# SEISMIC STRUCTURAL RESPONSE ANALYSIS USING CONSISTENT MASS MATRICES HAVING DYNAMIC COUPLING

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## SUMMARY

The classical treatment of seismic structural response has been primarily confined to simply coupled or close-coupled dynamic systems utilizing lumped-mass representation of the structural mass distribution. With the continued improvement of discrete or finite element methods of analysis, the use of consistent or equivalent mass matrices which are derived from the same displacement function used for the derivation of elemental stiffness matrices has become more popular, especially for equipment analysis and the analysis of continuum structures.

One of the benefits of consistent mass matrices is off-diagonal terms in the mass matrix. These terms offer a better approximation to the actual mass distribution of continuum structures than lumped-mass matrices and lead to a dynamic coupling in addition to the static coupling inherent in the stiffness matrix. However, the dynamic coupling leads to a different seismic inertial loading vector than that obtained in the absence of off-diagonal terms.

The basis for the theoretical development of this paper is the linear matrix equations of motion for an unconstrained structure subject to support excitation. The equations are formulated in terms of absolute displacement, velocity and acceleration vectors. By means of a transformation of the absolute response vectors into displacements, velocities and accelerations relative to the support motions, the homogeneous equations become non-homogeneous and the non-homogeneous boundary conditions become homogeneous, with relative displacements, velocities and accelerations being zero at support points. The forcing function or inertial loading vector is shown to consist of two parts.

The first part is comprised of the mass matrix times the support acceleration function times a vector of structural displacements resulting from a unit vector of support displacements in the direction of excitation. This inertial loading corresponds to the classical seismic loading vector and is indeed the only loading vector for lumped-mass systems.

The second part of the inertial loading vectors consists of the mass matrix times the support acceleration function times a vector of structural accelerations resulting from unit support accelerations in the direction of excitation. This term is not present in classical seismic analysis formulations and results from the presence of off-diagonal terms in the mass matrices which give rise to dynamic coupling through the mass matrix. Thus, for lumped-mass models, the classical formulation of the inertial loading vector is correct. However, if dynamic coupling terms are included through off-diagonal terms in the mass matrix, an additional inertial loading vector must be considered.

In general, off-diagonal terms may arise in mass matrices from two sources:

1. Consistent mass matrix formulations whereby mass elemental matrices are derived from the same assumed displacement functions as used to derive stiffness matrices.
2. Dynamic condensation of systems whereby a reduced set of dynamic degrees of freedom are to characterize the dynamic behavior of a structural system.

The presence of off-diagonal mass coupling terms in models using consistent mass matrices is obvious and a direct result of the mass matrix formulation. However, the presence of off-diagonal terms in a dynamically condensed reduced mass matrix is not obvious and off-diagonal terms may be present in the reduced mass matrix even though only lumped masses were used to formulate the dynamic model.

Several examples are included to show a comparison of the discrete element and closed-form solutions and the solution obtained using traditional methods. The result of consideration of the structural accelerations to unit support accelerations is shown to lead to larger modal participation factors than would be determined neglecting the dynamic coupling through the mass matrix.

1.0 Introduction

The development of the concepts involved in seismic analysis have been well documented in the literature so that no attempt will be made to derive the equations of motion for a linear structural system. Instead, the attention is focused on the conceptual development of seismic analysis techniques from the viewpoint of problems which arise when specific attention is not paid to the precise formulation of the equations of motion [1].

When formulating the equations of motion for a discrete multiple degree-of-freedom system having a total of N degrees of freedom, N equations of motion result. With the advent of high speed digital computers, it became convenient to express the N equations of motion in terms of a single matrix equation of motion whereby stiffness and mass elements of the system lead to N x N matrices and the unknown displacements at each degree of freedom become an N dimensional vector.

By virtue of Betti's theorem and Maxwell's reciprocity theorem [2], stiffness matrices are symmetric and contain off-diagonal elements for all but the most fundamental structural systems. The off-diagonal terms in the stiffness matrix arise from static coupling of degrees of freedom. Prior to the advent of finite element methods and even to the present, the mass distribution of the real structural system was incorporated in the N degree-of-freedom discrete model as lumped masses at points of connection of stiffness elements. Since the masses were lumped at the connection points of the stiffness elements, the inertial force at each degree of freedom resulted only from the acceleration at that degree of freedom. There was no coupling through the N x N mass matrix analogous to the coupling in the stiffness matrix. The N x N mass matrix consequently became a diagonal matrix, i.e., there are no off-diagonal terms in the mass matrix. This fact that mass matrices are diagonal for lumped mass sytems resulted in simplifications in the N x N matrix equation of motion since there is no dynamic coupling.

