STRONG GROUND MOTION SPECTRA FOR LAYERED MEDIA

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SUMMARY

This article presents an analytic method and calculations of strong motion spectra for the energy, displacement, velocity and acceleration based on the physical and geometric ground properties at a site.

Although earthquakes occur with large deformations and high stress intensities which necessarily lead to nonlinear phenomena, most analytical efforts to date have been based on linear analyses in engineering seismology and soil dynamics. There are, however, a wealth of problems such as the shifts in frequency, dispersion due to the amplitude, the generation of harmonics, removal of resonance infinities, which cannot be accounted for by a linear theory. In the study, the stress-strain law for soil is taken as \( \tau = G_0 \gamma + G_1 \gamma^3 + \eta \dot{\gamma} \) where \( \tau \) is the stress, \( \gamma \) is the strain, \( G_0 \) and \( G_1 \) are the elasticity coefficients and \( \eta \) is the damping and are different in each layer. The above stress-strain law describes soils with hysteresis where the hysteresis loops for various amplitudes of the strain are no longer concentric ellipses as for linear relations but are oval shapes rotated with respect to each other similar to the materials with the Osgood-Ramberg law. It is observed that even slight nonlinearities may drastically alter the various response spectra from that given by linear analysis. In fact, primary waves cause resonance conditions such that secondary waves are generated. As a result, a weak energy transfer from the primary to the secondary waves takes place, thus altering the wave spectrum. The mathematical technique that is utilized for the solution of the nonlinear equation is a special perturbation method as an extension of Poincare's procedure. The method considers shifts in the frequencies which are determined by the boundedness of the energy. The ratio \( G_1/G_0 \) (bedrock displacement/layer thickness)\(^2\) is used as the perturbation parameter.

First the general solution of shear waves in a typical layer is obtained. Based on this latter, transfer matrices for the various harmonics of the motion are constructed by relating the displacement and the stress at one face to those at the other. With the use of these transfer matrices, the solution of a multilayer system is obtained by requiring continuity in the nonlinear displacement and nonlinear stress.

Results are presented for case studies in 1, 2 and 3 layers as well as for actual data for a site with 5 layers at Las Vegas. Important changes in the spectra of weak motions (or microtremors) are observed for various amplitudes of strong motions. It is also observed that the nonlinear effects are more pronounced on the top layer. This latter has a much lower shear modulus as compared to the lower layers and thus undergoes higher strains. The formalism here in combination with a finite element method is readily extended to apply to irregular geometries in tow dimensions.

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1. Introduction

There have been efforts to establish the relation between the magnitude of an earthquake and the displacement at the source fault [1]. This displacement at the source determines in turn the amplitudes of the waves generated and consequently the amplitude of the motion at the bed rock is dependent on the magnitude of an earthquake. For strong ground motions the strains are large enough so that a linear analysis is insufficient. There is a wealth of phenomena such as shifts in frequency, dispersion due to amplitude, generation of harmonics, removal of resonance singularities, jump from one state to another which are primarily nonlinear in nature [2]. The higher harmonics generate a resonance condition which results in an energy flow from the fundamental mode. This causes important changes in the response spectra associated with the dynamics of the system. This paper studies the aforementioned phenomena as they pertain to the forced shear oscillations of ground layers with a nonlinear stress-strain law of polynomial type as derived from the Ramberg-Osgood relation [3]. The goal here is to account for the dependence of the response spectrum of layered soil deposits on the amplitude of strong ground motions at the bed rock.

The basic idea is an extension of Poincare's method for the nonlinear oscillator along the lines in [4, 5] and that of using the averaged energy (or work done in a cycle) which is an invariant of the dynamics.

2. Strain-Stress Relation

Several investigators have shown that the relationship between shear modulus and strain is nonlinear and that once a threshold strain is exceeded the modulus is a decreasing function of strain [3]. Among these we adopt the Ramberg-Osgood constitutive relation

$$G/G_{\text{max}} = [(1+\alpha(\tau/\tau_y)^R)^{-\frac{1}{R}}]^{-1}$$

where $\tau$ = shearing stress; $\tau_y$ = yield or reference shearing stress; and $\alpha$ and $R$ are parameters which determine the shape of the curve. It is found that for a large variety of soils the Ramberg-Osgood relationship fits the data quite reasonably with $\alpha = 1$, $R = 3$ and $\tau_y = 0.4$ $S_u$ where $S_u$ is the undrained shearing strength [3].

