A DIRECT METHOD OF STRUCTURAL ANALYSIS FOR CREEP IN HEATED CONCRETE STRUCTURES

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SUMMARY

Time-dependent stress analyses of massive concrete structures are often complex and costly. Here a method of calculation is presented which is direct, approximate to any desired degree, and economic if necessary: a compromise between cost and accuracy being within the control of the analyst or designer. Examples are presented to illustrate both the nature and the accuracy-dependence of the approximations.

In a Ritz type of approach, several self-equilibrating stress distributions are chosen and weighted along with one other which equilibrates the boundary loading. The weighting factors of the self-equilibrating stresses are initially unknown functions of time and these are evaluated from an optimization procedure based on minimising a total power quantity for the structure. This operation leads to a set of differential equations in time from which the time functions, or spot values of them, are evaluated. The equations are linear first order when non-recoverable strains only are considered and linear second order when recoverable strains are taken into account. Back substitution of the time functions into the stress equation then gives the time variation of the stresses due to creep.

For problems of sustained mechanical loads and temperatures as exist in PCPV's the method is known to generate good solutions when the number of self-equilibrating distributions is large. As this number is reduced the solution becomes more approximate and less costly. A compromise can thus be achieved between accuracy and cost.

A second simplifying approximation is possible. When each time function is represented by a series of exponential decay terms multiplied by separate coefficients, these coefficients become the unknowns and the problem collapses to the solution of an algebraic set of equations. For acceptable accuracy the number of exponential terms should exceed the number of self-equilibrating distributions chosen.

To demonstrate that the approximate methods can produce solutions of good engineering accuracy, a through-the-wall section of a PCPV is considered in a simple restrained example. The stresses used are those which would be computed normally, using an elastic procedure, for any real problem, and correspond to the thermal stresses only, and the difference between the total initial elastic solution and the limiting steady-state stress solution resulting from differential thermal creep. Solutions are presented for analyses in which both self-equilibrating distributions have been used and these are compared with cases where one distribution only is incorporated. Similar comparisons are made for a plane stress example where a two-dimensional time variation of the stresses exists.

It is concluded that approximate solutions of low order can produce acceptable accuracy at low cost and for little computational effort. For the example employing one self-equilibrating distribution of stress only, i.e. the crudest approximation possible, errors in stresses are everywhere less than 9% and generally better than 5% for all time: the maximum error reducing to less than 1% for the case in which two self-equilibrating distributions are used. Accuracy is little affected by the second simplifying assumption.

The views expressed in this paper are those of the authors and do not necessarily reflect those of the Nuclear Installations Inspectorate.
1. **Introduction**

Creep in engineering materials causes structures to exhibit time-dependent deformations when loads are sustained, or stress relaxation in situations where strains or displacements are imposed. A further feature is apparent when the creep properties of the material vary throughout the structure. When this situation exists, internal stresses and actions become time-variable quantities whatever the nature of the external loads and constraints.

Temperature is a parameter which can influence significantly the local creep behaviour in non-uniformly heated concrete structures.

In the following section theory is developed for determining the nature and extent of the stress redistribution which occurs with increase of time, in structures for which the material representation is consistent with the following criteria:

The creep rate is related,

(a) linearly to the applied stress, $\sigma$.

(b) to some function of temperature, $\Phi(T)$.

(c) to a suitable time parameter, $t'$, and is constant when the applied stress is time-invariant.

The uniaxial creep equation is then,

\[ \dot{\varepsilon}_c = \sigma \Phi(T) \]  

(1)

where the dot notation refers to differentiation with respect to $t'$.

When the elastic component of the material is introduced and the applied stress is permitted to vary in time, the total strain rate is represented by eq.(2), for which $E$ is the elastic modulus.

\[ \dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \sigma \Phi(T) \]  

(2)

Extension of eq.(2) for a three-dimensional state of stress leads to the following general tensorial equation for the strain rates.

\[ \dot{\varepsilon}_{ij} = \left( \frac{1}{K} + \frac{1}{G} \right) \frac{1}{2} \left( \frac{\sigma_{ij}}{E} + \frac{1}{G} \right) \left( \sigma_{ij} - \frac{\sigma_{kk} \delta_{ij}}{3} \right) \]  

(2a)

Here, $K$ and $G$ refer to the elastic bulk and shear moduli, $K'$ and $G'$ are the corresponding creep counterparts which will be related to $\Phi(T)$; $\sigma$ is the differential operator $\frac{d}{dt}$, and $\delta_{ij}$ is the Kronecker delta.