Typically, the lumped mass values were obtained by using the tributary mass of stiffness members connecting at any mass point. While conceptually simple, it was recognized that tributary mass was only an approximation to the actual mass distribution which indeed is dependent upon the true mode shape of the system. Also, for complex structural configurations, the choice of lumped masses often took on the aspects of more art than science. However, as long as the lowest frequency modes of a structural system were predominantly the important modes, lumped mass representations of the actual mass distribution were adequate. For higher modes, the frequency and mode shape results were more a matter of the skill of the analyst than the numerical methods.

With the advent of finite element methods of analysis, the means were available to more accurately represent the distributed mass as elements of a discrete mass matrix which contained off-diagonal terms representing coupling through the mass matrix. A fundamental and significant difference between consistent mass matrices and lumped mass matrices is the presence of off-diagonal terms in the consistent mass matrix. The off-diagonal terms account for dynamic coupling in the same manner that off-diagonal terms in the stiffness matrix account for static coupling.

Subsequent paragraphs discuss the implications of this dynamic coupling on analytical methods. It will be shown that dynamic response calculations based on techniques developed for lumped mass systems give erroneous results when applied directly to consistent mass

systems. With the increased reliance on finite element methods of analysis, it is considered extremely important to recognize these errors. This is especially true since the methods derived for lumped mass response calculations will execute with no difficulty for consistent mass systems. Thus, there is no internal checkpoint. The errors are shown to be a direct result of neglecting inertial forces arising from the dynamic coupling when using lumped mass numerical methods where dynamic coupling was not a consideration by virtue of the diagonal nature of the mass matrix.

2.0 Mathematical Background

The dynamic response of a linear structural system is, in general, given by the solution of the non-homogeneous initial value problem;

$$\frac{\partial^2 \bar{u}}{\partial t^2} - L[\bar{u}] = \bar{f}(\bar{x},t) , \tag{1}$$

where $\bar{u}$ is the vector of unknown dynamic response values, $\bar{u}\ (\bar{x},t)$, and $\bar{x}$ may be one, two or three dimensional. $L[\bar{u}]$ represents a differential operator involving no higher derivative with respect to time than $\frac{\partial \bar{u}}{\partial t}$. The nature of the operator $L[\bar{u}]$ is dependent upon the structural system. The function $\bar{f}(\bar{x},t)$ represents the dynamic forces applied to the structural system. To obtain a solution $\bar{u}(\bar{x},t)$, it is necessary to specify initial conditions $\bar{u}(\bar{x},0)$ and $\frac{\partial \bar{u}}{\partial t}\ (\bar{x},0)$ plus boundary conditions. The boundary conditions may be expressed in the terms of a differential operator $B[\bar{u}]$ evaluated at the boundary. The nature of $B[\bar{u}]$ depends on the structural system considered.

Typically forced vibration problems have zero initial conditions and zero boundary conditions such that;

$$\bar{u}(x,0) = \frac{\partial \bar{u}}{\partial t}\ (\bar{x},0) = 0 , \tag{2}$$

$$B[\bar{u}] = 0 . \tag{3}$$

The seismic response of a structural system poses a slightly different initial value problem since the only forces acting on the system are the inertial forces resulting from motion of the supports, or boundaries. Thus, the seismic response problem gives rise to a homogeneous initial value problem;

$$\frac{\partial^2 \bar{u}}{\partial t^2} - L[\bar{u}] = 0 , \tag{4}$$

with initial conditions that are typically zero;

$$\bar{u}(\bar{x},0) = \frac{\partial \bar{u}}{\partial t}\ (\bar{x},0) = 0 . \tag{5}$$

The boundary conditions, however, are non-homogeneous;

$$B[\bar{u}] = \bar{g}\ (\bar{x},t) \tag{6}$$

on the boundary. $\bar{g}(\bar{x},t)$ represents the prescribed motion of at least some of the boundary points which move with the seismic ground motion.

The homogeneous initial value problem with non-homogeneous boundary conditions which describes the seismic response problem may be transformed into a non-homogeneous initial value problem with homogeneous boundary conditions characteristic of forced vibration response problems [3,4].

