\sigma_r/H\sigma_c$, with subscripts standing for $i = b$ (bottom) and $i = t$ (top), respectively. Yield points of the material are: σ_c for the matrix (in compression) and σ_r for the reinforcement (in tension).

The non-dimensional parameterization of the load used in Eq.1 refers the current collapse load Q to the incipient rigid-plastic collapse load Q_{uo} in restrained bending of the structure, with its reinforcement neglected:

$$q = \frac{Q}{Q_{uo}} = \frac{M_{\max}}{2M_u} . \tag{3}$$

This non-dimensional load is equivalent to the ratio of maximum bending moment M_{max}, determined for the simple supported strip to the double ultimate moment (Fig.4b) for the no-tension matrix $M_u = \frac{1}{8}\sigma_c H^2$. Such representation ensures the same load-deflection curve for any load configuration resulting in a three-hinge collapse mechanism of a slab. The Fig.4 explains also the origin of the strength reserve due to restrained bending: the difference between flexural plastic moment M_f in the absence of the axial force and the ultimate moment M_u in the absence of in-plane membrane deformation rate ($\lambda = 0$).

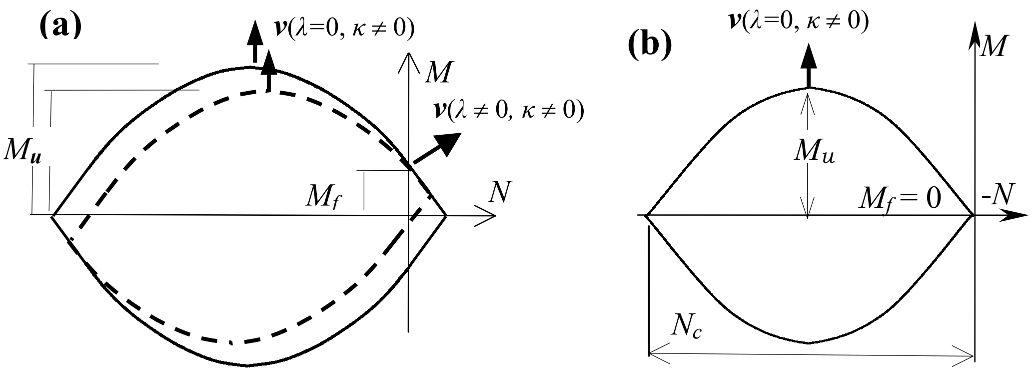
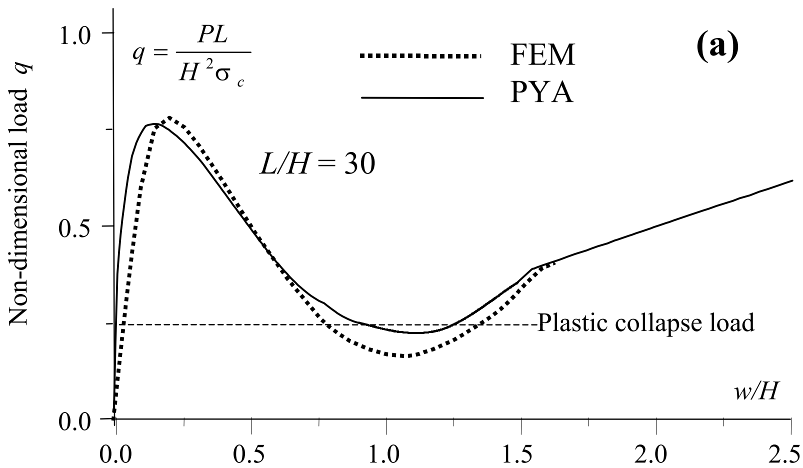


Fig.4. Yield Curve in one-way plastic bending; **(a)** no-tension matrix with 1% face reinforcement at one face (dashed line) and both faces (solid line); **(b)** unreinforced no-tension matrix.

The resulting compliance $C_{rs} = C_e + C_s$ is composed of the effective in-plane compliance of the slab span C_e and of the compliance of the supports C_s (if any). An essential drawback of the method consists of an arbitrary choice of the effective value of the slab compliance C_e. As a matter of fact, in the case of elastic-plastic deformation of a non-symmetric material the elastic axial compliance of the slab is different from its value in elastic behavior $C = L/E$;, it depends upon the distribution of the stress resultants and evolves in the deformation process. To furnish a reliable effective (average) value of C_e calibration by the FEM analysis for benchmark cases is necessary.

