A FINITE ELEMENT METHOD FOR SEISMIC RESPONSE ANALYSIS

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SUMMARY

In the mathematical modeling of a structural system or component, the support points of the structure can often be idealized by a single base point representation where the gross effect of the support motion is characterized by translational as well as rotational motions at the base point. For the response spectrum analysis of the structural system or component, the excitation due to the support motion is then expressed in terms of a response spectrum generated by the consideration of the translational base motion only. The effect of the base rotation is often neglected by assuming that it contributes only secondary effect on the overall structural response. For some structural models, the effect of the base rotation may have significant contribution on the structural response. The neglecting of the rotational effect may underestimate the result.

The purpose of this paper is to present a complete finite element method for the response spectrum analysis of a structural system subjected to a base excitation expressed in terms of both translational and rotational response spectra. The major complexity involved in the procedure is the formulation of the inertial forces associated with a non-inertial frame which is attached to and in motion with the base point of the structure. Throughout the discussion, the rotation of the base point is assumed to be small. The formulation is general and applied to any type of finite element discretization. The inertial forces are formulated on an element by element basis. The mass matrix of the element can be formulated by either the lumped or the consistant mass approach. The procedure on condensation is also utilized to reduce the solution matrix of a large structural system. The basic principle used in the condensation is to preserve the system kinetic energy, potential energy and the load potential.

The concept of the rotational base excitation in terms of a response spectrum is introduced. The physical meanings of participation factors and modal masses associated with the rotational response spectrum are closely examined. The concept of the modal mass defined by the Naval Research Laboratory for translational base motion is extended to the situation with base rotation. Thus, modal masses with respect to rotations and translation-rotation couplings are introduced. A simple structural system is selected to illustrate the use of the translational as well as the rotational response spectra. Results of the structural response due to translation and rotation are presented. The accuracy of the solution due to condensation is also examined. Finally, the method described in this presentation is general and can be implemented into any general purpose finite element computer code.

1. Introduction

Response spectrum analysis is generally considered to be a convenient mathematical method for the determination of maximum responses of a structural system under seismic excitation. In response spectrum analysis, the excitation at the base of a structural system is generally expressed in terms of response spectra generated by the translational base motion in different directions. For a mathematical model of a structural system or component, the base of the model often represents a collection of many structural supports. The gross effect of the base motion will generally produce a combined translational and rotational excitation. In most of the structural analyses using the response spectrum method, the structural responses due to the effect of base rotation are often neglected. For some structural models, the effect of the base rotation may have significant contribution on the overall responses. Neglecting the rotational effect may underestimate the magnitude of the structural responses. Thus, it is desirable to extend the computational procedure for response spectrum analysis to include the combined translational and rotational base motion.

In this discussion, a complete finite element procedure for the response spectrum analysis of a structural system under combined translational and rotational base excitation is presented. The mathematical formulations based on the finite element method are derived, and the use of the response spectra in conjunction with the combined translational and rotational excitation are presented in the subsequent sections. The concept of modal effective mass is closely examined and is generalized to the situation of base rotation. The technique of structural condensation in response spectrum analysis is also discussed in the text. Finally, a simple example is given to illustrate the application.

2. Mathematical Formulation

Consider a structural system which is subjected to both translational and rotational base motion. Assume that the motion of the base rotation is small and can be represented by a vector. The equations of motion for the finite element model of the structural system can be represented by

$$[M]\{\ddot{U}\} + [C]\{\dot{u}\} + [K]\{u\} = \{0\} \qquad (1)$$

where

- $[M]$, $[C]$ and $[K]$ are the system mass, damping, and stiffness matrices, respectively;
- $\{\ddot{U}\}$ is the absolute acceleration vector;
- $\{\dot{u}\}$ and $\{u\}$ are the velocity and displacement vectors relative to a coordinate system which is fixed to and in motion with the base of the structure.