The strain-stress relationship of Ramberg-Osgood with $\alpha = 1$ and $R = 3$ reads:

$$\frac{\tau}{\tau_y} = \frac{G_{\text{max}}}{\tau_y} \left[1 - (\frac{\tau}{\tau_y})^2\right]$$

This "form" of the strain-stress relationship is quite unusual in continuum mechanics. To put this equation in a more conventional form, substitute $\tau/\tau_y$ iteratively to get

$$\frac{\tau}{\tau_y} = [G_{\text{max}}] / [1 - (\tau/\tau_y)^2]$$

Neglecting $(\tau/\tau_y)^2$ on the right hand side and expanding the $(\tau^2)$ power by the binomial formula, as an approximation to the Ramberg-Osgood relationship, one obtains

$$\frac{\tau}{\tau_y} = G_{\text{max}} \left[1 - (\frac{\tau}{\tau_y})^2 + \frac{\tau}{\tau_y} \frac{\partial \tau}{\partial \tau} \right]$$

A simple study shows that the form of $\tau$ as proposed here is an excellent approximation of the Ramberg-Osgood relationship for a large range of strains corresponding to $\tau_y/G_{\text{max}} = 1$ while also being of convenient form for the applications.

The damping is introduced through a linear term in the strain rate as is experimentally suggested [6]. Thus one has:

$$\tau = G_{\text{max}} \left[1 - (\frac{\tau}{\tau_y})^2 \right] + \frac{\partial \tau}{\partial \tau} \gamma$$

(2)

The damping term introduces hysteresis into the strain-stress relationship. For the strain $\gamma = \gamma_0 \cos \omega t$, with the fundamental component of $\tau$ as $\tau_0 \cos \omega t$ one has: $\tau_0 = G_{\text{max}} \gamma_0$
\[
[1 - 3/4(G_{\text{max}} \gamma / \tau_y)^2] \cos wt - \omega \gamma \sin wt. \]

The compliance is obtained by substituting
\[
\cos wt = 1/2(e^{i\omega t} + e^{-i\omega t}) \quad \text{and} \quad \sin wt = 1/2(e^{i\omega t} - e^{-i\omega t})
\]
and defining \(G = \gamma / \tau_y\). Then, with
\(k = \tau / G_{\text{max}}\) as the relaxation time, for \(G = |G| e^{-i\delta}\)
\[
|G| = G_{\text{max}} [(1 - 3/4(G_{\text{max}} \gamma / \tau_y)^2 \gamma^2 \omega^2 \chi)^{1/2}]
\]
\[
\delta = \tan^{-1}(\omega \chi /[1 - 3/4(G_{\text{max}} \gamma / \tau_y)^2])
\]

It is seen that \(|G|\) decreases nonlinearly with the strain while the phase angle \(\delta\)
increases with it as is observed in the experiments [6]. In this study, the calculations
are presented first for the case without dissipation, i.e. with \(\zeta = 0\) and then for the case
with dissipation. In what follows, for brevity, use is made of the definitions:
\(G_o = G_{\text{max}}\) and \(G_1 = (G_{\text{max}} / \tau_y)^2 G_{\text{max}}\). Thus the nonlinear stress-strain relation in (3) yields the
field equation:
\[
\frac{\partial^2 v}{\partial x^2} + 3G_1 \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial t} \frac{\partial^2 v}{\partial x^2} = \rho \frac{\partial^2 v}{\partial t^2}
\]