It appears that for no-tension structures a satisfactory fit in ultimate-peak loads furnished by the PYA and the FEM analysis (Fig.5) is obtained if the effective slab compliance C_e is taken double of the compliance for the purely elastic slab strip $C = L/EH$, i. e. $C_{rs} = 2\,C + C_s$. For strongly reinforced structures a little better fit is obtained if this increase with respect to C is reduced a little depending upon the reinforcement intensity [4]. Nevertheless, even without this correction, the discrepancies are not excessive. As shown in Fig.5a, the PYA and FEM results differ qualitatively at the beginning of the deformation process, because the PYA analysis requires deformation commencing at the simple bending collapse load. The ultimate-peak load occurs at deformation equal about a half of that corresponding to the maximum axial force. At this phase the fit of both curves is the most satisfactory. The discrepancy becomes again more important near the curve minimum corresponding to the transition from the compressive arching action to the tensile membrane response. However, the latter phase is of little practical interest. Of course, in the case of no-tension unreinforced slabs (Fig.5b) the discrepancies discussed above are irrelevant, because the FEM and PYA curves (Fig.5b) start at zero load (simple bending collapse) and, moreover, there is no ascending membrane branch of the curve.



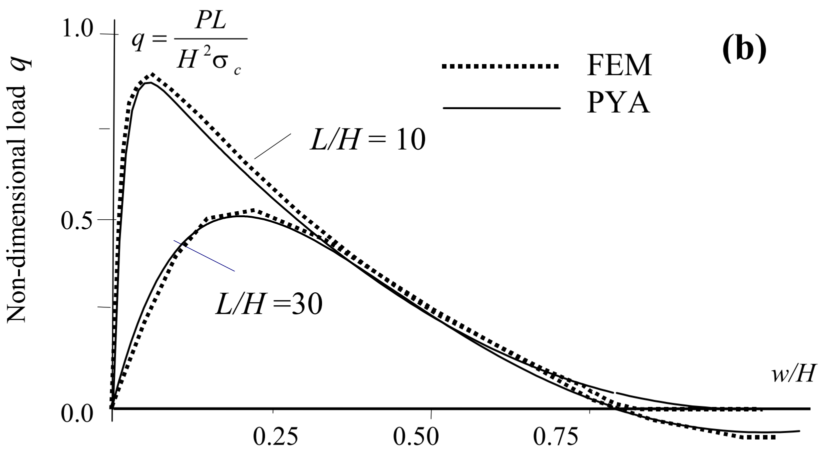


Fig.5. PYA and FEM results for one-way slabs; **(a)** slender both-faces reinforced slab; **(b)** unreinforced: slender and thick slabs

The ultimate-peak load q_u is attained at deflection α_u being nearly exactly half of the deflection corresponding to the maximum membrane force and, therefore, Eq.1 yields:

$$q_U = q_Y + (k - \frac{\alpha_U}{2}) - \varepsilon^{-2}(\sqrt{k\varepsilon + 1} - 1)^2. \quad \text{with} \qquad \alpha_U = \frac{1}{2\varepsilon}\ln(k\varepsilon + 1). \tag{4}$$

PRESSURIZED TUBES

The equilibrium equation for rotationally symmetric cylindrical shells is identical with that for beams on continuous Winkler-type foundation and kinematical relations at plastic collapse are the same as for beams. Therefore, the post-yield analysis of such structures will be the same as for beams on plastic foundation. If a brittle-matrix pressurized tube is built-in between massive walls (Fig.6a), restrained bending occurs along the cylinder generatrices and the formulae furnished in the preceding section remain valid if the current collapse load (Eq.1) is supplemented with a term resulting from the strength of circumferential reinforcement. Hence, an amazing effect appears: a short no-tension (or damaged/cracked) tube, with no reinforcement, may support a non-negligible pressure if only a soft lining ensures its air-tightness. Comparison of the PYA results with the FEM incremental analysis (Fig.6b) shows that the result of the calibration for the reduced compliance, as done for strips ($C_{rs} = 2C + C_s$), is satisfactory also in this case.

