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# Calculating latent frequencies of systems with local damping

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**ABSTRACT**

Modal analysis of damped systems mostly cannot be performed by common real-eigenvalue techniques. The system of equilibrium equations leads to a matrix with elements being quadratic functions of a $\lambda$ parameter. The values of that which make the matrix singular are the latent roots, while the solutions of the associated homogenous equation are the latent vectors. They are the (generally complex) characteristic frequencies and the mode shapes of the system. Although the theory is elaborated, a straightforward numerical application is rather unattractive because of complex arithmetics. A reduction to better-known real-domain subtasks needs attention. With a theorem of Popper and Gáspár, a *n x n* $\lambda$-matrix problem can be cut in two: into *n*-size asymmetric real matrices having as eigenvalues the *n* lower and *n* upper latent roots, ranked by absolute value. This approach is promising for systems with high number of DOF damped by a relatively few concentrated devices i.e. snubbers. It might be fit also for earthquake analysis, where the lower portion of eigenvalues is customarily what counts. The snubbers might be treated in the splitting algorithm as restricted-size modifications, using the Frobenius-Schur-Woodbury identity. The task is re-traced this way to a more usual real-asymmetric eigenproblem. A requirement of convergence is that the lower and upper *n*-set of latent values must be distinct. With odd-number DOF and neither overdamped, i.e. all latent roots being complex, this condition is surely violated. For such cases, a supplemental algorithm is proposed.

**KEY WORDS:** modal analysis, vibration, snubber, local damping, lambda matrix, latent root, complex eigenvalue.

## THE DAMPED VIBRATION AND THE POPPER-GÁSPÁR THEOREM

The harmonic analysis of vibrating systems with damping requires the solution of the system of equilibrium equations for characteristic frequencies and modal shapes connected with. Condensing out the DOF without inertia forces, reducing the coefficient damping and stiffness matrices, respectively, by multiplying them on both sides with the square root of the mass matrix and searching a solution in the usual $\mathbf{z}\exp(\lambda t)$ form, finally

$$(\mathbf{I}\lambda^2 + \mathbf{D}\lambda + \mathbf{Q})\mathbf{z} = \mathbf{0} \qquad (1)$$

has to be solved for $\lambda$, $\mathbf{z}$. Here, $\mathbf{I}$ is the *n x n* unit matrix, $\mathbf{D}$ (damping) a non-negative definite and $\mathbf{Q}$ (stiffness) a symmetric positive definite one. Eq.(1) yields generally $2n$ $\lambda$-s and $\mathbf{z}$-s pertaining to them. The theory of such "lambda matrices" is developed [1], establishing modal solutions elaborated [2], handling the generally complex interim results reviewed [3]. General algorithms and codes for application are presented [4]. There is an open question worth of trying, whether some specific features could be utilized in the solution process. If the damping originated by snubbers, taking into account cost and maintenance requirements, it is a quite realistic to presume that they are as few as possible. This means a very sparse, low-rank $\mathbf{D}$.

The Popper-Gáspár theorem [5] applied on Eq.(1) states that if

$$\mathbf{Y}^2 + \mathbf{DY} + \mathbf{Q} = \mathbf{0} \qquad (2)$$

is solved by the iteration

$$\mathbf{Y}_{n+1} = -(\mathbf{Y}_n + \mathbf{D})^{-1}\mathbf{Q}, \qquad (3)$$

the resulting asymmetric $\mathbf{Y}$ has eigenvalues equal to the *n* lower latent roots by absolute value of the lambda matrix in Eq. (1) and eigenvectors matching. The *n* higher latent quantities might be explored through $-(\mathbf{Y} + \mathbf{D})$. A necessary and sufficient condition of convergence is that the two sets of latent roots must be strictly distinct, i.e.

$$|\lambda_n| < |\lambda_{n+1}|. \qquad (4)$$

The iteration in Eq.(3) might be streamlined by uniting two consecutive steps.

