STRUCTURAL RESPONSE OF REINFORCED CONCRETE SLABS
TO IMPULSIVE LOADING

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SUMMARY

Structures subjected to localized dynamic loading may suffer only minor local damage
but still collapse by overall structural response. This situation can arise when the structure
is struck by crushable missiles, massive low velocity missiles, or penetrating but nonper-
forating missiles. Such structural response problems are of interest in the design of con-
tainment structures of nuclear reactor power plants.

The structure treated here is a clamped circular slab of reinforced concrete. The loading
is a rectangular pulse uniformly distributed over a central circular area. The practical value
of this problem is that it probably represents a most severe loading case for bending
response among more realistic cases, because it replaces the local loaded area with a cir-
cular area at the slab center, and because it replaces the pulse with a rectangular pulse
of the same peak pressure and impulse.

In the theoretical treatment the pulse is assumed to produce plastic deformations large
enough to neglect elastic deformation but small enough to neglect membrane action.
Yielding of the reinforced concrete slab is assumed to be governed by the Johansen cri-
teron and the associated flow rule. For simplicity, the analysis is restricted to isotropic
slabs with top and bottom steel reinforcement arranged to provide the same yield moment
magnitude for positive and negative curvature changes.

A consequence of the assumed rigid-perfectly plastic behavior is that the deformation
modes may be considered as simple mechanism governed by a yield circle. Moreover, the
yield circle is stationary while the constant pressure is being applied and expands to the
support once the pressure is removed. After the yield circle has arrived at the support,
the remaining deformation occurs in the static collapse mode.

The principal results are explicit simple formulas for permanent central deflection in
terms of pressure, duration, loaded area radius, and plate properties (radius, density, yield
moment). Convenient dimensionless forms of these formulas are given in terms of only
two parameters, a pressure parameter $\lambda = p_d/p_s$ and an area parameter $\alpha = a/R$. In these
parametric expressions, $p_d$ is the pressure applied uniformly over a central circular area of
radius $a$, and $p_s$ is the static collapse pressure acting on the same circular area of a clamped
circular slab of radius $R$.

The formulation is restricted here to ranges of the parameters $\lambda$ and $\alpha$ believed prac-
tical for impact calculations. A more general formulation is presented elsewhere. An ex-
ample is presented in which the analysis is adapted for calculating the central slab def-
lection caused by central impact of a crushable missile (utility pole).
1. **Introduction**

Structures subjected to localized dynamic loading may suffer only minor local damage but may still collapse as a result of overall structural response. Such collapse can occur when the structure is struck by crushable missiles, massive low velocity missiles, or penetrating but nonperforating missiles. Such structural response problems are pertinent to the design of containment structures of nuclear reactor power plants.

The structure treated here is a clamped circular slab of reinforced concrete. The loading is a rectangular pulse uniformly distributed over a central circular area (Figure 1). This problem probably represents the most severe loading case for bending response among more realistic cases, because it replaces the local loaded area with a circular area at the slab center and replaces the pulse with a rectangular pulse of the same peak pressure and impulse.

In the theoretical treatment of the problem, the pulse is assumed to produce plastic deformations large enough to neglect elastic deformation but small enough to neglect membrane action. Yielding of the reinforced concrete slab is assumed to be governed by the Johansen criterion and the associated flow rule (Figure 2). For simplicity, the analysis is restricted to isotropic slabs with top and bottom steel reinforcement arranged to provide the same yield moment magnitude for positive and negative curvature changes.

A consequence of the assumed rigid-perfectly plastic behavior is that the deformation modes may be considered as simple mechanisms governed by a yield circle (Figure 3). Moreover, the yield circle is stationary while the constant pressure is being applied and expands to the support once the pressure is removed. After the yield circle has arrived at the support, the remaining deformation occurs in the static collapse mode.

The principal results are explicit simple formulas for permanent central deflection in terms of pressure, duration, loaded area radius, and plate properties (radius, density, yield moment). Convenient dimensionless forms of these formulas are given in terms of only two parameters, a pressure parameter \( \lambda = \frac{p_d}{p_n} \) and an area parameter \( \alpha = a/R \). In these parametric expressions, \( p_d \) is the pressure applied uniformly over a central circular area of radius \( a \), and \( p_n \) is the static collapse pressure acting on the same circular area of a clamped circular slab of radius \( R \).

Here, the formulation is restricted to ranges of the parameters \( \lambda \) and \( \alpha \) believed practical for impact calculations. A more general formulation is presented elsewhere [1]. An example is presented in which the analysis is adapted for calculating the central slab deflection caused by central impact of a crushable missile (utility pole).