Before going into the solution for a multilayer system, the method is introduced in
some detail for a single layer of thickness \(d\) which is forced sinusoidally with frequency \(\Omega\) at \(x = d\) and is traction free at \(x = 0\). The problem is defined by the field equation (4)
with \(\zeta = 0\) and the following boundary conditions at \(x = 0\) and \(x = d\) along with the periodic
initial conditions in time:
\[
\begin{align*}
\tau &= 0 & v &= v(\Omega t) & v(x, \Omega t) &= v(x, \Omega t + 2\pi) \\
x &= d & v &= v(\Omega t) & v(x, \Omega t) &= v(x, \Omega t + 2\pi)
\end{align*}
\]
Introducing the dimensionless variables, with \(v_o\) as a reference frequency, as
\[
y = x/d \quad n = v/v_o \quad s = \Omega t \quad q = v_o d / \sqrt{G_o v_o} \quad \lambda = 3/2 \alpha
\]
the Equations (5) and (6) yield:
\[
\frac{\partial^2 n}{\partial y^2} - q^2 \frac{\partial^2 n}{\partial y^2} + \frac{\partial n}{\partial y} + \frac{\partial n}{\partial y} = 0
\]
\[
\frac{\partial n}{\partial y} + \frac{1}{3} \lambda \frac{\partial^3 n}{\partial y^3} \bigg|_{y=0} = \frac{\partial n}{\partial y} \bigg|_{y=1} = \cos s \quad \eta(y, s) = \eta(y, s + 2\pi)
\]
Introducing the expansions with \(\lambda\) as the perturbation variable,
\[
\eta(y; \lambda) = \eta_o(y, s) + \lambda \eta_1(y, s) + \ldots
\]
\[
\Omega(\lambda) = \Omega_o + \lambda \Omega_1 + \ldots
\]
into (7) and separating terms for the various powers of \( \lambda \), one finds for the first two orders,

\[
\begin{align*}
\frac{\partial^2 \eta_0}{\partial y^2} - q^2 \frac{\partial^2 \eta_0}{\partial s^2} &= 0 \\
\frac{\partial \eta_0}{\partial y} |_{y=0} &= 0 \\
\frac{\partial \eta_0}{\partial y} |_{y=1} &= \cos s
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 \eta_1}{\partial y^2} - q^2 \frac{\partial^2 \eta_1}{\partial s^2} &= 2 \Omega_1 \frac{\partial}{\partial y} \left( \frac{\partial \eta_0}{\partial y} \right) - 2 \frac{\partial^2 \eta_0}{\partial y^2} \\
\frac{\partial \eta_1}{\partial y} |_{y=0} &= \eta_1 |_{y=0} \\
\frac{\partial \eta_1}{\partial y} |_{y=1} &= \eta_1 |_{y=1}
\end{align*}
\]

(9)

The periodic solutions for (9) are of the forms:

\[
\eta_0(y,s) = \eta_{01}(y) \cos s
\]

\[
\eta_1(y,s) = \frac{\Omega_1}{2} \eta_{111}(y) + \eta_{112}(y) \cos s + \eta_{13}(y) \cos 3s
\]

(10)

By the use of (10) in (9), one has:

\[
\eta_{10}'' + q^2 \eta_{01} = 0 \\
\eta_{111}'' + q^2 \eta_{111} = -2q^2 \eta_{01} \\
\eta_{112}'' + q^2 \eta_{112} = -(3/4)(\eta_{111}'' - 2 \eta_{111}) \\
\eta_{13}'' + 9q^2 \eta_{13} = -(1/4)(\eta_{01}'') \eta_{01}'
\]

(11)

In the expression of \( \eta_1 \) for the term in \( \cos s \), the separation of the term \( \Omega_1/\Omega_0 \)

though not necessary, proves convenient in the calculation of the frequency shift \( \Omega_1 \).

Once \( \eta_{01} \) is obtained, it is employed on the right hand side of the equations for \( \eta_{111}, \eta_{112}, \eta_{13} \). Thus one has:

\[
\begin{align*}
\eta_{01} &= \cos qy/\cos q \\
\eta_{111} &= q(A_1 \cos qy + A_2 y \sin qy)/\cos q \\
\eta_{112} &= (1/16) - \frac{q}{\cos q} (A_3 \cos qy + A_4 y \sin qy + A_5 \cos 3qy) \\
\eta_{13} &= (1/16) - \frac{q}{\cos q} (A_6 \cos qy + A_7 \sin 3qy + A_8 \cos 3qy)
\end{align*}
\]

(12)

where

\[
\begin{align*}
A_1 &= \sin q \\
A_2 &= -\cos q/\eta \\
A_3 &= -3/2(\sin q+4 \cos q)/q \\
A_4 &= 3/2 \cos q \\
A_5 &= 3/8 \cos q/\eta \\
A_6 &= 1/8 \cos q/\eta \\
A_7 &= -1/6 \cos q \\
A_8 &= (1/6 \sin 3q -1/4 \cos q)/\cos q
\end{align*}
\]