Let X_1, X_2, and X_3 be the coordinate axes of an inertial frame, and let x_1, x_2, and x_3 be the axes of a moving frame which is fixed to the base of the structure. Assume that the two coordinate systems coincide initially and that the moving frame has a translational acceleration $\ddot{\underset{\sim}{s}} = (\ddot{s}_1, \ddot{s}_2, \ddot{s}_3)$, and a rotational acceleration $\ddot{\underset{\sim}{\theta}} = (\ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}_3)$ with respect to the inertial frame of reference. The absolute acceleration of a point in the structure can be expressed by

$$\begin{aligned}
\ddot{X}_1 &= \ddot{s}_1 + \ddot{\theta}_2 x_3 - \ddot{\theta}_3 x_2 + \ddot{u}_1 \\
\ddot{X}_2 &= \ddot{s}_2 + \ddot{\theta}_3 x_1 - \ddot{\theta}_1 x_3 + \ddot{u}_2 \\
\ddot{X}_3 &= \ddot{s}_3 + \ddot{\theta}_1 x_2 - \ddot{\theta}_2 x_1 + \ddot{u}_3
\end{aligned} \qquad (2)$$

where $\ddot{u}_i$'s are relative accelerations with respect to the moving frame.
Let the directional cosines of the translational and the rotational vectors be

$$\left.\begin{aligned} \gamma_i &= \ddot{s}_i/\ddot{s} \\ \beta_i &= \ddot{\theta}_i/\ddot{\theta} \end{aligned}\right\} \quad (i = 1,2,3) \tag{3}$$

where

$$\begin{aligned} \ddot{s} &= (\ddot{s}_1^2 + \ddot{s}_2^2 + \ddot{s}_3^2)^{1/2} \\ \ddot{\theta} &= (\ddot{\theta}_1^2 + \ddot{\theta}_1^2 + \ddot{\theta}_3^2)^{1/2} \end{aligned} \tag{4}$$

The equations of motion for the structural system can be expressed by

$$\begin{aligned} [M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} &= \{F\} \\ &= -[M]\{\Gamma\}\ddot{s} - [M]\{\lambda\}\ddot{\theta} \end{aligned} \tag{5}$$

where

$$\begin{aligned} \Gamma_i &= \alpha_{ij}\gamma_j \\ \lambda_i &= \alpha_{ij}\,\varepsilon_{jk\ell}\,\beta_k\,x_\ell \end{aligned} \tag{6}$$

$$\alpha_{ij} = \begin{cases} 1 & \text{If the ith degree of freedom (DOF) is a translational DOF parallel to the jth inertial axis.} \\ 0 & \text{Otherwise.} \end{cases}$$

x_ℓ = coordinates of the node.

The repeated index in eq. (6) implies summation. $\varepsilon_{jk\ell}$ is the permutation tensor of rank 3.
If $[\phi]$ is the matrix of eigenvectors associated with the free and undamped system of eq. (5), the mass, damping, and stiffness matrices can be diagonalized by the following similarity transformations;

$$\begin{aligned} [\phi]^T[M][\phi] &= [\diagdown I \diagdown] \\ [\phi]^T[C][\phi] &= [\diagdown 2\xi \diagdown] \\ [\phi]^T[K][\phi] &= [\diagdown w^2 \diagdown] \end{aligned} \tag{7}$$

where the eigenvectors are normalized with respect to the mass matrix.
By letting

$$\{u\} = [\phi]\{\mu\} \tag{8}$$

eq. (5) can be uncoupled into the following form

$$[\diagdown I \diagdown]\{\ddot{\mu}\} + [\diagdown 2\xi \diagdown]\{\dot{\mu}\} + [\diagdown w^2 \diagdown]\{\mu\} = -[\phi]^T[M]\{\Gamma\}\ddot{s} - [\phi]^T[M]\{\lambda\}\ddot{\theta} \quad (9)$$

For the ith mode, the equation of motion can be written as

$$\ddot{\mu}_i + 2\xi_i\dot{\mu}_i + w_i^2\mu_i = -p_i\ddot{s}(t) - q_i\ddot{\theta}(t) \quad (10)$$

where

ξ_i is the damping ratio

w_i is the circular frequency

p_i and q_i are defined by

$$p_i = \{\Gamma\}^T[M]\{\phi^{(i)}\}$$
$$q_i = \{\lambda\}^T[M]\{\phi^{(i)}\} \quad (11)$$

where $\{\phi^{(i)}\}$ denotes the eigenvector for mode i.