2. **Deformation Mechanisms**

Figure 3 shows two of the three mechanisms [1] postulated to form immediately after the pressure is applied, when the diameter of the loaded circle lies in the range \( 0 \leq \alpha \leq \hat{\alpha} \).

Figure 3a shows mechanism 2A with a hinge circle of dimensionless radius \( r_1 = \frac{r_1}{R} \) within the plate that forms when the pressure parameter lies in the range \( \lambda_1 \leq \lambda \leq \lambda_2 \). Figure 3b shows mechanism 1B with a hinge circle at the support that forms when \( \lambda \) lies in the range \( 1 \leq \lambda \leq \lambda_{1B} \). The limiting area parameter, \( \hat{\alpha} = 0.63 \), and the pressure parameter ranges are determined in the analysis. The third initial mechanism, which applies when \( \lambda > \lambda_{2B} \) and has two hinge circles, is not treated here, because the extremely high pressures would probably cause local damage. However, this case is treated in reference 1.
As indicated in Figure 3, the mechanisms are based on the use of plastic regimes A, AB, and B of the Johansen yield square in Figure 2. The flow rule associated with line AB requires no change of the radial component of curvature \( \kappa_r = 0 \). Consequently, the velocity field has the spatially linear form

\[
\dot{w}(r,t) = f(t) r + g(t)
\]

where \( w \) is the plate deflection and the dot above it denotes time differentiation. Hinge circles are represented by the corner regime B in Figure 2; the plate center is represented by the corner regime A.

2.1 Mechanism 2B: \( \lambda_{1B} \leq \lambda = \lambda_{2B} \)

The plate is partitioned into two regions by a hinge circle \( r_1 \) that remains stationary while under a constant pressure \( \lambda \). The inner region deforms as a cone in accord with eq. (1) while the outer region remains at rest. The velocity field meeting this physical description is

\[
\dot{w} = \begin{cases} 
V \left( 1 - \frac{r}{r_1} \right) & 0 \leq r \leq r_1 \\
0 & r_1 \leq r \leq R 
\end{cases}
\]

where \( V = V(t) \) is the central plate velocity. At the plate center and hinge circle, the radial components of the bending moment are \( M_r = M_p \) and \( M_r = -M_p \), where \( M_p \) is the fully plastic moment per unit arc length, corresponding to plastic regimes A and B in Figure 2.

In the region \( 0 \leq r \leq r_1 \), the radial moment lies in the range \( -M_p \leq M_r \leq M_p \), corresponding to regime AB. Also, the circumferential moment is \( M_\theta = M_p \) throughout. In the outer region \( r_1 \leq r \leq R \), the moments are \( M_r = M_p \) and \( M_\theta = M_p \), corresponding to regime B.

When the pressure is removed at time \( t_d \) (Figure 1b), the hinge circle expands to the support and the constant \( r_1 \) is replaced by \( r_1(t) \) in the velocity field (2) to give

\[
\dot{w} = \begin{cases} 
V \left( 1 - \frac{r}{r_1(t)} \right) & 0 \leq r \leq r_1(t) \\
0 & r_1(t) \leq r \leq R 
\end{cases}
\]

After the hinge circle arrives at the support, the remaining deformation occurs by mechanism 1 (Figure 3b with \( P_d = 0 \)).

The upper value \( \lambda_{2B} \) of the pressure parameter range is the value above which the applied pressure generates a second hinge circle \( r_2 \) within circle \( r_1 \). When \( \lambda = \lambda_{2B} \), the location is \( r_2 = 0 \), at the plate center, and \( \lambda_{2B} \) is found from the deflection solution by setting \( \partial^2 M_r / \partial r^2 = 0 \). The lower value \( \lambda_{1B} \) is the value of \( \lambda \) that gives \( r_1 = R \).

2.2 Mechanism 1: \( 1 \leq \lambda = \lambda_{1B} \)

In this mechanism, shown in Figure 3b, the entire plate deforms as a cone having the velocity field

\[
\dot{w} = V \left( 1 - \frac{r}{R} \right) \quad 0 \leq r \leq R
\]
At the plate center and support, the radial moments are \(M_r = M_p\) and \(M_r = -M_p\), corresponding to regimes A and B in Figure 2. The circumferential moment is \(M_0 = M_p\) throughout.