(13)

The frequency shift \( \Omega_1 \) is calculated from the energy averaged over a cycle which is an invariant of the dynamics. With the above nondimensionalizations the energy averaged over a cycle reads:

\[
\langle E \rangle = \mathcal{D} \omega_0 \left( \frac{\lambda}{4} \right)(1/2\pi) \left( \int_0^{2\pi} \int_0^1 \left[ \frac{1}{2} \left( \frac{\partial \eta_0}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial \eta_1}{\partial y} \right)^2 + \frac{1}{12} \lambda \left( \frac{\partial \eta_0}{\partial y} \right)^4 \right] dy ds \right)
\]

(14)
Introducing the perturbation expansions along with the time dependence as in (10) one finds:

\[
\frac{\bar{\Omega}_1}{\bar{\Omega}_o} = \left( \frac{I_3 + I_4}{I_1 + I_2} \right) \tag{16}
\]

with

\[
I_1 = \int_0^L \eta_{01}^2 \, dy \quad I_2 = \int_0^L \eta_{111} \, dy \quad I_3 = 2q^2 \int_0^L \eta_{01} \eta_{112} \, dy \quad I_4 = \frac{1}{12} \int_0^L (\eta_{13}^1)^2 \, dy \tag{17}
\]

Once the problem is solved in terms of the nondimensional variables, the physical quantities of interest can be easily derived. In particular, for the above non-dimensionalization the expression for the stress becomes:

\[
\tau_{01} = \sigma_0 \cos s + \left( \frac{\bar{\Omega}_1}{\bar{\Omega}_o} \right) (\tau_{111}, \tau_{112}, \tau_{13}) = \lambda G_o \frac{A}{d} \left( \eta_{111}, \eta_{112}, \frac{1}{4} \eta_{01}, \eta_{13}, \frac{1}{12} \eta_{01}^3 \right) \tag{18}
\]

In conclusion the energy spectrum as given by Eq. (15) is plotted in Fig. 1 for various values of \( (G_o / G_o)(A/d)^2 \).

4. Multi-Layer System

In the preceding section, the problem of a single layer has been formulated. In this section, the problem is being extended to a multi-layer system with \((N-1)\) layers and \(N\) interfaces where the displacements and stresses are taken to be continuous across the interfaces. The notation is presented in Fig. 2. The boundary conditions are prescribed on the 1st and \(N\)th face. For the \(k\)th layer, the upper face is the \(k\)th interface, and the lower face is the \((k-1)\)th interface. On the \(k\)th interface, the displacement and stress are labelled with the index \(k\) and are denoted respectively as \(v^k\) and \(T^k\). Conversely, the strain is discontinuous across the interfaces between the layers with different physical properties. The strain is labelled by the indices \(\varepsilon_L^k\) and \(\varepsilon_U^k\) for the lower and upper faces of the \(k\)th layer. Similarly, the nondimensional displacements are labelled as \(X_L^k\) and \(X_U^k\) for the lower and upper faces of the \(k\)th layer. With this
notation, the solutions of the field equations in (11) become:

\[ n_{01} = x_{L01} \cos qy + \frac{1}{q} \ y_{L01} \sin qy \]

\[ n_{111} = x_{L111} \cos qy + \frac{1}{q} \ y_{L111} \sin qy + [C_2 \ qy \ \sin qy + D_2 (qy \ \cos qy - \sin qy)] \]

\[ n_{112} = x_{L112} \cos qy + \frac{1}{q} \ y_{L112} \sin qy + [C_3 \ qy \ \sin qy + C_4 (\cos 3qy - \cos qy)] \]

\[ + D_3 (qy \ \cos qy - \sin qy) + D_4 (\sin 3qy - 3qy \ \sin qy)] \]

\[ n_{113} = x_{L113} \cos 3qy + \frac{1}{3} \ qy \ y_{L113} \sin 3qy + [C_5 (\cos qy - \cos 3qy) + C_6 qy \ \sin 3qy] \]

\[ + D_5 (qy \ \sin qy - \frac{1}{3} \ \sin 3qy) + D_6 (qy \ \cos 3qy - \frac{1}{3} \ \sin 3qy)] \]

where

\[ C_2 = -x_{L01} \]

