NUMERICAL ANALYSIS OF SOIL-STRUCTURE SYSTEMS
OF UNBOUNDED GEOMETRY

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SUMMARY

When the behavior of points in an engineering system far removed from the boundaries is of concern, the system can be regarded as unbounded.

This paper addresses finding equilibrium states of these systems using numerical analysis. It particularizes the discussion to the analysis of soil-structure systems, though the methodology is applicable to steady heat transfer fluid flow, and electromagnetic field problems as well.

The "usual" approach to analysis of these systems is to circumscribe the region of behavior concern by an artificial boundary and analyze the truncated domain, imposing zero displacements along the boundary. To obtain a measure of the quality of this approximate model, a larger truncated region is analyzed. If the difference of responses between the two analyses is negligible, predictions are assumed to be accurate.

This paper offers a non-linear iterative process which can determine the exact boundary conditions for the artificial boundary without recourse to analytical results. It describes the process and exhibits its effectiveness with illustrative problems.

The process involves an adaptive evaluation of boundary conditions using a relaxation solution process. It establishes conditions along the artificial boundary such that decay of deformations within the truncated region is consistent with their vanishing at infinity. Iteration is performed by relaxing displacements at each boundary point in turn. A second analysis with a larger domain produces absolute measures of analysis error.

Illustrations include linear systems in one, two and three dimensional space: Winkler’s beam-foundation problem and Boussinesq’s. Numerical analysis uses finite element models and produces data for comparisons with the exact solutions.

In each case, the exact responses are predicted, within the limitations of the finite element articulation and computer manipulation and process errors. When accuracy of results in the usual approach is unacceptable, the same truncated region produces results of negligible error using the iteration. Analysis running times are less than four times those of the usual approach. Adding condensation logic could reduce this time factor to less than 1.50.

The procedure is directly applicable to finite difference or boundary integral modeling. It can encompass certain types of nonlinear systems.
1. **INTRODUCTION**

Engineering systems are often idealized as unbounded. This occurs when the boundary where loads are applied or reacted is so far removed from the region of interest that it may be assumed to be an infinite distance away.

This idealization is most frequently used for field problems of stress, fluid flow, magnetism, temperature and electric potential. Here, we address analysis of stress in soils. We assume that a structure rests on a body of soil and we wish to determine settlement and stresses in the neighborhood of the structure.

The analysis approach to this class of problems may involve either truncation or direct representation of the unbounded region.

In the "usual" truncation analysis, an artificial boundary is defined at some distance from the region of interest. In stress analyses, the displacements are assumed to be zero along this boundary. Analysis of the now-bounded region produces the responses within the artificial boundary. If this boundary is sufficiently far removed from the region of interest, a small change in its location has a negligible effect on response predictions.

This approach simulates the region outside the artificial boundary as rigid. Thus, even if the change of deformations in relocating the boundary is small, the total settlement, relative settlement, and stiffness may be inaccurately modeled.

Currently the only approach for exactly representing behavior in the unbounded region is the boundary integral approach. In this approach singular response functions are used which preclude non-zero error integrals near the infinite boundary [1-3]. Techniques for combining this approach with finite element models are evolving [4,5].

Approximate representations using "infinite" elements are also available [4,5]. Here finite elements of large, but bounded extent, provide response functions which decay. Work on selecting the most suitable response functions and establishing the extent of the elements to obtain accurate behavior predictions is underway.

This paper examines a simpler approach: using the numerical model of the truncated region as the basis for evaluating displacements along the artificial boundary which imply the unboundedness. The approach includes finding displacements through an iterative process which adapts to the loading, geometry, and numerical model of the system.

Section 2 of this paper describes the analysis approach. Section 3 illustrates the capability to produce highly accurate response predictions using the approach for linear static analyses in one, two and three dimensions. The last section cites conclusions.