3. **Solution**

With the aid of Figure 4, the equations of translational and rotational motion,

\[
\frac{\partial}{\partial t} (rQ) = r \left(p - \frac{m^2 \omega}{\partial t^2}\right)
\]

(5)

and

\[
\frac{\partial}{\partial t} \left(rM_r\right) - M_0 + rQ = 0
\]

(6)

for a plate element can readily be derived (rotary inertia neglected); \(m\) is the areal mass density. Eqs. (5) and (6) may be combined by eliminating the shear force \(Q\) to give

\[
\frac{\partial}{\partial t} \left(rM_r\right) = M_p - \int_0^r \left(p - \frac{m^2 \omega}{\partial t}\right) r'dr'
\]

(7)

where the yield condition requirement \(M_0 = M_p\) throughout the plate has been used. In carrying out spatial integrations, the following radial moment conditions are used:

\(-M \leq M \leq M_p\) inside the hinge circle \(\bar{r}_1\) or \(r_1(t)\), \(M_r = M_p\) at the plate center \(r = 0\), 
\(M_r = -M_p\) at and outside the hinge circle and at the support.

Before solving the dynamic problem, the static collapse pressure, \(p_\infty\), is obtained by integrating eq. (7) without the inertial term. Static collapse occurs by mechanism 1, which has the velocity field (4). Integration of eq. (7), with the foregoing moment distribution applied to mechanism 1 leads to

\[
p_\infty = \frac{4M_p}{p_\infty} \frac{R^2}{a^2} \left(1 - 2a/1\right)
\]

(8)

3.1 **Mechanism 2B**: \(\lambda_{1B} \leq \lambda \leq \lambda_{2B}\)

3.1.1 **Phase 1**: \(p = p_d, 0 \leq t \leq t_d\).

During this constant pressure loading phase, the velocity field is given by eq. (2), where the hinge location \(\bar{r}_1\) is constant. Substitution of eq. (2) in eq. (7), spatial integration, and use of the radial moment conditions give

\[
4a/3\bar{r}_1 = 1 - (1 - 2a/1)/\lambda
\]

(9)

for the hinge location \(\bar{r}_1\) and

\[
V_d = 4p_d t_d (a/\bar{r}_1)^3/m \quad W_d = 2p_d t_d (a/\bar{r}_1)^3/m
\]

(10)

for the central velocity and deflection at time \(t_d\).

The smallest pressure for deformation by mechanism 2B produces a hinge circle at the support. Hence, substitution of \(p_\infty = 1\) in eq. (9) gives

\[
\lambda_{1B} = (1-2a/3)/(1-4a/3)
\]

(11)

Spatial integration of eq. (7) gives the radial moment

\[
M_r = M_p - \frac{p_d^2}{6} + m\bar{r}_1^2 (1-r/\bar{r}_1)/6 \quad 0 \leq r \leq a
\]
which gives at \( r = 0 \)
\[
\frac{3^2 M}{2 r^2} = \frac{1}{3} (a \gamma - p_d) = p_d \left[ \frac{4}{3} \left( \frac{\gamma}{p_1} \right)^3 \right] - \frac{1}{3}
\]

Equating this expression with zero and substituting eq. (9) leads to the largest pressure for deformation by mechanism 2B:
\[
\lambda_{2B} = \frac{(1-2a/3)}{1-4\alpha/3}
\]
where \( \alpha^3 = 1/4 \). If the loaded area parameter also gives \( p_1 = 0 \), then \( a = \bar{a} \) and mechanism 2B requires \( 0 < a < \bar{a} \).

3.1.2 Phase 2: \( p = 0 \quad t_d < t \leq t_{ba} \)
Removal of the pressure sets the hinge circle in motion toward the support, and the velocity field becomes eq. (3). Substitution of eq. (3) in eq. (7), spatial integration, and the moment conditions give equations
\[
\begin{align*}
\dot{V}_p^2 + V(p_1^2)^2 &= -3a^2(1-2a/3)(p_d/m)/\lambda \\
2\dot{V}_p^2 + 3V(p_1^2)^2 &= 0
\end{align*}
\]
for governing the hinge and plate motion. The solution of eqs. (13) and (14), satisfying the hinge and velocity conditions eq. (9) and eq. (10) at time \( t_d \), may be expressed in the form
\[
4a/3p_1 = 1 - (1-2a/3)(t/t_d)/\lambda
\]
\[
V = (p_1/p_1)^3 V_d
\]
Mechanism 2B ends when the hinge circle reaches the support, that is, when \( p_1 = 1 \).

