SEISMIC RESPONSE OF FLEXIBLE LIQUID CONTAINERS

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SUMMARY

Seismic analysis of liquid storage containers is based, in present design practice, on the method developed by Housner and subsequent modifications and extensions such as by Velestos, and more recently by Epstein. According to this method, part of the liquid contained moves rigidly with the excited container while another part at the top is flexible and participates in sloshing. The container walls are assumed to be rigid, so that the hydrodynamic forces created may be calculated as a product of the flexible mass by the maximum ground or input acceleration. This simple approximation of the exact solution of the wave equation is intuitively appealing and in most cases satisfactory.

The purpose of this work is to demonstrate that the errors introduced by the assumption that the container is rigid could be avoided in practical design, since they may not be on the conservative side. Indeed, the flexibility of the container should not be ignored in cases where the structural period (say 0.10 s) corresponds to a "peak" in the response spectra. On the other hand, these errors may yield excessively conservative results. In the method described herein the effects of liquid damping, in the order of 1/2%, is neglected. Effects of sloshing (non-horizontal free surface) and liquid compressibility are accounted for.

A practical analysis procedure is presented based on the solution of the two dimensional wave equation for the liquid, with time dependent boundary conditions, imposed by the dynamic response of the flexible container walls. The liquid and the walls are allowed to iteratively interact until the final common frequencies and hydrodynamic pressures are obtained. The method while still approximate is superior to other such methods where accounting for flexibility is important. It is also generally applicable to containers of any practical shape and structure, as standard dynamic analysis procedures can be applied iteratively.

The procedure begins by finding the dynamic response in the absence of the liquid. The frequeny and displacement of each mode are used in the time dependent boundary condition in solving the wave equation for the liquid. A pressure function is then obtained and converted to an equivalent mass, which is used to correct the displacement shape and frequency. The process is repeated until no significant correction of the frequency is found. The implementation is done on an open rectangular tank, containing water. Four different cases of analysis are presented, accounting for sloshing and compressibility.

1. Introduction

Seismic analysis of liquid-storage containers is based, in current design practice, on the method developed in 1957 by Housner [1] and subsequent modifications and extensions such as in ref. [2] and more recently in 1976 by Epstein [3]. According to this method, part of the contained liquid moves rigidly with the excited container while other parts of the liquid are considered as masses attached with springs to the walls and participate in sloshing. Since the walls are assumed to be rigid, the acceleration is uniform along the height and the main effect of the hydrodynamic pressure is calculated as a product of the rigid mass and the maximum ground acceleration. This simple and fast procedure yields analyses which are frequently good approximations of the exact solution and the resulting design is usually believed to be on the conservative side. Elaborate analyses accounting for dam flexibility, such as in the works by Bustamante et al. in 1963 [4] and Chopra in 1969 [5] have shown that the calculated hydrodynamic pressures exceed those based on the rigid walls assumption. Veletsos in 1974 [6] presented a simple procedure for liquid-filled cylindrical tanks, based on the assumption that the system behaves as a single-degree-of-freedom system.

This work presents a practical procedure for analyzing the response of liquid containers, taking into consideration the flexibility of the walls and the effects of liquid compressibility and the non-horizontal shape of the free surface, if it is so desired. The effect of damping, in the order of 1/2%, is neglected. Only the steady state part of the response will be of concern. The method is based on the solution of the two-dimensional wave equation for the liquid, on one hand, and the solution of the beam equation for the cantilever container wall on the other. The elastic response of the wall to a seismic lateral excitation, given by means of its response spectra, is used as a time-dependent boundary condition for the wave equation. The liquid and the walls are allowed to interact iteratively until the final common frequencies and hydrodynamic pressure are obtained. While still approximate, the procedure avoids errors due to the assumption that the container is rigid and hence also resulting designs, which may or may not be conservative.

2. Governing Equations

While the procedure presented is applicable in principle to covered or open containers of any shape, only an open rectangular container will be considered herein, and its walls, which are perpendicular to the direction of seismic excitation, will be assumed to behave as simple cantilevers.

Consider an open rectangular container symmetric about the vertical y axis, with walls at x=L and x=-L, perpendicular to the horizontal direction of excitation, the x axis. The free surface is initially at rest at level y=H. The seismic excitation is given by means of its response spectra at the bottom of the tank, y=0. The objective is to calculate the hydrodynamic pressure, p (x, y, t), in excess of the static pressure.