This paper is the outgrowth of work sponsored by the Electric Power Research Institute of Palo Alto, California. Conway Chen of EPRI was responsible for stimulating the development.
2. THE ANALYSIS PROCEDURE

Displacements along the artificial boundary can be established from the behavior of the region inside the boundary. This section describes the iteration process to evaluate these displacements for a particular loading.

The steps of the process are as follows:
1. Define an artificial boundary and discretize the truncated region.
2. Develop the load-deflection equations for the truncated region with the constraints that displacements along the artificial boundary are zero.
3. Solve the constrained load-deflection relations for the unknown displacements in the truncated region.
4. Using the calculated displacements, evaluate displacements along the artificial boundary so that responses near the boundary will comply with an assumed "decay function".
5. Replace the constraints on load-deflection relations with constraints prescribing that boundary displacements take on the values obtained in Step 4.
6. Repeat Steps 3 through 5 until the displacements of Step 4 match those of the previous cycle.

The "decay function" is an equation stating how displacements vary near the boundary as a function of the spatial coordinates. The choice of this function is critical to success of the process both in terms of convergence of the iteration and accuracy of the final solution. This function alone implies the unboundedness of the system.

The artificial boundary may be located as close to the region of interest as desired, as long as the decay function can be particularized by behavior within the truncated region and adequately imply behavior from the neighborhood of the boundary to infinity.

Consider the case in which the truncated region is modeled with finite elements. Then, from among those solutions which satisfy equilibrium in a macroscopic sense, the process defines one which complies with the infinity boundary conditions, as represented by the decay function. If the numerical analysis model and decay function are exact, the behavior predictions will be intolerant only to the extent of computer manipulation errors.

It is not critical that the region be modeled with finite elements. Finite difference, boundary integral, or series expansion techniques can be used. Furthermore, the process is appropriate to systems which can be partitioned into nonlinear and linear parts, where the linear part extends from the neighborhood of the artificial boundary to infinity. Finally, the process is appropriate for nonlinear systems for which a unique solution exists.

3. ILLUSTRATIVE PROBLEMS

This section summarizes results of application of the approach to foundation problems in unbounded one, two, and three dimensional domains. In all cases, comparisons of displacements with those expressed by analytical solutions indicate the exceptional accuracy of numerical analysis results.

Table I summarizes features of the problems and their numerical analyses. Each involves point loadings on an isotropic unbounded foundation.
In each, the truncated region is represented by a finite element model. The table cites the exact solution for each problem [6,7] and the decay function used for the numerical analysis. The artificial boundary has been located far enough away from the point of load application so decay is regular.

The finite element model for Boussinesq's problem uses 27 eight-node axisymmetric isoparametric solid elements [8]. An analysis of specific energy differentials for this mesh recommends that intermediate nodes on the radial side be located at the quarter points [9]. (This improves response strain energy by a factor of four over midside locations with no change in the number of elements). The decay function is a generalized form of the Green's function in three dimensions.

The location of the artificial boundary and finite element mesh for the reduced Boussinesq problem is the same as for the three dimensional application. Eight-node isoparametric quadrilaterals, constrained to behave in plane strain, provide the element models. The decay function is a generalized form which admits behavior in compliance with the exact solution.

The reduced Boussinesq problem was also treated with the same decay function when the loading involves two z direction concentrated forces on the free surface. In this case, one force is on the axis of symmetry, as shown in Table 1. The second force is applied 108 inches from the symmetry axis. This provides a two dimensional domain problem for which the chosen decay function can not match the analytic solution.

The artificial boundary for the Winkler problem falls 140 inches from the line of symmetry. This choice insures that at least a half wave of response will be included in the truncated domain. This defines the minimum region for measuring the decay by numerical analysis results. The decay function is a generalized expression of the analytic solution.

For simplicity only displacements at a single boundary node were relaxed in a cycle of iteration for each of the analyses. Reanalysis used influence vectors and superposition.