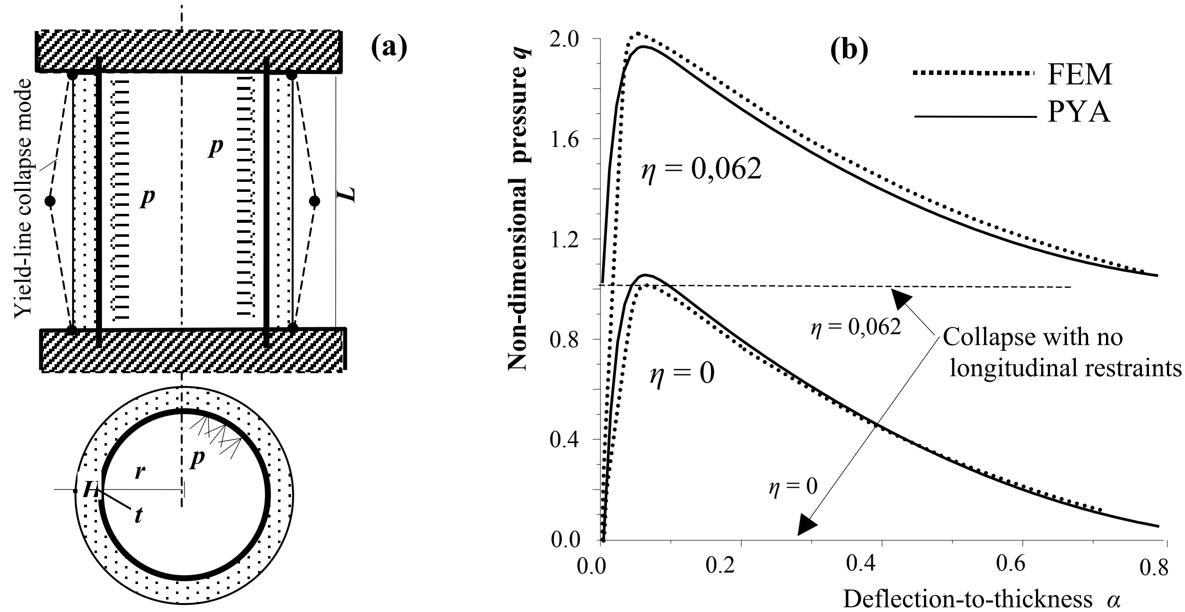


Fig.6. Pressurized built-in short no-tension tubes in restrained bending: with no reinforcement (η = 0) and with 1% lining t (η = 0,062). Dotted lines – FEM incremental analysis, solid lines - PYA approximation.

For relatively thick-walled tubes a correction should be introduced accounting for the difference of the inner and outer perimeters of the cylinder and, hence, of a trapezoid cross-section of the unit beam cut along the generatrix. With minor approximations (piece-wise constant width instead of the trapezium) Eq.1 takes now the form:

$$q = q'_Y + \frac{\eta}{2}\frac{L^2}{rH} + A(k' - \alpha)^2 - A[k' - (1 - e^{-\varepsilon\alpha})(k' + \varepsilon^{-1})]^2 \ , \tag{5}$$

with η standing for the intensity of the circumferential reinforcement (lining). Primed (') quantities differ from those in Eq.1 by the factors depending upon the thickness ratio H/r: q_Y is the simple bending collapse load of the trapezoidal beam; for inner longitudinal reinforcement at supports of intensity η_{in} the reinforcement parameter is $k' = 1 - \eta_{in}/A$, with $A = 1 + H/r$ approximately.

SENSITIVITY TO SHAPE AND SUPPORT IMPERFECTIONS

As mentioned before, the restrained bending response, being strongly non-linear, is extremely sensitive to shape and support imperfections, as well as to some other agents that engineers usually consider as irrelevant. Therefore, a particular attention should be done to these effects. First of all, the compliance of support restraints may strongly reduce the strength reserve. The PYA solution accounts for this effect reducing the rigidity parameter ε (Eq.2) with a supplementary term C_s. For one-way bending the comparison with the FEM analysis and tests [4] confirms pertinence of such approach. It appears from more detailed considerations that for the most frequent practical case: a slab clamped between two half-spaces, all made of the same material, supports may be considered as non-compliant.

Even a slight initial deflection of the slab strongly reduces the ultimate load, especially in one-way bending. For example, a sag of one tenth of the thickness results in decreasing the ultimate strength of about 20%. It may be remarked that this effect, although obvious when concerning restrained bending, is inverse to the known strengthening effect of initial deflections in the case of thin metal plates. Accounting for the shape imperfections in the framework of the PYA approach needs only a modification of the initial condition in the differential equation representing the kinematical compatibility of an instantaneous collapse mode [9]. The FEM analysis [10] confirms that the calibration of the rigidity parameter ε done for undeformed structures is still valid.

Imperfect support conditions, e. g., in the presence of unilateral constraints and clearances at supports strongly modify the structure response. It may be remarked, that unilateral in-plane restraints at supports generate compressive membrane forces also in the case of "symmetric" (metal-like) materials. However, this effect is of minor importance.

