NEW DEVELOPMENTS CONCERNING THE TWO-SURFACES THEORY OF PLASTICITY AND VISCOPLASTICITY

A. PHILLIPS
Department of Engineering and Applied Science,
Becton Center, Yale University, New Haven, Connecticut 06520, U.S.A.

SUMMARY

The classical theories of plasticity and viscoplasticity are based on the assumption that there exists a yield surface in stress space. Phillips and Sierakowski, Acta Mechanica 1(1965)29, proposed the concept that there exist simultaneously a yield surface and a loading surface. Thus, the concept of a two surface theory of plasticity and viscoplasticity was initiated. This concept was further elaborated by Eisenberg and Phillips, Acta Mechanica 11(1968)247, and it was used by Dafalias and Popov as well as by Krieg. Until recently there was a complete lack of experimental verification of the existence of a loading surface distinct from the yield surface. Recently, Phillips as well as Phillips and Moon have shown by experiments on pure aluminum that a loading surface distinct from the yield surface exists.

The concept of the loading surface and its relation to the yield surface was explored up to now only for one temperature, that is, in the six-dimensional stress space and not in the seven-dimensional stress-temperature space. In the present paper we introduce the concept of the loading surface as well as that of the yield surface for a set of temperatures in the stress-temperature space. We give experimental verification of the existence of the loading surface for a set of temperatures for pure aluminum, as well as for copper. On the basis of a large number of experiments we discuss, for a set of temperatures, the relation between the yield surface and the loading surface and the development of the plastic strains and creep strains.

From the experimental evidence and associated theoretical considerations we show the existence of a center of the yield surface and present a method of determining it. Then we show how this center is displaced with prestressing for both cases of a moving and a not moving loading surface. We then show how the yield surface is changing with prestressing and we introduce a model which approximates the motion of the center of the yield surface as well as the change in the yield surface. Thus, an experimentally verified hardening law is introduced. Finally, we investigate the relation between the development of the plastic strain and the prestressing for the cases of a moving and not moving loading surface.

1. The Yield Surface and the Loading Surface

In 1965 the author [1] proposed the concept that in the stress space there exist simultaneously a yield surface and a loading surface which correspond to a prestressing point. This concept was based on a generalization of the stress-strain diagram shown in Fig. 1. This diagram implies that unloading into the elastic region produces only elastic strains but that reloading produces plastic strains if carried beyond the level represented by the point D . The generalization of this diagram to combined stresses is shown in Fig. 2 and it shows two surfaces, one which corresponds to the point A (a in Fig. 2) and the other corresponds to the point D of Fig. 1 (d in Fig. 2). The first surface was called the loading surface, the other one the yield surface. The yield surface encloses the purely elastic region. Any stress path located within the region enclosed by the yield surface generates only elastic strains and leaves the two surfaces intact. The loading surface passes through the prestressing point. We conclude that the stress space is divided into three regions. An inner region enclosed by the yield surface; in this region only elastic strains can be generated. An intermediate region between the two surfaces; in this region only small plastic strains appear. Finally an external region outside the loading surface; in this region larger plastic strains appear.

The concept of the simultaneous existence of a yield and a loading surface corresponding to the points D and A of Fig. 1 should not be confused with Melan's concept of the simultaneous existence of a plastic potential surface and of a yield surface [2]. The plastic potential surface is connected with the normality concept and is not the same as our loading surface, although there is no prohibition to the possibility that the loading surface may coincide with the plastic potential surface. Indeed, if the plastic strain increment normal is always normal to the loading surface but it is not normal to the yield surface it is obvious that our loading surface will coincide with Melan's plastic potential surface. In general, however, these two surfaces are different.

The loading surface should also not be confused with the limit surface introduced by McLoughlin [3]. This limit surface is essentially the largest loading surface we can obtain. It is obviously different than our moving loading surface. Finally the loading surface should also not be confused with the concepts introduced by Mroz [4] which deal with a field of hardening moduli.

The concept of the simultaneous existence of a yield and a loading surface has been used analytically in [5,6]. Recently Dafalias and Popov [7] as well as Krieg [8] have used two surface theories to model cyclic loading.