Figure 1 exhibits the relation between the number of iterations and the solution error for the problems. The figure shows the error in the larger displacement component at the surface of the domain, on the artificial boundary. For each problem there is a dramatic improvement in accuracy over that of the usual solution (that of the first iteration). Convergence of the iterative process appears to be monotonic until high accuracy is obtained.

The rapid convergence for Winkler's problem is probably due to the existence of only two unknown boundary displacements. Since the Boussinesq problems have an equal number of unknown boundary displacements (thirteen), the more rapid convergence of the three dimensional case can be regarded as intrinsic.

The error of the final results for analyses which admit the analytical decay is extremely small. Tables II and III list response predictions at artificial boundary points for these analysis approaches. The errors in both cases can be attributed to discretization error. This was verified for the
Winkler problem by performing the usual analysis with 21 beam elements. This yields a centerline bending moment of $1.33 \times 10^5$ in. lb.

The second column of Table II cites the error in predicting the change of displacement rather than the total displacement. The numerical solution process produces exact values for vertical displacement within a rigid body translation in the vertical direction. This corresponds to the fact that in the analytical solution it is necessary to define vertical displacement at some point to particularize the constant B.

Tables IV and V define results of applying the technique to the Boussinesq problem and the reduced Boussinesq problem with two point loads on the half model. In neither of these problems does the decay function admit the exact solution. Nonetheless, both numerical analyses give results of good accuracy.

Table VI summarizes data describing computer solution times for each problem. In every case, the adaptive determination of displacement conditions for the artificial boundary increased computer times by less than a factor of four over that required for a usual analysis. The time to create influence vectors could be reduced to approximately one tenth those given by using matrix condensation, multiple back substitution, and avoiding stress calculations and print out. Iteration times of Boussinesq problems are excessive due to lack of extrapolation techniques and practical accuracy requirements. There is little doubt computer time for the process in a production code environment will be less than one and one half times those of the usual analysis.

4. CONCLUSIONS

A numerical approach for an approximate analysis of unbounded systems is presented. This approach involves truncation of the unbounded region where an artificial boundary is defined at some distance from the region of interest. An iterative process is adapted through which the exact conditions are established along the artificial boundary such that decay of deformations within the truncated region is consistent with their vanishing at infinity.

This analysis method was applied to one, two, and three dimensional linear systems in which the truncated regions were modeled with finite elements. In each case highly accurate response predictions were obtained.

Success of the iterative process both in terms of convergence and accuracy depends upon the choice of the "decay function". If this function is exact, the behavior predictions will be limited only to the extent of the numerical modeling and the computer manipulation errors. The proposed approach is also appropriate to finite difference or boundary integral modeling. It can accommodate nonlinear systems for which a unique solution exists.

REFERENCES


TABLE I
ILLUSTRATIVE UNBOUNDED SYSTEMS

Exact Solution (Boussinesq, 1874\(^{(6)}\))
\[
u = -\frac{2P}{Eh} \cos \theta \nu \theta + \frac{2(1-v)}{Eh} (\cos \theta - \sin \theta) - 8 \sin \theta
\]

Numerical Solution
\[
u = a_x + \beta_x \nu \theta; \quad \nu = a_z + \beta_z \nu \theta
\]

(b) Reduced Boussinesq Problem

Exact Solution (Boussinesq, 1874\(^{(6)}\))
\[
u = \frac{(1-2v)(1+v)P}{4Eh x^2 h^2} \left( \frac{z}{x^2 + h^2} \right)^{1/2} - 1
\]
\[
u = \frac{v (1+v) P}{4Eh} \left( \frac{z}{x^2 + h^2} \right)^{1/2} + \frac{2(1-v)}{Eh} \left( \frac{z}{x^2 + h^2} \right)^{1/2}
\]

Numerical Solution
\[
u = a_x \frac{x}{r A}; \quad \nu = a_z \frac{z}{r B}
\]