2. **Theory**

Figure 1 shows a continuum body or structure which is in equilibrium with the applied loads, $P$, and supporting reactions, $R$, and which is at all times compatible. The internal stresses and strain rates are denoted by $\sigma_{ij}$ and $\dot{\varepsilon}_{ij}$. By invoking the principle of virtual work at any time, it is possible to state that the integrated sum of the corresponding products of the internal strains, $\varepsilon_{ij}$, and support displacements, $u_s$, with an equilibrium set of internal stresses, $\sigma_{ij}$, and support reactions, $\delta R_s$, must equate to zero, i.e.
\[
\sum_{s=1}^{N} u_s \delta R_s = 0 \quad (3)
\]

Because eq. (3) is true for all times, it follows that,

\[
\sum_{s=1}^{N} \dot{u}_s \delta R_s = 0 \quad (4)
\]

Eq. (4) may be simplified when the supports do not exhibit time-dependent displacements, i.e. \( \dot{u}_s = 0 \) for all values of \( s \). For this case,

\[
\dot{\varepsilon}_{ij} \delta \sigma_{ij} dV = 0 \quad (5)
\]

Equation (5) forms the basis of an optimization procedure for the determination of stresses and their variation with time due to creep, provided \( \dot{\varepsilon}_{ij} \) may be specified in terms of the stresses and \( \delta \sigma_{ij} \) represents any self-equilibrating stress distribution.

From eq. (2) it is clear that \( \dot{\varepsilon}_{ij} \) may be specified in terms of the stresses and their time derivatives. Hence, when the total state of stress is defined in terms of a combination of independent distributions of the form of eq. (6), where the parameters \( a_1, a_2, \ldots a_n \), represent weighting functions to the individual self-equilibrating stresses, \( \sigma_1, \sigma_2, \ldots \sigma_n \), respectively, it becomes possible to define \( \dot{\varepsilon}_{ij} \) in terms of a set of time functions and their derivatives, \( a_1, \ldots a_n \), and distributions of stress which vary in space only. In eq. (6) the \( \sigma \) stresses satisfy the statical boundary conditions of the problem, and each stress distribution represents a set of stresses in three dimensions.

\[
\sigma_{ij} = \sigma_{o,ij} + a_1 \sigma_{1,ij} + \ldots + a_n \sigma_{n,ij} \quad (6)
\]

The stress variations \( \delta \sigma_{ij} \) of eq. (5) are then given by, \( \sigma_1, \sigma_2, \ldots \sigma_n \), of eq. (6) i.e.

\[
\delta \sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial a_r} \quad \text{for} \quad r = 1, 2, \ldots n \quad (7)
\]

Thus, substitution of \( \dot{\varepsilon}_{ij} \) from eq. (2a) and \( \delta \sigma_{ij} \) from eq. (7), into eq. (5) leads to a set of first order differential equations of the form,

\[
[A] \begin{bmatrix} a \\ \end{bmatrix} + [B] \begin{bmatrix} a \\ \end{bmatrix} + [C] = 0 \quad (8)
\]

For simplicity, the build up of these equations is illustrated below for the one-dimensional stress situation. Substitution of the stresses from eq. (6) into eq. (2) gives the strain rates as,

\[
\dot{\varepsilon} = \frac{1}{E} (\dot{a}_1 \sigma_1 + \ldots + \dot{a}_n \sigma_n) + \phi(T) (\sigma_o + a_1 \sigma_1 + \ldots + a_n \sigma_n) \quad (9)
\]

Then, after inclusion of \( \delta \sigma_{ij} \) from eq. (7), eq. (5) becomes,

\[
\left[ \frac{1}{E} (\dot{a}_1 \sigma_1 + \ldots + \dot{a}_n \sigma_n) + \phi(T) (\sigma_o + a_1 \sigma_1 + \ldots + a_n \sigma_n) \right] \sigma_r dV = 0 \quad (10)
\]