\[ C_3 = \frac{3}{32} q^2 x_{L01} (x_{L01}^2 + y_{L01}^2 / q^2) \]

\[ C_4 = \frac{3}{128} q^2 x_{L01} (x_{L01}^2 - 3y_{L01}^2 / q^2) \]

\[ C_5 = \frac{1}{128} q^2 x_{L01} (x_{L01}^2 + y_{L01}^2 / q^2) \]

\[ C_6 = -\frac{1}{96} q^2 x_{L01} (x_{L01}^2 - 3y_{L01}^2 / q^2) \]

\[ D_2 = y_{L01} / q \]

\[ D_3 = -\frac{3}{32} q y_{L01} (x_{L01}^2 + y_{L01}^2 / q^2) \]

\[ D_4 = -\frac{3}{128} q y_{L01} (-3x_{L01}^2 + y_{L01}^2 / q^2) \]

\[ D_5 = \frac{1}{128} q y_{L01} (x_{L01}^2 + y_{L01}^2 / q^2) \]

\[ D_6 = -\frac{1}{96} q y_{L01} (-3x_{L01}^2 + y_{L01}^2 / q^2) \]

In non-dimensional units, the displacements and strains on the upper face are obtained by setting \( y = q \) in the \( \eta \)'s and their derivatives respectively. The resulting expressions for the various cases with indices 01, 111, 112, 113 can be represented in matrix formalism as:

\[ x^k_u = A^k x^k + \xi^k \]

where with the appropriate indices 01, 111, 112 and 113.

\[ A^k = \begin{bmatrix} \cos q & \sin q \\ -q \sin q & \cos q \end{bmatrix} \]

\[ A^k_{113} = \begin{bmatrix} \cos 3q & \frac{1}{3} \sin 3q \\ -3q \sin 3q & \cos 3q \end{bmatrix} \]

\[ A^k_{01} = \begin{bmatrix} C_2 q \sin q + D_2 (q \cos q - \sin q) \\ C_2 q (\sin q \cos q - D_2 q^2 \sin q) \end{bmatrix} \]

\[ A^k_{111} = \begin{bmatrix} C_2 q \sin q + D_2 (q \cos q - \sin q) \\ C_2 q (\sin q \cos q - D_2 q^2 \sin q) \end{bmatrix} \]
The displacement and stress fields, in terms of the physical quantities, become:

\[ v^{k-1}_L = x^{k}_L \]
\[ \tau^{k-1} = \frac{1}{\lambda} G_o \left( \delta^{k}_L \right)^2 \left( \frac{1}{4} \left( v^{k}_L \right)^3 \right) \]
\[ v^{k}_U = x^{k}_U \]
\[ \tau^{k} = \frac{1}{\lambda} G_o \left( \delta^{k}_U \right)^2 \left( \frac{1}{4} \left( v^{k}_U \right)^3 \right) \]

where the indices 01, 011, 112 and 13 have to be placed and the \( \delta \)’s are given as:

\[ a_{01} = a_{011} = a_{112} = a_{13} = 0 \quad a_{011} = a_{112} = 1 \quad a_{13} = a_{113} = 1/3 \]

Defining the vector \( \mathbf{z}^k \) as \( \mathbf{z}^k = (v^k, \tau^k) \) with the appropriate indices 01, 11, 12 and 13, Eq. (24) expressed in matrix form becomes:

\[ x^k_L = \mathbf{z}^k \cdot \mathbf{Z}^{k-1} \cdot x^k_L \]
\[ x^k_U = \mathbf{z}^k \cdot \mathbf{Z}^{k-1} \cdot x^k_U \]

where

\[ \mathbf{Z}^k = \begin{bmatrix} 1/A & 0 \\ 0 & \lambda G_o (\delta^2) \end{bmatrix} \quad \mathbf{H}^k = \begin{bmatrix} 0 \\ H_L^k (\mathbf{Z}_U^k)^3 \end{bmatrix} \quad \mathbf{A}^k = \begin{bmatrix} 0 \\ H_U^k (\mathbf{Z}_U^k)^3 \end{bmatrix} \]

Utilizing (26) in (22) one has:

\[ \mathbf{E}^k \cdot \mathbf{Z}^{k-1} \cdot \mathbf{H}^k = \mathbf{A}^k \cdot \left( \mathbf{Z}^{k-1} \cdot \mathbf{H}^k \right) + \mathbf{E}^k \]