Introduce a function $\bar{v}(\bar{x},t)$ such that

$$B[\bar{v}] = \bar{g}(\bar{x},t) \tag{7}$$

on the boundary. Making the substitution;

$$\bar{w}(\bar{x},t) = \bar{u}(\bar{x},t) - \bar{v}(\bar{x},t) , \tag{8}$$

and applying the boundary conditions, we have;

$$B[\bar{w}] = B[\bar{u}] - B[\bar{v}] = \bar{g}(\bar{x},t) - \bar{g}(\bar{x},t) = 0 \tag{9}$$

on the boundary. Thus, the function $\bar{w}(\bar{x},t)$ satisfies homogeneous boundary conditions. Substituting for $\bar{u}(\bar{x},t)$ in the homogeneous equation (4);

$$\frac{\partial^2 \bar{w}}{\partial t} - L[\bar{w}] = - \frac{\partial^2 \bar{v}}{\partial t^2} + L[\bar{v}] , \tag{10}$$

we obtain a non-homogeneous equation subject to homogeneous boundary conditions. The initial conditions for the response function $\bar{w}(\bar{x},t)$ become:

$$w(\bar{x},0) = \bar{u}(x,0) - \bar{v}(\bar{x},0) = - \bar{v}(\bar{x},0)$$

and (11)

$$\frac{\partial \bar{w}}{\partial t} (\bar{x},0) = \frac{\partial \bar{u}}{\partial t} (\bar{x},0) - \frac{\partial \bar{v}}{\partial t} (\bar{x},0) = - \frac{\partial \bar{v}}{\partial t} (\bar{x},0) .$$

Since $\bar{v}(\bar{x},t)$ has the time dependence of the ground motion which has a zero initial displacement and velocity,

$$\bar{v}(\bar{x},0) = \frac{\partial \bar{v}}{\partial t} (\bar{x},0) = 0 ; \tag{12}$$

so that the initial conditions for $\bar{w}$ are the same as for $\bar{u}(\bar{x},t)$,

$$\bar{w}(\bar{x},0) = \frac{\partial \bar{w}}{\partial t} (\bar{x},0) = 0 . \tag{13}$$

For continuous systems, the mathematics of deriving the function $\bar{v}(\bar{x},t)$ can become quite involved and will not be discussed further. Reference [5] presents a detailed discussion of the function $\bar{v}(\bar{x},t)$ for the special case of a beam. In general, the function $\bar{v}(\bar{x},t)$ can be obtained by a means of Green's functions or influence functions, which give the time response of the system to the boundary excitation as if the excitation were applied statically. Thus, the response $\bar{v}(\bar{x},t)$ varies with time but does not include the effects of dynamic amplification. With the function $\bar{v}(\bar{x},t)$ derived in this manner,

$$L[\bar{v}] = 0 . \tag{14}$$

Thus, the non-homogeneous equation of motion (10) becomes;

$$\frac{\partial^2 \bar{w}}{\partial t^2} - L[\bar{w}] = -\frac{\partial^2 \bar{v}}{\partial t^2} . \tag{15}$$

Since $\bar{w} = \bar{u} - \bar{v}$ and $\bar{u}$ is the total or absolute response of the system and $\bar{v}$ is defined above, we see that the function $\bar{w}(\bar{x},t)$ is a relative displacement. This relative displacement is defined in a special way. It is the displacement of the system relative to the displacement of the system resulting from the application of the boundary displacements. Since $\bar{w}(\bar{x},t)$ must satisfy homogeneous boundary conditions, the displacement relative to the boundary points is zero at the boundary points.

The concept of the relative displacement $\bar{w}(\bar{x},t)$ is important to the consideration of discrete systems to follow. Thus, it is worthy of additional explanation.

If all boundary points move with exactly the same time dependence and amplitude, i.e.,

$$B[\bar{u}] = \bar{u} = \bar{A}f(t) \tag{15}$$

on the boundary, then the displacement $\bar{v}(\bar{x},t)$ will be the rigid body translation of the system and $\bar{w}(\bar{x},t)$ will simply be the elastic displacements relative to the rigid body motions.

If all boundary points move with exactly the same time dependence but different amplitudes, i.e.,

$$B[\bar{u}] = \bar{u} = \bar{A}(\bar{x})\, f(t) , \tag{16}$$

then the displacement $\bar{v}(\bar{x},t)$ will be the sum of rigid body plus elastic displacements owing to the distribution $\bar{A}(\bar{x})$ and $\bar{w}(\bar{x},t)$ will be additional elastic displacements relative to the displacements $\bar{v}(\bar{x},t)$. For the case where the time dependence of the boundary displacements varies from point to point on the boundary, the solution for $\bar{v}(\bar{x},t)$ becomes even more complex. However, $\bar{w}(\bar{x},t)$ can always be visualized as the displacement of the system relative to the displacement of an equivalent system with zero mass subjected to the boundary displacements as a function of time.