$\mathbf{Y}_{n+1}^{-1} = -\mathbf{Q}^{-1}(\mathbf{Y}_n + \mathbf{D})$ (5)

$\mathbf{Y}_{n+2} = -(\mathbf{Y}_{n+1} + \mathbf{D})^{-1}\mathbf{Q} = -(\mathbf{Y}_{n+1}^{-1} + \mathbf{W})\mathbf{Q} = -[-\mathbf{Q}^{-1}(\mathbf{Y}_n + \mathbf{D}) + \mathbf{W}]\mathbf{Q}.$ (6)

Here, **W** is the correction on the inverse of $\mathbf{Y}_{n+1}$ necessary for adding **D** to. If the latter contains but a few non-zero elements in relation to DOF, it is simpler to look for that correction than to invert the modified full matrix once again. The algorithm would override also the inversion of varying approaches in every step, **Q** being constant.

## ELEMENTARY STEP OF ADJUSTING AN INVERSE

The **W** correction might be calculated either at once, using the Frobenius – Schur – Woodbury identity [6], or step-by-step successively, with a skeleton form of that, Sherman – Morrison's rule. The latter says, if a matrix **Y** gets modified by adding a $d_{ik}$ to the $y_{ik}$ element, $\mathbf{Y}^{-1}$ has to be modified by adding

$$\mathbf{W}_{ik} = -\frac{\mathbf{y}_i \mathbf{y}^k}{\frac{1}{d_{ik}} + y_{ki}}. \qquad (7)$$

Here, $\mathbf{y}_i$ is the $i$-th column vector and $\mathbf{y}^k$ the $k$-th row vector of the matrix **Y**, respectively.

Snubbers might realize absolute damping, i.e. they connect some DOF to the ground. The addenda lay then along the diagonal of **D** – this is the most frequent situation. Sometimes, a relative damping arrangement might make sense, the snubber being positioned between two DOF. That means symmetric and equal off-diagonal elements $d_{ik}$ and $d_{ki}$ in the **D** matrix. Although a condensed formula can be created taking these two addenda at once into account, it shows no special advantage compared to successive elementary adjustments. The same is true for applying the FSW-identity at large, too.

## LOSING CONVERGENCE WITH ODD NUMBER OF DOF

A system without damping has pure imaginary latent roots in conjugate pairs. If there is an odd number of DOF $n$, the lower and upper set of latent roots as ranged by absolute value gets stuck together at the middle pair. The $n$-th and $n+1$-th latent roots are not distinct, the condition in Eq.(4) is violated. With appearing damping, all latent roots grow complex and the picture remains qualitatively the same. The middle conjugate complex pair (which must exist for an odd $n$) straddles the border between the lower and upper $n$-sets. The iteration of Eq.(6) does not converge. That would have a chance only then if some got overdamped and the straddling root would be split into two distinct real ones or shifted off. If the DOF of the model may be assigned freely e.g. while treating continua by discretization, an even number of them should and could be obtained. In systems, hovewer, where DOF are rather pre-determined by the structure, e.g. in those consisting of discrete components, convergence problems are rather unavoidable. Luckily, such systems mostly do not have too much DOF. For those, an "emergency exit" will be created.

## RECOVERING CONVERGENCE BY A PSEUDO-SUPPLEMENT

Be an one-DOF system of $\alpha$ frequency connected to the original one by weak coupling $\varepsilon\mathbf{q}$. Denoting the stiffness matrix of that by $\mathbf{Q}_1$, this will be modified to a *(n+1) x (n+1)*-size matrix of rimmed structure:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \varepsilon\mathbf{q} \\ \varepsilon\mathbf{q}^T & \alpha^2 \end{bmatrix}. \qquad (8)$$

Here, $\varepsilon$ is small and **q** sparse, respectively, and the upper $T$ means transpose. With that, the iteration of Eq.(6) converges, although the magnitude of elements on the rim strongly differs with that of the core, upper left minor. Be the result denoted with

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21}^T & y_{22} \end{bmatrix}. \qquad (9)$$