In seismic response analysis, the motion of the base excitation is generally considered in three orthogonal directions. The coefficients p_i and q_i for each mode will not change with time. Eq. (10) is the basic equation used for the determination of the maximum response of the structural system. The method generally used for translational base excitation and the techniques which can be used to incorporate the rotational base excitation are discussed in the following section.

2.1 Determination of Maximum Structural Responses

In the case of translational base motion, the concept of response spectrum is often utilized to estimate the maximum responses of a structural system. The response spectrum is a plot of the maximum responses (in terms of acceleration or displacement or stress, etc.) of all possible one DOF systems with a certain damping ratio subjected to a given translational base excitation. Once the response spectrum has been constructed, the (maximum) spectral acceleration for the ith mode is computed by

$$\{\ddot{u}^{(i)}\} = p_i a_i \{\phi^{(i)}\} \quad (12)$$

and the (maximum) spectral displacement for mode i is determined to be

$$\{u^{(i)}\} = \frac{p_i a_i}{w_i^2}\{\phi^{(i)}\} \quad (13)$$

where a_i is the maximum acceleration at frequency, w_i from the plot of the response spectrum, and p_i is the modal participation factor [1].

The maximum structural response is generally estimated by taking the square root of the sum of the squares (SRSS) of the spectral responses for each mode.

For the combined translational and rotational base motion, the generalized force for each mode is a linear combination of the translational and the rotational motions as shown in eq. (10). Since the ratio between p_i and q_i is generally different for each mode, the construction of response spectra which can be used for all structures requires the consideration of all possible combinations of the translational and the rotational motions. In practice this complexity can be avoided by constructing only those points in a spectrum table which are really needed in a response spectrum analysis of a given structure.

One of the simplest and most conservative ways of considering the translational and rotational excitations in response spectrum analysis is to treat the two motions independently. The two response spectra corresponding to the translational and rotational base excitations can be constructed separately. Then following the procedures of response spectrum analysis for the translational base excitation discussed earlier, the maximum structural responses can be determined for the translational as well as the rotational base excitations. The maximum structural responses under the combined translational and rotational excitation can be estimated by taking the absolute sum of their individual maximum responses. Under this consideration, the coefficients p_i and q_i in eq. (11) can be defined as the modal participation factors for translational and rotational excitations respectively.

2.2 Modal Effective Mass

The modal effective mass as defined by the Naval Research Laboratory [2] for the translational base motion is the mass of a one DOF system which will exert the same reaction force on the system base as that exerted by the structural mode under consideration. Modal mass is commonly used as a variable in determining the effective shock transmitted to the structural base.

Consider an undamped structural system subjected to a translational base motion. The uncoupled equation of motion for a mode, say mode i, can be represented by

$$\ddot{\mu} + w^2\mu = -p\,\ddot{s}(t) \qquad (14)$$

Construct a single DOF system of mass m^* and stiffness k^* and subjected to the base motion $\ddot{s}(t)$. The equation of motion is expressed by

$$m^*\ddot{x} + k^*x = -m^*\ddot{s}(t)$$

or (15)

$$\ddot{x} + w^2x = -\ddot{s}(t)$$

Assume that the single DOF system has the same frequency as the mode under consideration, i.e.,

$$w = \sqrt{k^*/m^*} \qquad (16)$$

The displacement and velocity of the two systems are related by

$$\mu = px$$
$$\dot{\mu} = p\dot{x} \tag{17}$$

The kinetic energy and the potential energy of the single DOF system are expressed by

$$K.E. = \frac{1}{2} m^* \dot{x}^2 = \frac{1}{2} \left(\frac{m^*}{p^2}\right) \dot{\mu}^2$$
$$P.E. = \frac{1}{2} k^* x^2 = \frac{1}{2} \left(\frac{m^*}{p^2}\right) w^2 \mu^2 \tag{18}$$

By letting

$$m^* = p^2 \tag{19}$$

the kinetic energy and the potential energy for the single DOF system will be equal to that of the structural mode under consideration.