## 3.0 Discrete Systems

For discrete structural systems subject to boundary excitation, such as for seismic ground motion, the equation of motion may be written [2];

$$[M]\{\ddot{U}\} + [K]\{U\} = 0 , \tag{18}$$

subject to the boundary conditions;

$$[B]\{U\} = \{U_G(t)\} . \tag{19}$$

The displacement vector $\{U\}$ may be partitioned as;

$$\{U\} = \begin{Bmatrix} U_x \\ \hline U_y \end{Bmatrix} , \tag{20}$$

where $U_x$ are the displacements of the non-boundary nodal points and $U_y$ are the displacements of the boundary nodes. Thus, the boundary condition operator has the form;

$$[B] = \left\{ \begin{array}{c|c} 0 & 0 \\ \hline 0 & I \end{array} \right\} , \tag{21}$$

where I is the unit matrix. The boundary displacement vector is:

$$\{U_G(t)\} = \left\{ \begin{array}{c} 0 \\ \hline U_{yG}(t) \end{array} \right\} . \tag{22}$$

Equation (18) then reduces to:

$$\{U_y\} = \{U_{yG}(t)\} . \tag{23}$$

The stiffness and mass matrices may be similarly partitioned:

$$[M] = \left\{ \begin{array}{c|c} M_{xx} & M_{xy} \\ \hline M_{xy}^T & M_{yy} \end{array} \right\} ; \tag{24}$$

$$[K] = \left\{ \begin{array}{c|c} K_{xx} & K_{xy} \\ \hline K_{xy}^T & K_{yy} \end{array} \right\} ; \tag{25}$$

where the subscript x refers to the non-boundary displacements and y refers to the boundary displacements. The off-diagonal terms $K_{xy}$ in the stiffness matrix account for static coupling between the boundary and interior displacements. Similarly, the off-diagonal terms $M_{xy}$ in the mass matrix account for dynamic coupling between the boundary and interior points. Note that if a lumped mass matrix were used, the partitioning would be:

$$[\tilde{M}] = \left[ \begin{array}{c|c} \tilde{M}_{xx} & 0 \\ \hline 0 & \tilde{M}_{yy} \end{array} \right] , \tag{26}$$

so that $M_{xy} = 0$ and there is no dynamic coupling.

Using the partitioned matrices, the equation of motion (17) becomes;

$$\left[ \begin{array}{c|c} M_{xx} & M_{xy} \\ \hline M_{xy}^T & M_{yy} \end{array} \right] \left\{ \begin{array}{c} \ddot{U}_x \\ \hline \ddot{U}_y \end{array} \right\} + \left[ \begin{array}{c|c} K_{xx} & K_{xy} \\ \hline K_{xy}^T & K_{yy} \end{array} \right] \left\{ \begin{array}{c} U_x \\ \hline U_y \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ \hline 0 \end{array} \right\} , \tag{27}$$

subject to the boundary conditions;

$$\{U_y\} = \{U_{yG}(t)\} . \tag{28}$$

Define a new displacement vector;

$$\{V\} = \left\{ \begin{array}{c} V_x \\ \hline U_{yG}(t) \end{array} \right\} , \tag{29}$$

and solve the "static" problem of a massless system given by;

$$[K]\ \{V\} = \{0\} . \tag{30}$$

Substituting (29) into (27) with a zero mass matrix, we have:

$$\left[\begin{array}{c|c} K_{xx} & K_{xy} \\ \hline K_{xy}^{T} & K_{yy} \end{array}\right] \left\{\begin{array}{c} V_x \\ \hline U_{yG}(t) \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline 0 \end{array}\right\} . \tag{31}$$

Solving for $\{V_x\}$;

$$\{V_x\} = -[K_{xx}^{-1}]\ [K_{xy}]\ \{U_{yG}(t)\} ; \tag{32}$$

so that $\{V\}$ becomes:

$$\{V\} = \left\{\begin{array}{c} -[K_{xx}^{-1}]\ [K_{xy}]\ \{U_{yG}(t)\} \\ \hline U_{yG}(t) \end{array}\right\} . \tag{33}$$

Thus, $\{V\}$ represents the time history response of the massless system to the prescribed boundary motions $U_{yG}(t)$.

The relative displacement function $\{W\}$ is defined by:

$$\{W\} = \{U\} - \{V\} . \tag{34}$$

Or, in partitioned form:

$$\left\{\begin{array}{c} W_x \\ \hline W_y \end{array}\right\} = \left\{\begin{array}{c} \{U_x\} + [K_{xx}^{-1}]\ [K_{xy}]\ \{U_{yG}(t)\} \\ \hline 0 \end{array}\right\} . \tag{35}$$

Thus, $\{W_x\}$ are the displacements of the non-boundary nodes relative to the displacements of the massless structure resulting from application of the prescribed boundary displacements.