Substituting the aboves into Eq.(2),

$$\begin{bmatrix} \mathbf{Y}_{11}^2 + \mathbf{y}_{12}\mathbf{y}_{21}^T + \mathbf{D}\mathbf{Y}_{11} + \mathbf{Q}_1 & (\mathbf{Y}_{11} + \mathbf{I}y_{22})\mathbf{y}_{12} + \varepsilon\mathbf{q}_{12} \\ \mathbf{y}_{21}^T(\mathbf{Y}_{11} + \mathbf{I}y_{22}) + \varepsilon\mathbf{q}_{12}^T & \mathbf{y}_{21}^T\mathbf{y}_{12} + y_{22}^2 + \alpha^2 \end{bmatrix} = \mathbf{0}. \tag{10}$$

The augmented system must have the approximate eigenvalues $\pm i\alpha$. Trying the vectors

$$\mathbf{u} = \begin{bmatrix} \mathbf{y}_{12} \\ y_{22} \pm i\alpha \end{bmatrix}, \mathbf{v}^T = \begin{bmatrix} \mathbf{y}_{21}^T & y_{22} \pm i\alpha \end{bmatrix} \tag{11}$$

by multiplying with **Y** from left, the results turn out to be as if multiplied by $\pm i\alpha$, with an error $[-\varepsilon\mathbf{q}_{12}^T\ 0]$ as row or column as necessary. This means that if $\varepsilon$ is small enough, $\mathbf{v}^T$ and **u** might be considered as left- and right-side eigenvectors, respectively. The two dyads pertaining to can be subtracted from the matrix of (10), the rank being diminished by 2 of.

$$\mathbf{Y} - i\alpha\mathbf{u}_{+i\alpha}\mathbf{v}_{+i\alpha}^T + i\alpha\mathbf{u}_{-i\alpha}\mathbf{v}_{-i\alpha}^T = \begin{bmatrix} \mathbf{Y}_{11} - \dfrac{y_{22}}{y_{22}^2 + \alpha^2}\mathbf{y}_{12}\mathbf{y}_{21}^T & 0 \\ 0 & 0 \end{bmatrix}. \tag{12}$$

The zero rims might be left, the *n*-size remainder must have a zero eigenvalue and *n-1* ones of the original system.

**REDUCTION OF THE REMAINDER**

Let the upper left minor in Eq.(12) be denoted with R and its structure as

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{r}_{12} \\ \mathbf{r}_{21}^T & \mathbf{r}_{22} \end{bmatrix}. \tag{13}$$

This is necessarily singular. The right-side eigenvector pertaining to the zero eigenvalue is

$$\mathbf{x} = \begin{bmatrix} -\mathbf{R}_{11}^{-1}\mathbf{r}_{12} \\ 1 \end{bmatrix}, \text{ provided } -\mathbf{r}_{21}^T\mathbf{R}_{11}^{-1}\mathbf{r}_{12} + r_{22} = 0. \tag{14}$$

Here, $\mathbf{r}_{21}^T\mathbf{R}_{11}^{-1}$ is the unique vector consisting of constants necessary to linearly combining the rows of $\mathbf{R}_{11}$ in order to obtain $\mathbf{r}_{21}^T$, supposing that $\mathbf{R}_{11}$ is not singular. Because **R** itself is singular, the whole last row must be yield by the same linear combination and the condition inEq.(14) shows just that. A Wielandt deflation [7] leads to

$$\mathbf{R}_1 = (\mathbf{I} - \mathbf{x}\,\mathbf{e}_2^T)\mathbf{R} = \mathbf{R} - \mathbf{x}[\mathbf{r}_{21}^T\ \ \mathbf{r}_{22}], \tag{15}$$

where $\mathbf{e}_2$ is an n-size vector having 1 at the “2” minor position and 0 otherwise. From $\mathbf{R}_1$, the rim consisting of zero “2” row and of “2” column, might be crossed out and the remainder will have the nonzero eigenvalues of **R**.