Let R and R^* be the component of the reaction forces on the base in the direction of the shock exerted by the ith spectral mode and the single DOF system respectively. The two reaction forces can be proven to be identical through the following manipulations.

$$R = \{\Gamma\}^T [K] \{u^{(i)}\}$$
$$= \mu\{\Gamma\}^T [K] \{\phi^{(i)}\} = \mu w^2 \{\Gamma\}^T [M] \{\phi^{(i)}\} \tag{20}$$
$$= \mu w^2 p = k^* x = R^*$$

Thus the single DOF system can be considered as the equivalent system of the structural mode under consideration. The mass m^* as defined by eq. (19) is generally referred to as the modal effective mass [2]. It should be noted that the sum of all the modal effective masses is equal to the total mass of the structural system.

The concept of modal effective mass can be generalized to include the situation of rotational base excitation. Consider an uncoupled equation of motion for a mode

$$\ddot{\mu} + w^2 \mu = -p\ddot{s}(t) - q\ddot{\theta}(t) \tag{21}$$

Decompose the generalized coordinate into the translational and rotational components.

$$\mu = \mu_t + \mu_r$$

such that

$$\ddot{\mu}_t + w^2 \mu_t = -p\ddot{s}(t)$$
$$\ddot{\mu}_r + w^2 \mu_r = -q\ddot{\theta}(t) \tag{22}$$

and construct an equivalent two DOF system with the equation of motion

$$\begin{bmatrix} m^*_{tt} & m^*_{tr} \\ m^*_{rt} & m^*_{rr} \end{bmatrix} \left(\begin{Bmatrix} \ddot{x}_t \\ \ddot{x}_r \end{Bmatrix} + w^2 \begin{Bmatrix} x_t \\ x_r \end{Bmatrix} \right) = \begin{bmatrix} m^*_{tt} & m^*_{tr} \\ m^*_{rt} & m^*_{rr} \end{bmatrix} \begin{Bmatrix} -\ddot{s}(t) \\ -\ddot{\theta}(t) \end{Bmatrix} \tag{23}$$

Assume that the equivalent two DOF system has the same kinetic and potential energies as the structural mode under consideration. The modal effective mass matrix can be determined by

$$\begin{bmatrix} m^*_{tt} & m^*_{tr} \\ m^*_{rt} & m^*_{rr} \end{bmatrix} = \begin{bmatrix} p^2 & pq \\ qp & q^2 \end{bmatrix} \tag{24}$$

where m^*_{tt}, m^*_{rr}, and m^*_{tr} are modal effective masses with respect to translation, rotation, and translation-rotational coupling.

Let R and B be the base reaction force and bending moment exerted by the spectral mode under consideration, and R^* and B^* be the same variables for the equivalent two DOF system. The two sets of variables can be shown to be identical from the following manipulations:

For the base reactional force in the direction of the translational shock

$$\begin{aligned} R &= \{\Gamma\}^T [K] \{u^{(i)}\} \\ &= w^2 \{\Gamma\}^T [M] \{\phi^{(i)}\}(\mu_t + \mu_r) \\ &= w^2 p(\mu_t + \mu_r) = w^2 p(px_t + qx_r) \\ &= w^2(m^*_{tt}x_t + m^*_{rt}x_r) = R^* \end{aligned} \tag{25}$$

and the bending moment in the plane of rotation

$$\begin{aligned} B &= \{\lambda\}^T [K] \{u^{(i)}\} \\ &= \{\lambda\}^T [K] \{\phi^{(i)}\}(\mu_t + \mu_r) \\ &= w^2 q(\mu_t + \mu_r) = w^2 q(px_t + qx_r) \\ &= w^2(m^*_{rt}x_t + m^*_{rr}x_r) = B^* \end{aligned} \tag{26}$$

It should be noted that the sum of the modal effective masses for all the modes will be equal to the structural masses (or inertia) with respect to the base translational and rotational shock. This information provides an important check on the results obtained from the finite element analysis.