From (35);

$$\{W_x\} = \{U_x\} + [K_{xx}^{-1}]\ [K_{xy}]\ \{U_{yG}(t)\} . \tag{36}$$

Or;

$$\{U_x\} = \{W_x\} - [K_{xx}^{-1}]\ [K_{xy}]\ \{U_{yG}(t)\} . \tag{37}$$

Substituting (37) into the equation of motion (27) and using only the first equation since $\{\ddot{W}_y\} = \{W_y\} = 0$, we have:

$$[M_{xy}]\ \{\ddot{W}_x\} + [K_{xx}]\ \{W_x\} = [M_{xx}]\ [K_{xx}^{-1}]\ [K_{xy}]\ \{\ddot{U}_{yG}(t)\} - [M_{xy}]\ \{\ddot{U}_{yG}(t)\}. \tag{38}$$

Noting that:

$$[M_{xy}] = [M_{xx}]\ [M_{xx}^{-1}]\ [M_{xy}] , \tag{39}$$

the equation of motion for the relative displacements $\{W_x\}$ becomes:

$$[M_{xx}]\ \{\ddot{W}_x\} + [K_{xx}]\ \{W_x\} = [M_{xx}]\left[[K_{xx}^{-1}]\ [K_{xy}] - [M_{xx}^{-1}]\ [M_{xy}]\right]\ \{\ddot{U}_{yG}\}\ . \qquad (40)$$

The boundary conditions for $\{W\}$ are homogeneous by virtue of $\{W_y\} = 0$, as given by (35).

Equation (40) may be written in the form;

$$[M_{xx}]\ \{\ddot{W}_x\} + [K_{xx}]\ \{W_x\} = [M_{xx}]\ [I_{dx}]\ \{\ddot{U}_{yG}(t)\}\ , \qquad (41)$$

where;

$$[I_{dx}] = [K_{xx}^{-1}]\ [K_{xy}] - [M_{xx}^{-1}]\ [M_{xy}]\ . \qquad (42)$$

The right-hand side of Equation (41) is the inertial loading vector exciting the response of the structure relative to the response produced by the boundary displacements as if the structure were massless. The matrix $[I_{dx}]$ given by Equation (42) provides the inertial load distribution. It is composed of a component due to a unit displacement of the support nodes, plus a second component due to a unit acceleration of the support points. The second term accounts for the dynamic coupling through the mass matrix.

If lumped masses are used for the discrete system representation, the mass matrix is diagonal. Thus, the term $[M_{xy}]$ is identically zero. The load distribution matrix $[I_{dx}]$ then becomes:

$$[\tilde{I}_{dx}] = [K_{xx}^{-1}]\ [K_{xy}]\ , \qquad (43)$$

which is just the matrix of structural displacements due to a unit displacement of the support nodes. The additional term resulting from unit support accelerations is zero since there is no coupling through the mass matrix.

The definition of the matrix $[I_{dx}]$ given by Equation (43) is precisely that defined for lumped mass multiple degree-of-freedom systems subject to seismic excitation [1]. However, this definition is often applied to discrete representations of structural systems which use consistent mass matrices, popular in finite element computer programs.

If the support excitations $\{U_{yG}(t)\}$ are given in the form;

$$\{U_{yG}(t)\} = \{U_G\}\ G(t)\ , \qquad (44)$$

Equation (41) can be solved by the method of modal superposition. The solution $\{W_x(t)\}$ is then given as:

$$\{W_x\} = \sum_{n=1}^{N} q_n(t)\ \Gamma_n\ \{\psi_n\}\ , \qquad (45)$$

where $q_n(t)$ is the response of the generalized coordinate for the $n^{th}$ mode, $l_n$ is the participation factor of the $n^{th}$ mode and $\{\psi_n\}$ is the $n^{th}$ mode shape. N is the total number of degrees of freedom in the discrete model. The mode shapes $\{\psi_n\}$ and natural frequencies $\omega_n$ are found as solutions of the Eigenvalue problem:

$$\left[[K_{xx}] - \omega^2\ [M_{xx}]\right]\ \{\psi\} = 0\ . \qquad (46)$$

For an undamped structural system, the generalized coordinates are given by Duhamels Integral [6] as;

$$q_n(t) = \frac{1}{\omega_n} \int_0^t \ddot{G}(\tau) \sin \omega_n (t-\tau) \, d\tau \; ; \tag{47}$$

and the participation factors for each mode are given by;

$$\Gamma_n = \frac{\{\psi_n\}^T [M_{xx}] [I_{dx}] \{U_G\}}{\{\psi_n\}^T [M_{xx}] \{\psi_n\}} \; . \tag{48}$$

For discrete systems having off-diagonal terms in the mass matrix such that dynamic coupling exists, $[I_{dx}]$ is given by Equation (42) when determining the modal participation factors. For lumped mass systems having no dynamic coupling, $[I_{dx}]$ is given by Equation (43).