Rearranging (28), the transfer matrix going from interface \( k-1 \) to \( k \) is obtained as:

\[ \mathbf{z}^k = \mathbf{B}^k \cdot \mathbf{z}^{k-1} + \mathbf{F}^k \]

where

\[ \mathbf{B}^k = (\mathbf{E}^k)^{-1} \cdot \mathbf{A}^k \cdot \mathbf{F}^k \]
\[ \mathbf{F}^k = (\mathbf{E}^k)^{-1} \cdot \mathbf{E}^k \cdot \mathbf{H}^k - \mathbf{B}^k \cdot \mathbf{H}^k \]

The transfer matrix is utilized by carrying the information from the face 1 to the face 2, from face 2 to face 3, and so on until the face N. Thus for the \( k \)th face

\[ \mathbf{z}^k = C^k \cdot \mathbf{z}^{k-1} + \mathbf{C}^k \]
where:
\[
\mathbf{G}^k = \mathbf{G}^{k-1} \mathbf{G}^k_1 \quad \mathbf{G}^k = \mathbf{G}^{k-1} \mathbf{G}^k_2
\]
\[k = 2, 3, \ldots N\]  \hspace{1cm} (32)

with \( \mathbf{G}^1 = \mathbf{I} \) and \( \mathbf{G}^1 = \mathbf{0} \) where \( \mathbf{I} \) is the identity matrix and \( \mathbf{0} \) is the zero vector.

The problem for a system with \( N \) interfaces, \((N-1)\) layers) has \( 2(N-1) \) unknowns, which are determined by the \( 2(N-2) \) continuity conditions at the inner interfaces and the two prescribed conditions, one at each of the outer faces. However, the transfer matrix method reduces the problem to the solution of a single equation. In fact, for \( k = N \), written explicitly (31) reads:
\[
\begin{bmatrix}
V^N \\
T^N
\end{bmatrix} = \begin{bmatrix}
G_{11}^N & G_{12}^N \\
G_{21}^N & G_{22}^N
\end{bmatrix} \begin{bmatrix}
V^1 \\
T^1
\end{bmatrix} + \begin{bmatrix}
G_{11}^N \\
G_{12}^N
\end{bmatrix} \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]  \hspace{1cm} (33)

If \( V^1 \) and \( V^N \) are the prescribed conditions, \( T^1 \) is given by the first of the two equations in (33). Similarly, if \( V^1 \) and \( T^N \) are the prescribed conditions, \( T^1 \) is found from the second of the two equations in (33). Once, the component of \( V^1 \) not given as a boundary condition is determined, the complete solution is generated by (31).

As for the energy, and the frequency shift, they are calculated from relations similar to (15) by summation over the layers as:
\[
\langle E \rangle = (1/2) \sum_{k=2}^{N} \left( G_o \frac{d}{d\xi} I_o \right)^k
\]
\[
\frac{\partial}{\partial \xi} = - \sum_{k=2}^{N} \left( G_o \frac{d}{d\xi} \right)^k \frac{1}{2} \sum_{k=2}^{N} \left( G_o \frac{d}{d\xi} \right)^k
\]
\[= \sum_{k=2}^{N} \left( G_o \frac{d}{d\xi} \right)^k \]
\[
\langle E \rangle = \sum_{k=2}^{N} \left( G_o \frac{d}{d\xi} \right)^k \]
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where substitution of (36) into (35) yields

\[
\eta_{11} = \frac{1}{2} \cos \gamma_y / \cos \gamma \quad \eta_{111} = \cos \gamma_y + \eta_{2} y \sin \gamma_y
\]

\[
\eta_{112} = \xi \cos \gamma + \eta_{2} \cos (2\gamma - \gamma_y) + \eta_{3} \cos (2\gamma + \gamma_y)
\]

\[
\eta_{13} = \xi \eta_{1} + \eta_{2} \cos 3\gamma_y
\]

with

\[
Q^2 = \frac{\gamma^2}{(1 + 3\gamma_y \eta_{1})} \quad P^2 = \frac{\gamma^2}{(1 + 3\gamma_y \eta_{1})} \quad \eta_y = \xi \eta_{1}
\]