3. Finite Element Procedure and Structural Condensation

The finite element procedure for response spectrum analysis is generally implemented into a computer program as an extension of the modal analysis capability [3] with certain additional computations. At the element level, the inertia forces associated with a unit translation ($\ddot{s} = 1$) and a unit rotation ($\ddot{\theta} = 1$) in certain directions are formulated separately according to eqs. (5) and (6). The damping matrix and the history of the base motion are not needed in the analysis since they are considered during the response spectrum generation stage. After the eigenvalue/eigenvector solution the generalized forces corresponding to the right hand side of eq. (9) can be obtained. They are the participation factor p_i's and q_i's of eq. (10) if the translational and rotational base motions are treated independently. The spectral response for each mode can be determined from the given response spectra which are input to the response spectrum analysis. For the case where only translational base motion is considered, the spectral acceleration and displacement responses are determined by eqs. (12) and (13) respectively. Similar procedures are followed for the rotational base motion. In the case of combined translational and rotational base motion the response spectra corresponding to the correct combination of the translational and rotation base motion must be generated for each significant mode under consideration. The maximum structural responses are estimated by the combination of each spectral response as discussed in Section 2.1.

For a finite element model of a large structural system, frequencies and modes of significant importance can often be represented by a small number of selected degrees of freedom. The remaining dynamic DOF's are interpolated based on the pattern of their static deformation. This process of structural condensation is often utilized in a computer program to reduce the size of the eigen-space solution. The condensation technique in reference [4] can be applied to response spectrum analysis through the following processes.

The system stiffness matrix and the displacement vector are partitioned into the following form

$$\left[\begin{array}{c|c} K_{AA} & K_{AB} \\ \hline K_{BA} & K_{BB} \end{array}\right] \left\{\begin{array}{c} u_A \\ \hline u_B \end{array}\right\}$$

and the following displacement relationship can be imposed

$$[K_{BA}]\{u_A\} + [K_{BB}]\{u_B\} = \{0\}$$

Let $\{u_A\}$ be the selected dynamic DOF's. Then $\{u_B\}$ can be expressed in terms of $\{u_A\}$ by

$$\{u_B\} = -[K_{BB}]^{-1}[K_{BA}]\{u_A\} \tag{27}$$

The system displacement vector is then reduced by the following transformation

$$\begin{aligned} \{u\} &= \left\{\begin{array}{c} u_A \\ \hline u_B \end{array}\right\} \\ &= [Q]\{u_A\} \end{aligned} \tag{28}$$

where

$$[Q] = \left[\begin{array}{c} I \\ \hline -[K_{BB}]^{-1}[K_{BA}] \end{array}\right] \tag{29}$$

Assuming that the system potential energy is being conserved during the process of condensation, the stiffness matrix corresponding to the reduced system can be determined to be

$$[\tilde{K}] = [Q]^T[K][Q] \tag{30}$$

Using the same interpolation relation of eq. (22) for the velocity, i.e.,

$$\{\dot{u}\} = [Q]\{u_A\} \tag{31}$$

and by preserving the system kinetic energy and the damping dissipation energy, the reduced mass and damping matrices can be determined to be

$$\begin{aligned} [\tilde{M}] &= [Q]^T[M][Q] \\ [\tilde{C}] &= [Q]^T[C][Q] \end{aligned} \tag{32}$$

By preserving the system load potential

$$\{F\}^T\{u\} = \{\tilde{F}\}^T\{u_A\}$$

the reduced force vector can be determined by

$$\{\tilde{F}\} = [Q]^T\{F\} \tag{33}$$

The equations of motion in the reduced system can be written as

$$\begin{aligned} [\tilde{M}]\{\ddot{u}_A\} + [\tilde{C}]\{\dot{u}_A\} + [\tilde{K}]\{u_A\} &= \{\tilde{F}\} \\ &= -[Q]^T[M](\{\Gamma\}\ddot{s} + \{\lambda\}\ddot{\theta}) \end{aligned} \tag{34}$$

The response spectrum analysis is performed on the condensed system of eq. (34). The response for the entire system is obtained by eq. (28).