It should be noted that the prescribed support motions do not have to be given by Equation (44) for the method of modal superposition to be valid. In fact, different time functions can be specified for different support points. A more detailed treatment of the more general support excitation is given in Reference [7] and will not be treated further. Also, under restrictions on the form of a damping matrix that it be proportional to the mass and/or stiffness matrices, the method of modal superposition remains valid. The only difference is that the solution for the generalized coordinate $q_n(t)$ is given by:

$$q_n(t) = \frac{1}{W_{nd}} \int_0^t G(\tau) \exp \{-\zeta_n \omega_n (t-\tau)\} \sin \omega_{nd} (t-\tau) \, d\tau \; , \tag{49}$$

where $\omega_{nd}$ is the damped natural frequency;

$$\omega_{nd} = \omega_n \sqrt{1 - \zeta_n^2} \; , \tag{50}$$

and $\zeta_n$ is the viscous damping factor for the $n^{th}$ mode. The definition of the participation $\Gamma_n$ remains unchanged for either damped or undamped systems. The effect of the $[I_{dx}]$ matrix given by Equation (42) on the Participation Factors, $\Gamma_n$, given by Equation (48), is the concern of the present work. No further discussion of the effects of damping will be made.

Since many seismic analysis problems are solved using Response Spectra and the Square Root of the Sum of the Squares (SRSS) method of combining modal responses without recourse to the time response of the generalized coordinate, $q_n(t)$, it is the participation factors that are of greatest concern. The subsequent section presents an example illustrating the effect of the $[I_{dx}]$ matrix on the dynamic response of a simple system, including the effect on the participation factors.

## 4.0 Example Problem

To illustrate the above concepts, consider the dynamic response of a uniform shear beam to a prescribed motion of one end, as shown on Figure 1. It was demonstrated above that the time dependence of the boundary displacement only affects the time history response of the generalized coordinates, $q_n(t)$, of each mode. Therefore, the support motion will be taken to be:

$$U_G(t) = bt^2 \; , \tag{51}$$

so that the solution can be compared to an exact solution given in Reference [2] for the case of the axial dynamic response of a uniform bar. The equations which follow may be

compared with the solutions presented in Reference [2] by replacing the Shear Modulus, G, by the Modulus of Elasticity, E.

4.1 Continuous Solution

The continuous equation of motion for a uniform shear beam subject to a prescribed displacement at x = 0 of:

$$u(0) = bt^2 , \tag{52}$$

is;

$$\frac{\partial^2 u}{\partial t^2} - \frac{G}{\rho}\frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 < x < L . \tag{53}$$

The boundary conditions are given by Equation (52) at x = 0. At x = L, there is no force such that;

$$\frac{\partial u}{\partial x}(L) = 0 . \tag{54}$$

The initial conditions corresponding to no motion at time, t = 0, are:

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 . \tag{55}$$

Equations (52) through (55) comprise a homogeneous initial value problem subject to non-homogeneous boundary conditions and zero initial conditions. Using the solution presented in Reference [2], the response of the free end, u(L), of the shear beam is:

$$u(L) = bt^2 - \frac{\rho bL^2}{G} + \frac{32}{\pi^3}\frac{\rho bL^2}{G}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{(2n-1)^3}\cos\left[\frac{2n-1}{2}\pi\sqrt{\frac{G}{\rho L^2}}\,t\right]\right\} . \tag{56}$$

Evaluating the first few terms of the series solution given in Equation (52), we have:

$$u(L) = bt^2 - \frac{\rho bL^2}{G}\left\{1.000 - 1.032\cos\left(1.571\sqrt{\frac{G}{\rho L^2}}\,t\right) + 0.038\cos 4.713\sqrt{\frac{G}{\rho L^2}}\,t - \ldots\ldots\right\} . \tag{57}$$

4.2 Discrete Approximation

Now consider a finite element approximation to the previous solution. For simplicity, two elements will be used. Figure 2 shows the finite element model and node numbers. Node 1 corresponds to the support where the prescribed motion is applied. Node 3 is the free end of the shear beam.