and

\[
B_1 = \frac{1}{4} \left( \frac{\gamma^2}{(1 + 3\gamma_y \eta_{1})} \right) \frac{Q \sin Q}{\cos^2 Q}
\]

\[
B_2 = \frac{1}{4} \left( \frac{\gamma^2}{(1 + 3\gamma_y \eta_{1})} \right) \frac{Q}{\cos Q}
\]

\[
B_3 = -\left[ \xi \cos \gamma_y + \eta_{2} \cos (2\gamma - \gamma_y) + \eta_{3} \cos (2\gamma + \gamma_y) \right] / \cos \gamma
\]

\[
B_4 = \frac{\gamma^2}{(1 + 3\gamma_y \eta_{1})} \left( \frac{Q^2 - \gamma^2}{(1 + 3\gamma_y \eta_{1})} \right)
\]

\[
B_5 = \left( \frac{\gamma^2}{(1 + 3\gamma_y \eta_{1})} \right) B_0 / \left[ Q^2 - (2\gamma - \gamma_y)^2 \right]
\]

\[
B_6 = -\frac{\gamma^2}{(1 + 3\gamma_y \eta_{1})} \left( \frac{Q^2 - \gamma^2}{(1 + 3\gamma_y \eta_{1})} \right)^2
\]

\[
B_7 = -\left[ \xi \cos \gamma_y + \eta_{2} \cos 3\gamma_y \right] / \cos \gamma
\]

\[
B_8 = -\frac{\gamma^2}{(1 + 3\gamma_y \eta_{1})} \left( \frac{Q^2 - \gamma^2}{(1 + 3\gamma_y \eta_{1})} \right)^2
\]

\[
B_9 = -\frac{\gamma^2}{(1 + 3\gamma_y \eta_{1})} \left( \frac{Q^2 - 9\gamma^2}{(1 + 3\gamma_y \eta_{1})} \right)^2
\]

\[
A_o = \frac{1}{8} \cos^3 \eta_y
\]

\[
B_o = \frac{Q^2}{(1 + 3\gamma_y \eta_{1})^2} \cos \gamma
\]

The integral of the energy over a cycle and the shift in the frequency are calculated according to (15) by using the expressions in (37). Similarly, the extension to multilayer systems is achieved in the same way as in Sec. 4 through the use of transfer matrices. The only difference is that the transfer matrices in this case are complex valued.

6. Discussion

Fig. 1 shows the energy spectrum for a single elastic layer and Fig. 2 that for a 4 layer system for a site in Las Vegas based on actual data [7]. In all cases the results are expressed in terms of the parameter \((G_1/G_0)(A/d)^2\). For example, the case where \(G_1/G_0\) is taken equal to \(-1.4 \times 10^6\), corresponds to a stress-strain relationship where for a strain \(\gamma = 10^{-3}\), the shear coefficient has decreased by 25%. Because the shear coefficient decreases with increasing strains, the energy response spectrum is bent to the left as compared to that of the linear analysis. This behavior makes the response spectrum multivalued and consequently the resonance singularities of the elastic case are eliminated. Furthermore because of the multivalued nature of the solution, one of the branches becomes unstable and the "jump" phenomenon occurs. To obtain the figures, the solution is considered to be expressed in terms of the parameter \(q\) or equivalently \(\eta_0 = c_q/d\). Consequently eliminating \(\eta_0\) between the two expressions \(\eta_0 = E(\eta_0)\) in Eq. (15) one obtains \(E(\eta)\) which is the desired relationship for \(\eta\), the prescribed frequency.

Study of these figures shows that even a very small nonlinearity alters the results significantly in eliminating the singularities predicted by a linear analysis. This is
accomplished as a result of some of the modes becoming inaccessible due to the multi-valuedness of the spectrum.

References


![Fig. 1. Energy Response Spectrum for A Single Elastic Nonlinear Layer for Various Values of the Displacement $\omega_n = c/d$](image)

(c=shear velocity; d=thickness of the layer)
(a) Notation and Data used for the Calculations [7]. $G_1/G_o$ is taken as $-0.25 \times 10^6$ for all layers.

(b) Energy Response Spectrum for the Data in (a). Bed rock displacement amplitude 10 cm. $
\omega^c = c/d = 0.456 \text{ sec}^{-1}$ (c=averaged shear velocity$= 1592 \text{ m. sec}^{-1}$; d=total thickness=3489m.)