If this event occurs at time \( t_{ba} \), eqs. (15) and (16) give
\[
t_{ba}/t_d = \lambda_{1B}/\lambda \\
V_{ba} = p_1^3 V_d
\]
and temporal integration of eq. (16) gives the central deflection
\[
W_{ba} = W_d + (p_d t_d^2/2m) 8a^4 \lambda (1/p_1^4-1)/3(1-2a/3)
\]

3.1.3 Phase 3: \( p = 0 \), \( t > t_{ba} \).
The remaining deformation occurs by mechanism 1, which has the velocity field of eq. (4). Substitution of eq. (4) in eq. (7), spatial integration, and the moment conditions give the equation
\[
V = V_{ba} - 6a^2(1-2a/3)(p_d/m)(t-t_{ba})/\lambda
\]
for the central plate velocity satisfying the velocity condition eq. (17) at time \( t_{ba} \). If \( t_f \) is the time when motion ceases, substitution in eq. (19) of \( V(t_f) = 0 \), \( V_{ba} \) and \( t_{ba} \) from eq. (17), and \( V_d \) from eqs. (10) and (9), lead to the duration of motion
\[
t_f/t_d = \lambda
\]
Temporal integration of eq. (19) from \( t_{ba} \) to \( t_f \) results in the additional deflection
\[
W_f - W_{ba} = (p_d t_d^2/2m) 8a^4 \lambda/3(1-2a/3)
\]
Deflection eqns. (10), (18), and (21) then give the final central deflection as
\[ W_f = \frac{(p_d t_d^2/m)}{2(a/\pi_1)^3[1+\lambda/(1-2\alpha/3)]} \] (22)

3.2 Mechanism 1: \( 1 \leq \lambda \leq \lambda_{1B} \)
3.2.1 Phase 1: \( p = p_d \ 0 \leq t \leq t_d \).

Substitution of the velocity field eq. (4) in eq. (7), spatial integration, and satisfaction of the moment conditions, give
\[ V_d = 6a^2(1-3a/2)1-1/\lambda) p_d t_d /m \] (23)
\[ W_d = 6a^2(1-3a/2)(1-1/\lambda) p_d t_d^2 /2m \] (24)

for the central plate velocity and deflection at time \( t_d \).

3.2.2 Phase 2: \( p = 0 \ 0 > t_d \)

Continuation of the analysis but with \( p = 0 \) gives the central plate velocity.

If motion ceases at time \( t = t_f \), this velocity expression gives
\[ t_f/t_d = \lambda \] (25)

which is the same formula as eq. (20) for the duration of motion. Temporal integration of the velocity expression from \( t_d \) to \( t_f \) along with eq. (24) leads to the final central deflection
\[ W_f = \frac{(p_d t_d^2/2m)}{6a^2(1-2a/3)(\lambda-1)} \] (26)

4. Soft Missile Impact

4.1 Impact Model

As an example of pulse loading over a small central area of the circular plate, consider the crushable cylindrical missile of Figure 5 impacting normally with a velocity \( u_1 \). A simple approximate description of the impact is afforded by assuming an incompressible rigid-plastic behavior for the missile (Figure 6) and a fixed rigid surface for the plate. This description is a special case of that described in reference [2] treating impact of steel cylinders against rigid targets.

On impact, a plastic wave front forms and travels with a velocity \( v \) away from the impact surface, leaving the material behind it at rest. The rigid back part of the rod originally of length \( L \) and currently of length \( x \) approaches the plate with a velocity \( v \). At the plastic wave front, \( A \) and \( \sigma \) are the area and nominal (engineering) stress behind it and \( A_0 \) and \( \sigma_0 \) are the values in front of it. The stress \( \sigma_0 \) is the yield stress because the material in front of the plastic wave front is about to become plastic.

Conservation of mass and momentum across the wave front requires
\[ (u + v) A_0 = \rho A \quad \rho(u + v) A_0 u = A_0 (\sigma - \sigma_0) \]
where \( \rho \) is the mass density. The kinematic condition and equation of motion for the undeformed portion of the missile are
\[- \frac{dx}{dt} = u + v \quad \rho x \frac{du}{dt} = -\sigma_0 \]

The linear strain hardening law for the missile material (Figure 6) taken to approximate the constitutive equation is
\[ \sigma = \sigma_0 + \sigma_h e \]

where \( e \) and \( \sigma_h \) are the strain and strain-hardening modulus. Immediately behind the plastic wave front, the strain is

\[ e = (A - A_0)/A \]

The above six governing equations lead to the results:

- Initial strain \( e_1 = \frac{u_1}{c_p} \)
- Initial stress \( \sigma_1 = \sigma_0 + \sigma_h e_1 \)
- Uncrushed length \( x_d = L e^{-\frac{(\sigma_1 - \sigma_0)}{\sigma_0}} \)
- Impact duration \( t_d = \frac{L}{c_p} \left[ 1 - \frac{1}{e^{\frac{(\sigma_1 - \sigma_0)}{\sigma_0}}} \right] \)
- Stopping distance \( S_d = e_1 L - \frac{\sigma_0 (L - x_d)}{\sigma_h} \)
- Plastic wave velocity \( c_p = \left( \frac{\sigma_h}{\rho} \right)^{\frac{1}{2}} \)

### 4.2 Example

As an example consider the normal impact of a utility pole with the following specifications:

<table>
<thead>
<tr>
<th>Specification</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>44 ft</td>
</tr>
<tr>
<td>Diameter</td>
<td>13.5 in.</td>
</tr>
<tr>
<td>Density</td>
<td>34 lb/ft³</td>
</tr>
<tr>
<td>Weight</td>
<td>1500 lb</td>
</tr>
<tr>
<td>Crush (yield) stress</td>
<td>4000 lb/in²</td>
</tr>
<tr>
<td>Hardening modulus</td>
<td>1000 lb/in²</td>
</tr>
<tr>
<td>Impact velocity</td>
<td>350 ft/sec</td>
</tr>
</tbody>
</table>

The formulas of Section 4.1 then give the following results:

- Plastic wave velocity \( c = 11,300 \text{ cm/sec} \) (370 ft/sec)
- Initial strain \( e_1 = 0.95 \)
- Initial stress \( \sigma_1 = 4950 \text{ lb/in}^2 \)
- Uncrushed length \( x_d = 34.7 \text{ ft} \)
- Impact duration \( t_d = 25 \text{ msec} \)
- Stopping distance \( S_d = 4.6 \text{ ft} \)

The force corresponding to a uniformly distributed contact pressure of 4950 psi is 708,000 lb. For a cylinder of a diameter \( D = 13.5 \text{ in.} \), impacting a slab of depth \( H = 18 \text{ in.} \), a reasonable effective contact area has a radius of \( a = (H + D)/2 = 15.75 \text{ in.} \), (area of 780 in²) resulting in an effective contact pressure of \( p_d = 908 \text{ psi} \). For a slab with a plastic moment of \( M_p = 25,000 \text{ psi} \), the static collapse pressure for \( a = 15.75 \text{ in.} \) \( (a = a/R = 0.13) \), according to eq. (8), is \( p_g = 443 \text{ psi} \). The value of the pressure parameter is then \( \lambda = \frac{p_d}{p_g} = 2.05 \). Also, for \( a = 0.13 \), the pressure range for initiation of mechanism 2B is \( 1.1 < \lambda < 5.7 \), for which the appropriate deflection formula is eq. (22) with \( a/R_1 \) given by eq. (9).
Substitution of $a = 0.13, \lambda = 2.05$ in eq. (9) gives $\alpha/\beta_1 = 0.42$. The deflection of a free plate element of areal mass $m$ at time $t_d$ caused by a constant pressure $p_d$ is $p_d \frac{t_d^2}{2m}$. For $p_d = 908$ psi, $t_d = 23$ msec, and an 18-inch-thick slab, $p_d \frac{t_d^2}{2m} = 71$ in. The final deflection is therefore $W_f = 34$ in. If the slab has a second moment of area $I = 55$ in.$^4$/in., the stiffness is $k = 0.69 \times 10^6$ lb/in. and the natural period of the equivalent single-degree-of-freedom system [3] is $T = 2\pi \sqrt{M/k6(1-2a/3)} = 108$ msec, where $M$ is the plate mass; to find the formula for $T$, the fundamental plastic mode was used (mechanism 1). The resulting design parameters are $C_R = 1/\lambda = 0.49$ and $C_T = t_d/T = 0.23$, giving a low ductility ratio of $\mu \approx 4.0$. The ductility ratio corresponding to deflection eq. (22) is $\mu \approx 7.1$

5. Conclusions

The rigid-plastic theory has been developed as a preliminary structural design aid for missile-plate impact problems. The method is potentially useful where structural modes higher than the fundamental plastic mode are excited, where the equivalent single-degree-of-freedom or resistance-function method is too approximate. The usefulness of the method is currently being assessed by comparison with finite-element code predictions. If the method is promising other more practical structures, such as rectangular plates, will be investigated. If possible, predictions will be compared with experimental results.

References


1. Circular Plate Problem

2. Johannsen's Yield Criterion

3. Initial Mechanisms
4. Plate Element with Forces and Moments

5. Crushable Cylinder Striking Rigid Surface

6. Assumed Stress-Strain Law for Cylinder