(a) Boussinesq Problem

Exact Solution (Winkler, 1967\(^{(7)}\))
\[
u = \frac{P f}{2k} e^{-fx} (\sin f x + \cos f x)
\]

Numerical Solution
\[
u = ae^{8x} \sin x
\]

(C) Winkler Problem
TABLE II
RESULTS FOR REDUCED BOUSSINESQ PROBLEM

<table>
<thead>
<tr>
<th></th>
<th>( u_{90}^* )</th>
<th>( \Delta w_0^{**} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact Solution</td>
<td>-2.13333</td>
<td>0.38018</td>
</tr>
<tr>
<td>400th Cycle Result</td>
<td>-2.13117</td>
<td>0.38005</td>
</tr>
<tr>
<td>Usual Result</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

* \( u_{xx} \) = x displacement of surface \((\theta = xx^0)\) 708 inches from the centerline

** \( \Delta w_0 \) = change of z displacement between boundary and first interior node along centerline \((\theta = 0)\)

TABLE III
RESULTS FOR WINKLER PROBLEM*

<table>
<thead>
<tr>
<th></th>
<th>End deflection</th>
<th>End moment</th>
<th>( \xi ) moment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact Solution</td>
<td>-5.531(^{-6})</td>
<td>-0.960(^3)</td>
<td>1.333(^5)</td>
</tr>
<tr>
<td>3rd Cycle Result</td>
<td>-3.108(^{-6})</td>
<td>-0.654(^3)</td>
<td>1.302(^5)</td>
</tr>
<tr>
<td>Usual Result</td>
<td>0</td>
<td>-2.404(^3)</td>
<td>1.302(^5)</td>
</tr>
</tbody>
</table>

*Exponents imply a power of ten. Thus 1.\(^3\) = 1 \times 10^{3}

TABLE IV
RESULTS FOR BOUSSINESQ PROBLEM*

|             | \( w_0 \) \( w_{30} \) \( u_{30} \) \( w_{90} \) |
|-------------|---------------------|---------------------|---------------------|
| Exact Solution | +.115095\(^{-2}\) | .169046\(^{-2}\) | .195837\(^{-3}\) | .187030          |
| 100th Cycle Result | +.112133\(^{-2}\) | .166499\(^{-2}\) | .194760\(^{-3}\) | .184952          |
| Usual Result    | 0                  | 0                   | 0                   | 0                 |

*Exponents imply a power of ten

TABLE V
RESULTS FOR TWO-LOAD REDUCED BOUSSINESQ PROBLEM

<table>
<thead>
<tr>
<th></th>
<th>( u_{90} )</th>
<th>( \Delta w_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact Solution</td>
<td>-.0533333</td>
<td>.092425</td>
</tr>
<tr>
<td>300th Cycle Result</td>
<td>-.607223</td>
<td>.089742</td>
</tr>
<tr>
<td>Usual Result</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Table VI

COMPUTER SOLUTION TIMES*

<table>
<thead>
<tr>
<th></th>
<th>Usual Analysis</th>
<th>Influence Vectors</th>
<th>Iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winkler</td>
<td>64.3</td>
<td>9.6</td>
<td>1.4</td>
</tr>
<tr>
<td>Reduced Boussinesq</td>
<td>113</td>
<td>232</td>
<td>98.2 (400 cycles)</td>
</tr>
<tr>
<td>Reduced Boussinesq, 2 Loads</td>
<td>113</td>
<td>232</td>
<td>69.3 (300 cycles)</td>
</tr>
<tr>
<td>Boussinesq</td>
<td>108</td>
<td>226</td>
<td>31.0 (100 cycles)</td>
</tr>
</tbody>
</table>

*CDC System Billing Units on the Cyber 175 Computer. All times include both application and non-application units.

![Graph showing the number of iterations versus error in percent for different problems](image)

Figure 1

ERROR IN SOLUTIONS IN RELAXATION PROCESS