The familiar wave equation and boundary conditions governing the response of the liquid are:

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} \qquad (1)$$

$$\frac{\partial \Phi}{\partial y} = 0 \quad , \text{ at } y=0 \tag{2}$$

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial y} = 0, \text{ at } y=H \tag{3}$$

$$\Phi = 0 \quad , \text{ at } x=0 \tag{4}$$

$$\frac{\partial \Phi}{\partial X} = - \frac{\partial W}{\partial t} \quad , \text{ at } X=L \tag{5}$$

where Φ is a velocity potential; W is the wall displacement in the X direction, varying along the vertical $W = W(y, t)$; c is the velocity of sound and g is the gravity acceleration. Denoting the liquid density and bulk modulus by ρ and B respectively then c, ρ and B are related by

$$c^2 = B/\rho \tag{6}$$

Given the initial conditions Φ can be solved for analytically, or numerically, and the hydrodynamic pressure p obtained from

$$p = \rho \frac{\partial \Phi}{\partial t} \tag{7}$$

It should be noted that the above assume the liquid to be compressible and the free surface not to remain horizontal. For incompressible liquid the differential equation (1) becomes

$$\nabla^2 \Phi = 0 \tag{8}$$

and for horizontal free surface, the boundary condition in eq. (3) becomes

$$\frac{\partial \Phi}{\partial y} = 0 \quad , \text{ at } y=H \tag{9}$$

The equation governing displacement of a wall with constant EI and mass per unit length m is

$$EI\, W^{IV} + m \ddot{W} = -p\ (y) \tag{10}$$

where p is the solution from eq. (7) evaluated at X=L. The negative sign accounts for the fact that compressive pressure is considered positive and it acts away from the positive direction of the deflection W. The boundary conditions are the usual ones for cantilevers and rest conditions are assumed. The solution of this equation is assumed to be of the general form

$$W = \Sigma\, \Psi_i\ (y) \sin \Omega_i t \quad ; i = 0, 1, 2, \ldots \tag{11}$$

where Ψ_i and Ω_i are the displacement shape and frequency (common to the wall and the liquid) associated with the i^{th} vibration mode and

$$\Psi_o (y) = d_{max} \tag{12}$$

where d_{max} is the maximum ground acceleration, so that if only Ψ_o is considered, the case of rigid walls is obtained.

For the purpose of substituting in the right hand side of eq. (5), the modes will be considered separately and the subscript i will be dropped for clarity. It should be emphasized that the function Ψ is the actual displacement shape of the wall for a certain mode rather than the mode shape itself.

3. Solutions for p

In solving for the steady state part of eq. (1), with a harmonically time-dependent boundary condition at X=L, the solution was assumed to have the form [7]

$$\Phi = G (x, y) \cos \Omega t$$

The following are four cases for which expressions for p were obtained. In all of them p has the form

$$p = \rho\Omega \sin\Omega t \ f (y) \tag{13}$$

3.1 Compressible liquid, non-horizontal free surface

$$p = \rho\Omega \sin\Omega t \sum_k C_k \sin \alpha_k x \cosh\beta_k y \tag{14}$$

where β_k is obtained from the solution of

$$(\beta_k H) \tan (\beta_k H) = \Omega^2 H/g \tag{15}$$

$$\alpha_k^2 = \Omega^2/c^2 + \beta_k^2 \tag{16}$$

$$C_k = \frac{\Omega}{\alpha_k \cos \alpha_k L} \frac{\int \Psi (y) \cosh \beta_k y \, dy}{\int \cosh^2 \beta_k y \, dy} \tag{17}$$

In the above and in what follows the limits of integration are from y=0 to y=H.

3.2 Compressible liquid, horizontal free surface

$$p = \rho \Omega \sin\Omega t \ \Sigma C_k \sin \alpha_k x \cos \frac{k\pi y}{2H} \tag{18}$$

where α_k is obtained from

$$\alpha_k^2 = \frac{\Omega^2}{c^2} - \frac{(k\pi)^2}{(2H)^2} \ ; \ k = 1, 3, 5, \ldots \tag{19}$$

and the constant C_k is calculated from

$$C_k = \frac{2\Omega}{H\ \alpha_k \cos \alpha_k L} \int \Psi\ (y) \cos \frac{k\pi y}{2H}\ dy \tag{20}$$

3.3 Incompressible liquid, non-horizontal free surface

$$p = \rho\Omega \sin\Omega t\ \Sigma\ C_k \sin \alpha_k x \cosh \alpha_k y \tag{21}$$

where α_k is obtained by solving the equation

$$(\alpha_k H) \tanh (\alpha_k H) = \Omega^2\ H/g \tag{22}$$

and the constant C_k is calculated from

$$C_k = \frac{\Omega}{\alpha_k \cos \alpha_k L} \frac{\int \Psi\ (y) \cosh \beta_k y\ dy}{\int \cosh^2 \alpha_k\ y\ dy} \tag{23}$$

3.4 Incompressible liquid, horizontal free surface

$$p = \rho\Omega \sin\Omega t\ \Sigma\ C_k \sinh \frac{k\pi x}{2H} \cos \frac{k\pi y}{2H}\ ;\ k\ \ 1, 3, 5, \ldots \tag{24}$$

where the constant C_k is calculated from

$$C_k = \frac{4\Omega}{k\pi \cosh \frac{k\pi L}{2H}} \int \Psi\ (y) \cos \frac{k\pi y}{2H}\ dy \tag{25}$$

4. The Wall Response

To calculate the common vibration frequency Ω_i corresponding to the i-th vibrational mode the diffrential equation for the wall will be written in a free-vibration form

$$EIW^{IV} + M\ddot{W} = 0 \tag{26}$$

which is obtained by substituting the value of p from eq. (13) into eq. (10) using the definition of the equivalent mass per unit length M

$$M\ (y) = M + \frac{p\ (y)}{g} = M + \rho \frac{\Omega}{g} f\ (y) \tag{27}$$

The expression for f (y) varies, depending on which of the cases in section 3.1 through 3.4 is considered.