For no-tension structures the unilateral contact conditions are standard cases and the problem of clearances at support became particularly important. Difficulties with the FEM analysis increase. Fortunately, it appears that the PYA approach may be applied also in the presence of support clearances. It became identical with the analysis initially deformed structures if the initial deflection w_0, at which the contact occurs is determined from the rigid-plastic collapse mode (Fig.3), i. e.:

$$w_0 = kH\left(1-\sqrt{1-\Delta/\Delta_{\max}}\right) \qquad \text{and} \qquad \Delta_{\max} = \frac{LH^2k^2}{2l_l l_r} \; . \tag{6}$$

with Δ being the clearance (difference of the support span and the slab length) and Δ_{max} - the maximum clearance at which the contact may appear, following the plastic collapse mode (Fig.3). Then, the PYA analysis for the sagged slab may be performed, as described in the preceding paragraph. However, the comparison with the FEM analysis of the contact problem shows that such behavior differs from the response of a slab with a natural initial sag. If the deflection w_0 results from the elastic-plastic bending (inducing residual stresses) a satisfactory fit in ultimate loads may be obtained if the calibration of the rigidity parameter ε is updated. A quite good fit is obtained (Fig.7) if the reduced compliance is taken dependent upon the clearance: $C_r = 2C / (1 - \Delta/\Delta_{max})$. The PYA and FEM results for centrally loaded concrete slabs with different support clearances Δ are compared in Fig.7. The intensity of the bottom reinforcement is ($\eta_b = 0.062$, $\eta_t = 0$) corresponding to about 1% mild-steel reinforcement.

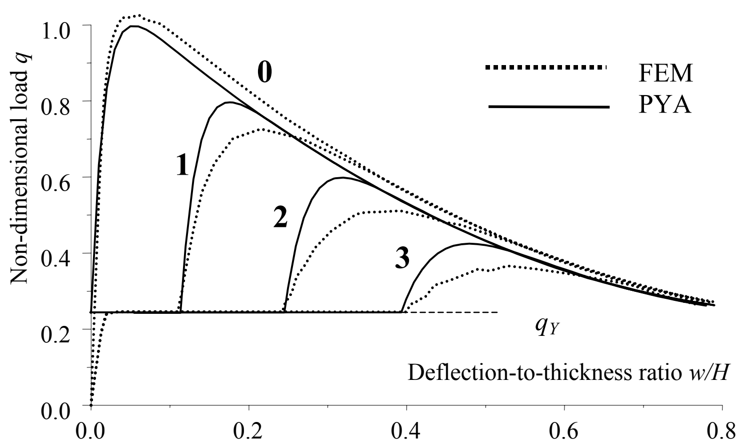


Fig.7. Thick ($L/H = 10$) bottom reinforced strip with clearances at supports:
0. $\Delta = 0$ (perfect contact), **1**. $\Delta = 0{,}23\ \Delta_{max}$, **2**. $\Delta = 0{,}46\ \Delta_{max}$, **3**. $\Delta = 0{,}66\ \Delta_{max}$

The sensitivity to clearances is particularly important for slender structures. In this case even a gap of the order of only 5% of the slab thickness may practically annihilate the strengthening effect of the arching action. It is obvious that also a drop in the temperature may increase the clearance. Therefore, one should be very cautious when accounting for the arching action in presence of unilateral restraints, when any wedging/prestraining is absent. This fact was well known to ancient builders of stone-skeleton structures.

FINAL REMARKS

The strengthening effect (arching action) due to restrained bending of flat slabs and some other structural elements with no-tension or brittle matrix (as concrete) may have an important impact on the structure safety, especially when the reinforcement is weak or absent. However, because of a strongly unstable character of this effect and its sensitivity to “secondary“ effects it is rarely taken into account in the engineering practice.

Numerical analysis of the problem is possible, at least in principle, using FEM codes, but it is cumbersome and frustrating potential users. Therefore, such procedures are of a very limited practical utility, when dealing with restrained bending; approximate methods are still needed.

A method based on the post-yield rigid-plastic approach (PYA) completed with an in-plane elastic spring model, gives simple and reasonably correct description of the load-deflection response of elastic-perfectly plastic structures with no-tension matrix. A correct estimation of the ultimate load of the structure may be obtained if the characteristic in-plane rigidity parameter is appropriately calibrated with the FEM analysis for benchmark cases. For one-way slabs and similar structural elements (e. g., pressurized tubes) data concerning such calibration are already available. The method is applicable also to two-way slabs of regular shapes, but in this case a more extensive parametric studies concerning calibration are needed and sufficient experimental evidence is still lacking.

The method permits to account easily for the supports compliance and imperfections, as well as for shape imperfections. It results in simple formulae for the ultimate supportable load and needs no more knowledge than the elementary limit analysis (plastic hinges, yield line method).

The ultimate goal of this study is furnishing a tool at the level of complexity of strength of materials. The approach is, in some manner, similar to the Eulerian buckling theory for straight bars. The strength surplus due to the arching action (the dome effect) is derived from general post-yield formulae, with an appropriately reduced compliance parameter. The reduction coefficients, determined with a more sophisticated methods (FEM analysis for benchmark cases), may be given in tables depending upon the geometry of the problem. Similar methodology concerns Eulerian buckling theory, where lengths of the buckling waves for non-standard cases have been derived from a more detailed analysis.

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