As \( r \) ranges from 1 to \( n \), equations (10) take the following form, for \( r = 1, \ldots n \).
\[
\begin{bmatrix}
A_{11} & \cdots & A_{1n} \\
A_{r1} & \cdots & A_{rn} \\
A_{n1} & & A_{nn}
\end{bmatrix}
\begin{bmatrix}
\dot{a}_1 \\
\dot{a}_r \\
\dot{a}_n
\end{bmatrix}
+ \begin{bmatrix}
B_{11} & \cdots & B_{1n} \\
B_{r1} & \cdots & B_{rn} \\
B_{n1} & & B_{nn}
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_r \\
a_n
\end{bmatrix}
+ \begin{bmatrix}
C_1 \\
C_r \\
C_n
\end{bmatrix} = 0
\]  
(11)

The general terms of the A, B and C matrices are,

\[
A_{rs} = \int \frac{1}{\phi(\tau)} \frac{\sigma_{rs}}{r} d\mathbf{V}; \quad B_{rs} = \int \phi(\tau) \frac{\sigma_{rs}}{r} d\mathbf{V}; \quad C_r = \int \sigma_{r} \phi(\tau) d\mathbf{V}
\]  
(12)

The solution of eq.(8) has the form,

\[
\begin{bmatrix}
a(t')
\end{bmatrix} = e^{-t'} \begin{bmatrix}
A^{-1}B
\end{bmatrix} \begin{bmatrix}
\mathbf{a}(0) \\
\mathbf{b}
\end{bmatrix} - \begin{bmatrix}
\mathbf{b}
\end{bmatrix}^{-1} \begin{bmatrix}
c
\end{bmatrix}
\]  
(13)

where ||D|| is a vector dependent upon the initial condition at \( t' = 0 \). In many problems eq.(13) may be simplified by choosing the stress distribution, \( \frac{\sigma_{o,t'}}{o} \) of eq.(6) to represent the total actual state of stress at \( t' = 0 \). When this is done, \( \mathbf{a}(0) = 0 \) and \( \mathbf{D} = \mathbf{B}^{-1} \mathbf{C} \). Eq.(13) then reduces to,

\[
\begin{bmatrix}
a(t')
\end{bmatrix} = e^{-t'} \begin{bmatrix}
A^{-1}B
\end{bmatrix} \begin{bmatrix}
\mathbf{1}
\end{bmatrix} - \begin{bmatrix}
\mathbf{1}
\end{bmatrix} \begin{bmatrix}
B^{-1}C
\end{bmatrix}
\]  
(14)

Although this equation is readily solved for small values of \( n \), the number of separate differential equations, a general numerical approach has been adopted in order that the solution process may be systematic and suitable for any value of \( n \). It consists of expanding in series form the exponential matrix for a small value of the exponent \( A^{-1}B t' \), by suitably selecting a value of \( t' \), and then summing the series in truncated form to a predetermined degree of convergence. Solutions for other values of \( t' \) are readily obtained for all such values as \( 2^n t' \) by simply squaring and resquaring the sum of the truncated series. The time-dependent nature of the \( a_r \) functions is then determined.

2.1. Extension of method to include delayed elastic strain response

The theory developed so far allows for time-dependent strain behaviour of the viscous flow type. Some materials, however, exhibit a delayed strain response following both the application and removal of stress. For materials which exhibit this delayed elastic behaviour in addition to a flow response, it is often possible when the delayed component is small compared to flow, to approximate the delayed effect as an instantaneous response and build this behaviour into the elastic moduli of eqs.(2) and (2a), and to then use the above theory without further modification. For cases where this approach is considered to be inappropriate the theory may be extended to include second derivatives of the strains and to make the following statement for structures resting on time-invariant supports,

\[
\int \varepsilon_{ij} \sigma_{ij} d\mathbf{V} = 0
\]  
(15)

Equations (15) and (5) may, with certain restrictions [1], be used together to develop a solution procedure for the weighting functions of eq.(6) when the stresses are specified as in the previous analysis. In this case, however, it is necessary to solve a set of
simultaneous second order differential equations in time for the \( a_r \) functions, thus,

\[
[r][a] + [Q][a] + [R][a] + [S] = 0
\]