The stiffness and consistent mass matrix for each of the two elements are given by:

$$[m] = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} ; \tag{58}$$

$$[k] = \frac{2GA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} . \tag{59}$$

The global stiffness and mass matrices for the system thus become:

$$[M] = \frac{\rho AL}{12}\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & 4 & 1 \\ \hline 0 & 1 & 2 \end{array}\right] ; \tag{60}$$

$$[K] = \frac{2GA}{L}\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 2 & -1 \\ \hline 0 & -1 & 1 \end{array}\right] ; \tag{61}$$

corresponding to the displacement vector $\{U\}^T = \{U_3, U_2, U_1\}$. The boundary displacement is $U_y = U_1$. Thus, $\{U_x\}^T = \{U_3, U_2\}$. The stiffness and mass matrices given by (60) and (61) have been partitioned to correspond to the boundary and internal displacements. Thus,

$$[M_{xx}] = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} , \quad [K_{xx}] = \frac{2GA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} , \tag{62}$$

and;

$$[M_{xy}] = \frac{\rho AL}{12}\begin{bmatrix} 0 \\ 1 \end{bmatrix} , \quad [K_{xy}] = \frac{2GA}{L}\begin{bmatrix} 0 \\ -1 \end{bmatrix} . \tag{63}$$

Taking the inverses of $[K_{xx}]$ and $[M_{xx}]$ and substituting into the definition of the inertial load distribution matrix;

$$[I_{dx}] = [K_{xx}^{-1}]\,[K_{xy}] - [M_{xx}^{-1}]\,[M_{xy}] , \tag{42}$$

we have:

$$[I_{dx}] = \left(\frac{L}{2GA}\right)\left(\frac{2GA}{L}\right)\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} - \left(\frac{\rho AL}{12}\right)\left(\frac{12}{7\rho AL}\right)\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} , \tag{64}$$

$$[I_{dx}] = \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} -1 \\ 2 \end{bmatrix} . \tag{65}$$

The first term in Equation (65) corresponds to the static displacements of the system due to a unit displacement of the support, $U_1$. The second term represents additional displacements resulting from a unit acceleration applied at the support, $\ddot{U}_1$. The frequencies and mode shapes of the system are given by:

$$\omega_1 = 1.611\sqrt{\frac{G}{\rho AL^2}} , \quad \omega_2 = 5.629\sqrt{\frac{G}{\rho AL^2}} , \tag{66}$$

and;

$$\{\psi_1\} = \begin{Bmatrix} 1 \\ \frac{\sqrt{2}}{2} \end{Bmatrix} , \quad \{\psi_2\} = \begin{Bmatrix} 1 \\ -\frac{\sqrt{2}}{2} \end{Bmatrix} . \tag{67}$$

The participation factors are given by Equation (48);

$$\Gamma_n = \frac{\{\psi_n^T\}\ [M_{xx}]\ [I_{dx}]}{\{\psi_n^T\}\ [M_{xx}]\ \{\psi_n\}} \quad . \tag{48}$$

Substituting from (62), (65), and (67) into (48), the participation factors are found to be:

$$\Gamma_1 = -1.3377 \ ; \qquad \Gamma_2 = 0.4806 \ . \tag{68}$$

If the second term arising from the dynamic coupling is neglected in the calculation of $[I_{dx}]$, the participation factors are given by:

$$\tilde{\Gamma}_1 = -1.207 \ ; \qquad \tilde{\Gamma}_2 = 0.207 \ . \tag{69}$$

If only the static displacements resulting from a unit support displacement are used in determining the participation factors, the error is found to be approximately -10 percent for the first mode participation factor and -60 percent for the second mode participation factor. Thus, the calculated response would be less than the exact solution.

The generalized coordinates for each mode are given by Equation (47);

$$q_n(t) = \frac{1}{\omega_n} \int_0^t \ddot{G}(\tau) \sin \omega_n (t-\tau)\, d\tau \ , \tag{47}$$

where $\ddot{G}(\tau) = \ddot{U}_1(\tau) = 2b$. Performing the integration and substituting for $\omega_1$ and $\omega_2$ from Equation (66), the generalized coordinates are given by:

$$\begin{aligned} q_1(t) &= 0.7706 \frac{\rho L^2}{G} \left[1 - \cos \left(1.611 \sqrt{\frac{G}{\rho L^2}}\, t\right)\right] \quad \text{(i)} \ ; \\ q_2(t) &= 0.0631 \frac{\rho L^2}{G} \left[1 - \cos \left(5.629 \sqrt{\frac{G}{\rho L^2}}\, t\right)\right] \quad \text{(ii)}. \end{aligned} \tag{70}$$