**NUMERICAL EXAMPLE, CONCLUSIONS**

Let a chain of consecutive springs and point masses be considered, with stiffness 372, 108 and 72 and mass 4, 9 and 1, respectively, fixed at beginning and free at the other end. The stiffness **Q** and latent roots $\lambda$ are

$$\mathbf{Q} = \begin{bmatrix} 120 & -18 & 0 \\ -18 & 20 & -24 \\ 0 & -24 & 72 \end{bmatrix}, \quad \begin{aligned} \lambda_{1,2} &= \pm 2{,}844615\ i, \\ \lambda_{3,4} &= \pm 8{,}966543\ i, \\ \lambda_{5,6} &= \pm 11{,}11347\ i. \end{aligned}$$

If a snubber with a reduced constant δ is attached on the middle mass, **D** will be a diagonal <0 δ 0>. The **Y** from Eqs (5)(6) has the 3 lower latent roots of the problem as eigenvalues. Its 3-degree real characteristic equation has either three real i.e. overdamped roots or one real and two conjugate complex ones. It cannot have three complexes and in this case, even an **Y** cannot be extracted. The original problem has then six pair-wise conjugated (i.e. 3×2) latent roots, the pairs being of same absolute value. The middle root just straddles the upper-lower border, the (4) condition is not satisfied. Let's try δ = 15 as substantial enough for an overdamping:

$$\mathbf{Y} = \begin{bmatrix} 6{,}52679027 & -1{,}272389 & 0{,}6015995 \\ 21{,}0924497 & -4{,}311584 & 1{,}5292924 \\ -225{,}66704 & 34{,}605422 & -3{,}292321 \end{bmatrix}, \qquad \begin{aligned} &\lambda_1 = -0{,}652118 \,, \\ &\lambda_{2,3} = -0{,}212498 \pm 8{,}597727\, i, \\ &|\lambda_{2,3}| = 8{,}600353 \,. \end{aligned}$$

A slight damping, say δ = 1, does not cange yet the all-complex scene Be the system augmented with an unit $\alpha^2$ connected by a spring of reduced stiffness $10^{-11}$ to the outermost one, see Eq.(8) . The **Y** extracted, **R** remnant, $\mathbf{R}_1$ deflated matrices and λ latent roots will be then:

$$\mathbf{Y} = \begin{bmatrix} 89{,}719649 & -18{,}185025 & 9{,}996232 & -8{,}884E\text{-}13 \\ 1769{,}1177 & -196{,}51851 & -234{,}44722 & 3{,}59E\text{-}11 \\ 6827{,}0497 & -495{,}76203 & -1604{,}9954 & 2{,}253E\text{-}10 \\ 4{,}98E+16 & -3{,}412E+15 & -1{,}225E+16 & 1710{,}9359 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 115{,}57856 & -19{,}956928 & 3{,}63502353 \\ 724{,}20923 & -124{,}91934 & 22{,}5968903 \\ 268{,}11055 & -46{,}330733 & 8{,}48238997 \end{bmatrix},$$

$$\mathbf{R}_1 = \left[\begin{array}{cc|c} 164{,}82469 & -182{,}55638 & 11{,}8728366 \\ 149{,}01851 & -164{,}89478 & 10{,}6959178 \\ \hline 0 & 0 & 0 \end{array}\right], \qquad \begin{aligned} &(\lambda_{1,2} = \pm i), \\ &\lambda_{3,4} = -0{,}429196 \pm 2{,}814242\, i, \\ &|\lambda_{3,4}| = 2{,}846782 \,. \end{aligned}$$

The emergency exit seems viable, however, the rather unbalanced structure of **Y** deserves further attention.

A series of calculations shows that while δ tends to 0, $\lambda_{3,4}$ is nearing the undamped $\lambda_{1,2}$ but convergence gets lost. With growing δ, its imaginary part declines and the real one grows. Between 6,456…6,457 the complex root splits into two real ones (about -2,92), they gradually drift apart. This does not concern the convergence, the border over the 4-th root is not touched. The fake roots are at least of no harm, presently even needed, without them the iteration does not converge. Most likely, the 5. root is part of a complex and then, while the border got over the 3.root, it would slide down to the straddling 3-4. place. At δ = 10,3, $\lambda_3$ = -1,016445, $\lambda_4$ = -8,631382 result and futher on, the latter turns to complex. That must be calculated without supplenment, the fake roots would press it into straddling position. At δ = 10,4, $\lambda_1$ = -1,003922, $\lambda_{2,3}$ = -0,246153 ± 8,678473*i* are obtained. Convergence is decaying on both sides.

Thus, with judicious applying enlargement, the straddling root always could be shifted off from the middle position. Augmented systems were checked against smaller supplements, this did not influence the results.

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