(16)

2.2. Reduction of differential equations to algebraic form

The differential equations of the previous sections yield expressions for the weighting functions, \( a_r \), which are characterised by exponential terms with negative exponents. This form of solution results from the quasi-static nature of the creep problems. It is thus possible to pre-empt the form of solution and to specify each of the weighting functions as a series of terms containing exponential functions, thus,

\[
a_r = A_{r0} + A_{r1} e^{\lambda_1 t'} + A_{r2} e^{\lambda_2 t'} + \ldots + A_{rm} e^{\lambda_m t'}
\]

(17)

for \( r = 1, \ldots, n \)

Here, the exponent terms, \( \lambda_s, s = 1, \ldots, m \), are specified and the coefficients, \( A_{rs} \), represent the unknowns in the problem.

Eliminating \( a_r \) between eqs. (17) and (6) and using eqs. (2) and (7) to substitute for \( \dot{z} \) and \( \ddot{z} \) in eqs. (5) and (15) allows the problem to be formulated in terms of the coefficients \( A_{rs} \) of eq. (17). For the general problem, the three-dimensional form of the constitutive relationships must be incorporated.

By using eq. (17), the number of unknowns in the problem has been increased from \( n \), the number of \( a_r \) functions, to \( n(m+1) \), while the number of available equations remains at \( n \). It is therefore necessary to generate additional equations to form a solvable set. These are produced by selecting \( (m+1) \) distinct values of \( t' \) in the range \( 0 \leq t' \leq \infty \). Of these, the two limiting values of \( t' \) correspond to the initial elastic solution at \( t' = 0 \) and the steady-state stresses \( [2] \) at \( t' = \infty \). It is necessary therefore to select only \( (m-1) \) values of \( t' \) to obtain a solution. The method by which a solution is achieved is demonstrated in the worked example of the following sections.

3. Application of theory to concrete structures.

The creep behaviour of concrete is characterised by a creep rate which decreases continually with time under stress and in this respect differs markedly from the linear, with respect to time, creep rate of eq. (1). However, when the non-recoverable component of creep of concrete is normalised with respect to stress and temperature\(^1\) and a new time parameter is defined as the normalised creep itself and termed here pseudo-time, the creep behaviour of concrete then appears in a form which is consistent with the theory of the previous sections. The delayed elastic strain is then related also to pseudo-time in the analysis. Its properties are time-invariant with respect to the new time parameter and the normal ageing which occurs in relation to real time is taken account of automatically by the time transformation. In many practical situations it will be satisfactory to include the delayed elastic component of strain as an instantaneous modification to the modulus term of eq. (2).

This is possible because at raised temperature the temperature-dependent flow portion of the time-dependent strain behaviour dominates over the delayed strain component, which is
essentially temperature independent [3]. The effect of the latter component on the stress redistribution process in non-uniformly heated concrete structures is thus small.

4. Example problems

4.1. Stresses in flexurally restrained member with temperature crossfall

The first example relates to the stresses in a prestressed member and the redistribution which results from differential creep under the influence of a sustained temperature crossfall, figure 1b. This example, although simple, does approximate to the behaviour of many structural situations where by reason of structural or geometrical constraints in the statically indeterminate structure, curvatures are restrained. For this example three analyses have been carried out in order to determine the influence of the approximations made on the accuracy of the solution. Figure 2 shows the predicted variations of stress with pseudo-time for the example of figure 1 when a temperature crossfall of 40°C exists. Conversion to real time may be made by reference to figure 3 where creep data are displayed in the familiar logarithmic form. Table 1 compares the predictions made by the separate analyses:

(i) A numerical step-by-step solution for small time intervals
(ii) A direct calculation based on the optimization procedure of section 2, and incorporating two self-equilibrating stress distributions. The solution results from a pair of differential equations.
(iii) An approximate calculation based on the formulation of eq.(17) and resulting from a set of algebraic equations.

The analysis (i) is taken to provide an 'exact' solution for the purpose of comparing the more approximate solutions from (ii) and (iii). The method of obtaining a step-by-step solution in time is well understood and will not be described here.