The solution is given by the superposition of the modal responses:

$$\{W\} = \sum_{n=1}^{N} \Gamma_n\, q_n\, (t)\, \{\psi_n\}, \tag{45}$$

where $\{W\}$ are the displacements relative to the displacement of an equivalent massless structure subject to the boundary motions. For the present case:

$$\{U\} = \{W\} + \{bt^2\} \ . \tag{71}$$

Substituting for the participation factors from (68), the generalized coordinates from (70) and the mode shapes from (67) and evaluating the displacement as the top of the beam, $W_3$, we have:

$$W_3(t) = -\frac{\rho L^2}{G} \left\{1.000 - 1.030 \cos \left(1.611 \sqrt{\frac{G}{\rho L^2}}\, t\right) + 0.030 \cos \left(5.629 \sqrt{\frac{G}{\rho L^2}}\, t\right)\right\}. \tag{72}$$

The total displacement of the top of the beam, $U_3$, is then given by:

$$U_3(t) = bt^2 - \frac{\rho L^2}{G}\left\{1.000 - 1.030 \cos\left(1.611\sqrt{\frac{G}{\rho L^2}}\,t\right) + .030 \cos\left(5.629\sqrt{\frac{G}{\rho L^2}}\,t\right)\right\}; \qquad (73)$$

which is practically identical to the exact solution given by (57) with the exception of the differences in natural frequencies and lack of higher mode responses in the finite element solution. The use of more elements would, of course, improve the accuracy.

If the participation factors $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ given by Equation (69) were used, the total displacement at the top of the beam would be:

$$\tilde{U}_3(t) = bt^2 - \frac{\rho L^2}{G}\left\{0.917 - 0.930 \cos\left(1.611\sqrt{\frac{G}{\rho L^2}}\,t\right) + 0.013 \cos\left(5.629\sqrt{\frac{G}{\rho L^2}}\,t\right)\right\}. \qquad (74)$$

The rigid body displacement is, of course, exactly the same. However, the elastic displacements which give rise to stresses are approximately 10 percent less for the first mode response and 60 percent less for the second mode response. These errors are a direct consequence of the errors in the participation factors.

## 5.0 Summary and Conclusions

A derivation of a matrix to distribute seismically induced inertial forces throughout a structural system has been presented. It was shown that the matrix is composed of two parts. The first part is equivalent to the structural displacements resulting from a unit displacement of the supports and is present for all structural systems, regardless of the type of mass matrices used. The second part is equivalent to the structural accelerations resulting from a unit support acceleration and results from the dynamic coupling represented by the off-diagonal terms in consistent mass matrices. If lumped mass matrices are used, there is no dynamic coupling and the second part of the inertial load distribution matrix is zero.

An example has been presented which demonstrates that the neglect of the dynamic coupling part of the inertial load distribution matrix leads to predicted responses which are less than actual. One problem with some finite element programs is that the matrix $[I_{dx}]$ is not calculated from matrix operations. Instead, it is often constructed by setting the elements of $[I_{dx}]$ equal to 1 for displacements in the direction of support motion. For the portion of $[I_{dx}]$ arising from the static support displacements, this method is valid. However, it does not account for the contribution due to dynamic coupling when consistent mass matrices are used. Calculation of the matrix $[I_{dx}]$ based on Equation (42) can readily be accomplished as a few extra steps in a computer program and should be implemented in programs using consistent mass matrices.

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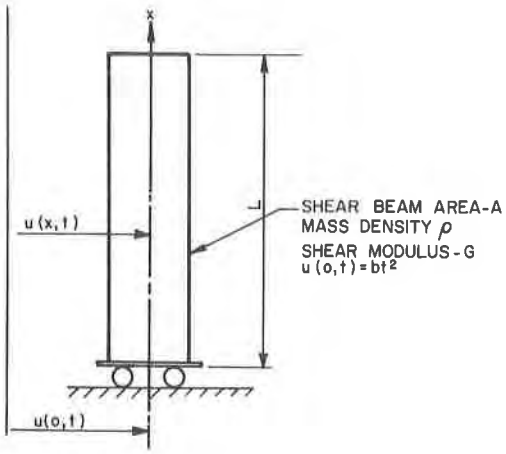


FIG. 1 SHEAR BEAM SUBJECTED TO BASE EXCITATION

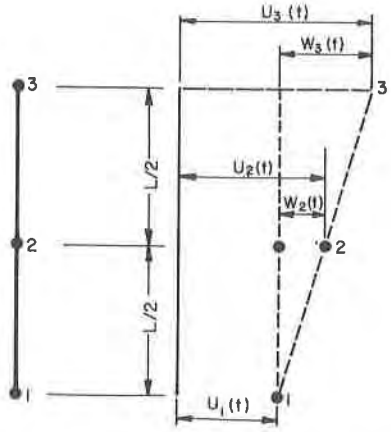


FIG. 2 FINITE ELEMENT MODEL AND DEFORMED POSITION