For a computer program which employs the wave front technique of Gaussian elimination [3] as an equation solver, the reduced system matrices and force vector are obtained by retaining those selected dynamic DOF's from the elimination.

4. Examples and Results

Consider a simple structural model which is free at the top and partially restrained to the ground at the bottom as shown in Fig. 1. The structural system is modeled by eight beam elements and three springs for translational and rotational restraints at the bottom. The mass is assumed to be uniformly distributed along the beam elements and there is a 5% damping for each mode. The computer program WECAN is used for the analysis. Let the structural system be subjected to a translational base acceleration as shown in Fig. 2. A time history analysis was performed on the system and the result indicates that the motion is determined predominately by the first natural mode of vibration. The first natural mode is a cantilever mode with a frequency of 6.65 Hz. Figs. 3

through 5 show the displacements at the top and mid-point of the beam. The maximum bending moment at the mid-point (or point B) of the beam is recorded to be 124 in-kip, which will be used for comparison with the response spectrum analysis.

To illustrate the use of the response spectrum analysis for a combined translational and rotational base motion, a free body of the top half of the beam is isolated from the system. The isolated component is shown in the upper right corner of Fig. 1 where point B is considered as the base point for this model. The motion at point B is obtained from the previous time history analysis.

By performing the response spectrum analysis on the component model as discussed in the earlier sections, two approaches were used to determine the maximum structural responses. In the case of treating the translational and rotational base excitations as independent motions, two response spectra describing the base excitations were generated as shown in Figs. 6 and 7. The maximum bending moment at point B is found to be 93 and 38 in-kip respectively for translational and rotational base motions. By taking the absolute sum of the two responses, the maximum moment at B is estimated to be 131 in-kip, which is 5.6% higher than the time history value. In the case of considering the combined or coupled translational and rotational base motion, response spectra are only constructed at two points for two significant modes (frequencies 23.6 and 83.0 Hz) at their appropriate combinations of translational and rotational motions. The maximum base moment determined in this case is 130 in-kip, which is 5% higher than the time history value.

In parallel to the time history run, the corresponding response spectrum analysis is also made for the complete structural system. Without using the condensation technique, the DOF's at all nine nodes are taken to be independent, and the maximum bending moment at point B is found to be 134 in-kip. By condensing the DOF's to three points (at A, B, and C), the maximum bending at point B is computed to be 126 in-kip. If the structure is condensed further to only one point (at A), the maximum bending moment is 151 in-kip.

5. Conclusions

The response spectrum analysis provides a simple and efficient mathematical method for the computation of upper bounds on a structural response. By the extension of the response spectrum analysis technique to include the rotational base excitation, a wider range of structural problems can be applied. This presentation provides a complete documentation on the computational procedure which can easily be implemented into any general purpose computer code. The modal effective mass is an important concept in response spectrum analysis. The mathematical postulates based on this concept are systematically derived in the text.

References

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[3] WECAN - Westinghouse Electric Computer Analysis, User's Manual, Westinghouse Research Laboratories, Pittsburgh, Pa. (February, 1973).

[4] GUYAN, R., "Reduction of Stiffness and Mass Matrices", AIAA Journal, Vol. 3, p. 380 (1965).