4.1.1. Solution from differential equations

The stresses are represented in the form of eq.(6) as,

\[ \sigma = f + a_1X + a_2(x^2 - 1/3) \]  (18)

The first distribution satisfies the applied prestress and the two remaining sets of stresses in eq.(18) give rise to zero longitudinal force when integrated over the beam section of figure 1, between the limits of \( X = \pm 1 \).

Thus, introducing the stresses from eq.(18) into eq.(2), yields an expression for the longitudinal strain rates as,

\[ \dot{e} = \frac{1}{E} \left[ a_1X + a_2(X^2 - 1/3) \right] + (\alpha + SX) \left[ f + a_1X + a_2(x^2 - 1/3) \right] \]  (19)

In the above equation for \( \dot{e} \) it has been assumed that the temperature-creep function \( \phi(T) = T \), and \( T = \alpha + SX \).
Introducing the virtual power equation (5) and substituting for $\delta a_{ij}$, as $X$ and $(X^2 - 1/3)$, the following equations result,

$$\int_{-1}^{1} \epsilon X dX = 0$$
$$\int_{-1}^{1} \epsilon (X^2 - 1/3) dX = 0$$

(20)

Substitution of $\epsilon$ from eq. (19) into eq. (20) leads to the following matrix equation in $a_1$ and $a_2$, after the spatial integrations of eq. (20) have been performed.

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + E \begin{bmatrix} \alpha & \frac{\beta}{15} \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = F$$

(21)

Equation (21) may be solved readily in closed form, but here has been solved using the numerical procedure described earlier. The results are listed in Table 1.

4.1.2. Solution from a set of algebraic equations

The functions $a_1$ and $a_2$ are specified in the form,

$$a_1 = A_{10} + A_{11} e_{1}^{\lambda_1 t'} + A_{12} e_{2}^{\lambda_2 t'} + A_{13} e_{3}^{\lambda_3 t'}$$

$$a_2 = A_{20} + A_{21} e_{1}^{\lambda_1 t'} + A_{22} e_{2}^{\lambda_2 t'} + A_{23} e_{3}^{\lambda_3 t'}$$

(22)

and the expression for the strain rate becomes,

$$\dot{\epsilon} = fT + A_{10} X + A_{20} (X^2 - 1/3) + \sum_{r=1}^{r=3} \left(T - \frac{\lambda r}{E} \right) \left[ A_{1r} X + A_{2r} (X^2 - 1/3) \right] e^{\lambda r t'}$$

(23)

After applying the variational principle of eq. (5) and carrying out the integrations over the depth of the beam, the following two equations are obtained.

$$\left[ \beta f + a A_{10} + \frac{48}{15} A_{20} \right] + \sum_{r=1}^{r=3} \left( a - \frac{\lambda r}{E} \right) A_{1r} e^{\lambda r t'} + \frac{48}{15} \sum_{r=1}^{r=3} A_{2r} e^{\lambda r t'} = 0$$

$$\left[ \beta A_{10} + a A_{20} \right] + \sum_{r=1}^{r=3} A_{1r} e^{\lambda r t'} + \sum_{r=1}^{r=3} \left( a - \frac{\lambda r}{E} \right) A_{2r} e^{\lambda r t'} = 0$$

(24)

Because eqs. (24) are valid for all $t'>0$, it follows that $t'\rightarrow\infty$ requires the terms in the square brackets to be zero. This is the long-term steady-state solution and corresponds to,

$$A_{10} = \frac{-f \alpha \beta}{a^2 - 48 \frac{2}{15}}; \quad A_{20} = \frac{f \beta^2}{a^2 - 48 \frac{2}{15}}$$

(25)

Eq. (24) then reduces to,
\[
\sum_{i=1}^{k} \frac{\lambda}{E} A_i e^{-\lambda t_i} + \frac{4\beta}{15} - 1 = 0
\]

The solution of the elastic problem, corresponding to \(t' = 0\), is obtained from the complementary energy statement, which for this example, is,

\[
\int \epsilon \sigma dV = 0
\]

in which the strain, \(\epsilon\), is represented by,

\[
\epsilon = \frac{\sigma}{E} - \alpha (T - T_0)
\]

where, \(\alpha\) is the coefficient of thermal expansion for the concrete and \((T - T_0)\) is the temperature rise. When the integrations of eqs.(27) are carried out, two further equations are generated; these are,

\[
\sum_{i=1}^{k} A_i \epsilon = E\alpha \beta; \quad \sum_{i=1}^{k} A_i \lambda = 0
\]

Because eqs.(26) and (29) contain six unknowns it is necessary to generate four equations from eqs.(26) in order to produce a solvable set. It is thus necessary to select two values of pseudo-time, \(t'\), within the range \(0 < t' < \infty\).