Letting A_i and g_i (y) be the arbitrary constant and characteristic shape, respectively, of mode i the displacement shape may be expressed as

$$\Psi_i\ (y) \doteq A_i\ g_i\ (y) \tag{28}$$

The total deflection W is obtained as a summation as shown in eq. (11).

The following are well known solutions of eq. (28), see for example ref. [8].

An upper bound on the frequency is given by Raleigh's quotient Q_i

$$Q_i = \frac{EI \int (g'')^2\ dy}{\int Mg^2\ dy} \geq \Omega^2{}_i \tag{29}$$

The modal participation factor is

$$\gamma_i = \frac{\int M\ g_i\ dy}{\int M\ g_i^2\ dy} \tag{30}$$

The maximum value of A_i is given by

$$A_{i_{max}} = \gamma_i\ S_i/\Omega_i^{\ 2} \tag{31}$$

where S_i is the spectral acceleration. If $A_{i_{max}}$ is substituted in eq. (28), the maximum of the modal displacement Ψ_i is obtained.

5. Iterative Procedure

Obtaining the exact theoretical response of the container and the liquid involves the simultaneous solution of eqs. (1-10), a rather difficult task. Since the quantities Ψ (y), Ω and p cannot be solved independently, the following iterative procedure is presented to allow for their interaction.

a. Start by establishing initial values for Ω and Ψ corresponding to the seismic response of the container, with the liquid absent (p=o). Knowing Ω and Ψ an initial pressure function p will be determined, depending on the case at hand as shown in section 3.

b. Using p from the preceding step, obtain a new value for the mass M from eq. (27) This will determine corrected values for Ω, γ, A and Ψ from eqs. 29, 30, 31 and 28 respectively.

c. Knowing Ω and Ψ from the preceeding step a corresponding value of p can be calculated as in "a". Go to "b".

These steps are repeated for each of the modes being considered. In each cycle a convergence test (in step b) is performed, and the process is stopped when the change in the frequency is found to be insignificant. The final values of Ω, Ψ and p constitute then the desired response, and forces required for design can be calculated.

6. Implementation and Example

The procedure above was implemented on a simple cantilever with a unit width and constant stiffness and given values of M, EI, H. The liquid was assumed to be water. Only the first three modes of vibration were considered.

The initial value of Ω can be readily, and exactly, calculated from any handy reference. The characteristic mode shape g was assumed to be a polynomial for expediency. The participation factor γ can then be calculated with M=m. Corresponding to Ω, on the acceleration response spectra of the given earthquake, the spectral value S is obtained and hence the maximum amplitude A_{max} can be established from eq. (31). This will determine the maximum displacement response Ψ.

Solving for p from the wave equation involves the solution of the characteristic equation, which may be transcendental, and the calculation of the constant C_k as shown in section 3. For a simple cantilever only the first few values of k were considered in the summation.

7. Conclusion

The method presented involves the repetition of standard dynamic analysis procedures As such it is a general method and can be applied to obtain the hydrodynamic and seismic response in containers of any practical shape and structure. In particular, the walls need not behave as cantilevers, as assumed here to illustrate the principle.

References

[1] HOUSNER, G. W., "Dynamic Pressures on Accelerated Fluid Containers", Bulletin of the Seismological Society of America 47 (1), Jan. 1957, pp. 15-35.

[2] "Nuclear Reactors and Earthquakes", TID 7024, prepared by Lockheed Aircraft Corp. and Holmes and Narver Inc., U. S. Atomic Energy Commission, Aug. 1963.

[3] EPSTEIN, H. I., "Seismic Design of Liquid-Storage Tanks", J. of the Structural Division, ASCE, Vol. 102, No. ST9, Proc. Paper 12380, Sept. 1976, pp. 1659-1673.

[4] BUSTAMANTE, J. I., ROSENBLUETH, E. and FLORES, A., "Presion Hidrodinamica en Presas y Depositos," Bol. Soc. Mex. Ing. Sism., 1 (2), 1963, pp. 37-54.

[5] CHOPRA, A. K., WILSON, E. L. and FARHOOMAND, "Earthquake Analysis of Reservoir - Dam Systems", Proc. Fourth World Conf. Earthquake Engnrg., Santiago, Chile 1969, 2, B-4, pp. 1-10.

[6] VELETSOS, A. S., "Seismic Effects in Flexible Liquid Storage Tanks", Proc. of the International Assoc. for Earthquake Engrg; Fifth World Conference, Rome, Italy, 1974, Vol. 1, pp. 630-639.

[7] MINDLIN, R. D. and GOODMAN, L. E., "Beam Vibration with Time-Dependent Boundary Conditions", J. Appl. Mechs., Dec. 1950, pp. 377-380.

[8] BIGGS, J. M., Introduction to Structural Dynamics, McGraw Hill, 1964.