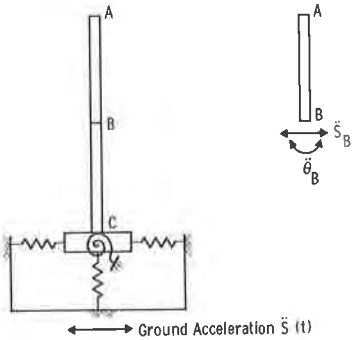

Fig. 1 - Dynamic model

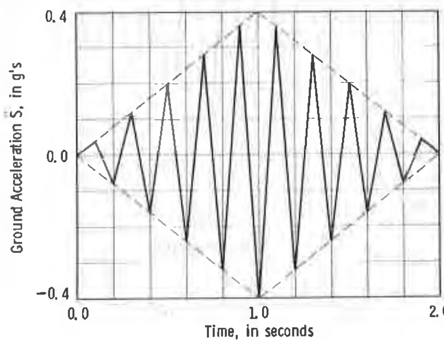

Fig. 2 - Ground acceleration

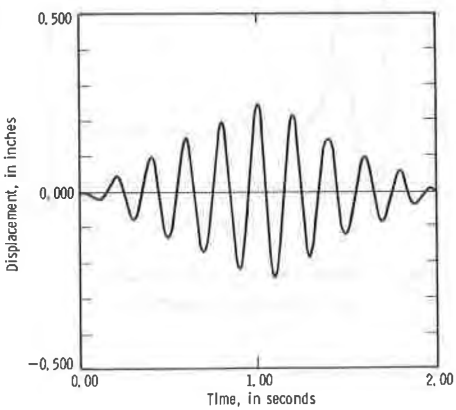

Fig. 3 - Lateral displacement at point A

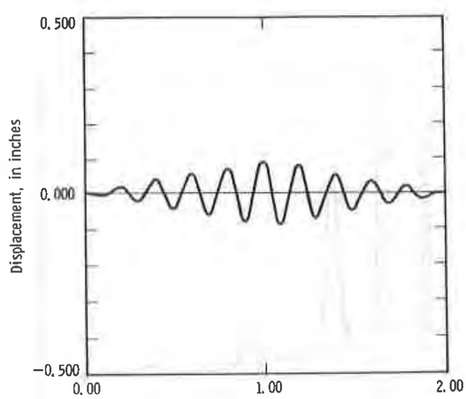

Fig. 4 - Lateral displacement at point B

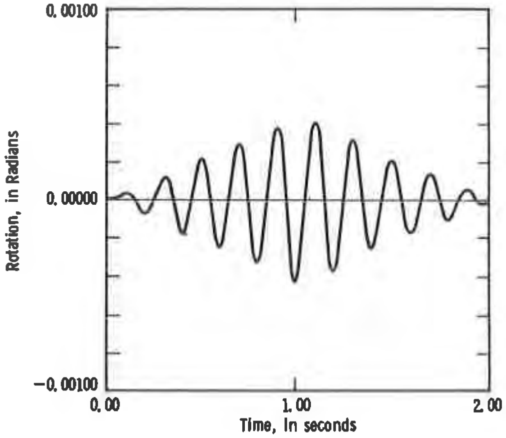

Fig. 5 - Rotation at point B

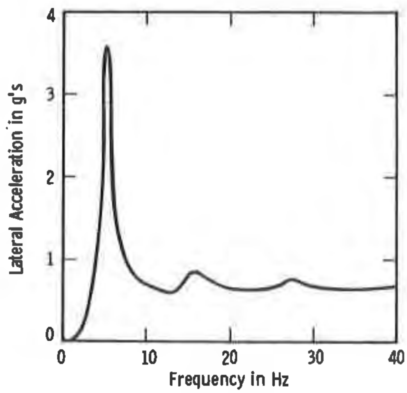

Fig. 6 - Response spectrum for lateral acceleration at point B

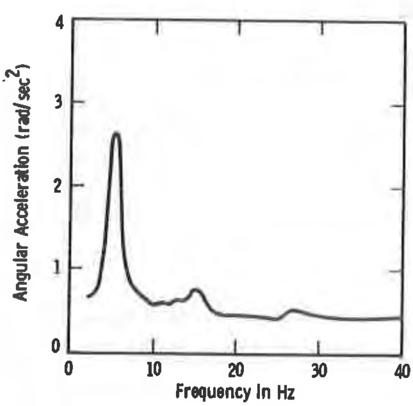

Fig. 7 - Response spectrum for angular acceleration at point B