Experience has shown that the majority of the redistribution of stress takes place while \(t'\) is within the range \(0 < t' < 10.10^{-9}\) in/in per 1b/in² per °C, and that the exponents, \(\lambda\), are of the order ET. With the additional knowledge that the stress redistribution takes place faster close to zero time the following values of pseudo-time and \(\lambda\) have been selected,

\[
t'_1 = 2.10^{-9}; \quad t'_2 = 5.10^{-9}
\]

\[
\lambda_1 = 100.10^6; \quad \lambda_2 = 300.10^6; \quad \lambda_3 = 500.10^6
\]

The stresses which have been deduced from the six algebraic equations are listed in Table 1 alongside values from the differential equations and the step-by-step numerical analysis.

4.2. Containment vessel

The second illustrative example relates to a prestressed containment structure which has been analysed by a step-by-step approach and the variational method of section 2. In this example, the effectiveness of the method has been checked in relation to a first stage - albeit crude - approximation to the time-dependent state of the stresses. Two self-equilibrating distributions have been incorporated and these were derived in the following manner.
(i) The thermal stresses corresponding to the actual temperature distribution were calculated from a finite element analysis.

(ii) The limiting stress distribution caused by creep was computed from a standard elastic analysis using the correspondence that exists between the normal elastic modulus and the reciprocal of the temperature-creep function in the determination of steady-state stresses for the creep problem. The difference between these stresses and those of the initial elastic solution at \( t' = 0 \), represents the second self-equilibrating distribution.

Figure 4 shows the sections through the vessel analysed at which stresses have been determined and compared in figures 5 and 6 for a selection of times up to 10,000 days.

5. Discussion and concluding remarks

The results of table 1 and figures 5 and 6 indicate the ability of the approximate solution techniques to produce results of acceptable engineering accuracy. The errors in the stress predictions for the containment structure — based on two self-equilibrating stress distributions only — are illustrated in figures 5 and 6 by the shaded areas. These errors, although perhaps unacceptable in a final design calculation are acceptable for assessing the early features of a new design, for which computational costs need to be kept to a minimum. The method reveals satisfactorily the stress redistribution features associated with creep at non-uniform temperatures. Greater accuracy is readily achieved by incorporating additional self-equilibrating distributions of stress into the analysis.

Numerical step-by-step solutions to creep problems of real and complex structures are costly. The method outlined here allows approximate solutions to be obtained at a cost little greater than is required for the elastic stress analyses of the same problem.

6. Acknowledgement

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7. References


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Stresses in lb/in², calculated from three separate analyses,

(a) Exact solution from a step-by-step analysis, Col.'Ex'.
(b) Solution from approximate analysis and differential equations.
(c) Solution from approximate analysis and algebraic equations.
Fig. 1. (a) Continuum body and equilibrium force system
(b) Flexurally restrained beam subjected to sustained temperature crossfall and axial prestress of 2000 lb/in².

Fig. 2. Variation of concrete stress with time, due to creep, for flexurally restrained beam.

Fig. 3. Normalised creep data, $t'$, plotted against real time.

Fig. 4. Prestressed concrete containment vessel analysed by optimization procedure.
Fig. 5. Comparison of vessel stresses for sections illustrated in figure 4. Shaded areas represent errors between exact and approximate solutions.
(a) Circumferential stresses on section C-C
(b) Circumferential stresses on section B-B
(c) Radial stresses on section B-B

Fig. 6. Comparison of vessel stresses on section A-A of figure 4. Shaded areas represent errors between exact and approximate solutions.
(a) Longitudinal stresses
(b) Circumferential stresses