2. The Equilibrium Stress-Strain Line

The concept of the loading surface can be better understood if we introduce the auxiliary concept of the equilibrium stress-strain line in tension. We shall assume that in performing our tensile test we use a deadload tensile machine. The equilibrium tensile stress-strain curve at a given temperature is a sequence of equilibrium positions due to successively larger values of stress. Each increment of stress is assumed to be applied after the total permanent strain due to the previous stress increment had the time to appear fully. Therefore, the equilibrium stress-strain curve is in reality an incremental stress-strain curve where the infinitesimal increments are in the stress. To each increment of stress $\Delta\sigma$ at a given temperature corresponds an increment $\Delta\varepsilon^{e\ell}$ in the elastic strain and an increment $\Delta\varepsilon^{p\ell}$ in the plastic strain; the latter needs time to appear. When the temperature and stress are smaller than certain limiting values the time needed for the increment $\Delta\varepsilon^{p\ell}$ to appear is finite.

If the material is forced to extend faster than it needs to express the entire permanent strain, as for example, when the deformation is with a constant stress rate, the stress-strain curve appears elevated, as curve OK instead of curve OL in Fig. 3, since the permanent portion of strain AB does not have the time to develop fully before the stress increases even further. Thus, only the portion AC of the permanent strain appears. The equilibrium stress-strain curve OL is the lowest stress-strain curve possible at a given temperature. The curve OL could be obtained, in principle, if we could add small increments of stress in a deadweight machine but at each increment of stress we would wait considerable time before the next increment of stress would be applied.

Suppose that in Fig. 4 curve OA represents the equilibrium stress-strain curve. A moderate decrease of the stress below the value σ_A at the permanent strain ε_A will produce only elastic strain. Subsequent increase of the stress to σ_A will produce only elastic strains. Only after σ_A is exceeded plastic strain will appear again. On the other hand if OB is a stress-strain curve obtained under nonequilibrium conditions, for example, under constant finite stress rate, then OB will always be higher than OA , and both a slow decrease in stress from σ_E to σ_D as well as a slow increase again of the stress from σ_D to σ_E will produce some plastic strains.

The equilibrium stress-strain curve can be obtained by first loading at a constant stress rate then unloading at selected strains followed by very slow reloading. In this way the equilibrium stress-strain curve will be the locus of all the points where the first deviation from proportionality during reloading occurs.

Let us again consider the equilibrium stress-strain curve OA and the non-equilibrium stress-strain curve OB obtained with a constant stress rate. While the stress is increasing with the prescribed stress rate plastic strain develops. If at a stress σ_E , point E , we unload abruptly, below the equilibrium stress-strain line the plastic strain is frozen to some value $\varepsilon_D^{p\ell}$, and the corresponding stress at the equilibrium stress-strain line is σ_D , at point D . If instead of decreasing the stress we keep the stress at σ_E then creep strain will develop under the constant stress σ_E . The total creep strain ε_A^{cr} is equal to the segment EA of the line parallel to the strain axis but between the two stress-strain curves OA and OB at the stress level σ_E . This creep strain which develops gradually, stops at A , and it can be assumed to develop with a rate which is a function of the distance between the line EA and the equilibrium stress-strain line OA . If the stress σ_E is so high that there is no intersection between the line parallel to the strain axis but intersecting the line OB at E and the equilibrium stress-strain line OA then creep strain develops without limit. This unlimited creep will occur also when the temperature is sufficiently high because the equilibrium stress-strain line becomes flatter as the temperature increases so that no intersection A will occur at higher temperatures.

Let us return to the model shown in Fig. 2. When the stress point moves from a point on the loading surface to a point on the yield surface very fast there will be no plastic strains developed. Similarly, a very fast motion from a point on the yield surface to a point on the loading surface will produce no plastic strains. If, however, the motion of the stress point between the two surfaces is slow enough then plastic strains will be developed. When plastic strains develop then the yield surface changes since as seen from Fig. 4 we are moving to a higher level of the equilibrium stress-strain line. The loading surface will be disturbed only if we move to a stress level higher than the one corresponding to the original loading point.

3. Some Experimental Results Regarding the Yield Surface

We obtained experimentally a large number of subsequent yield surfaces due to prestressing. Most of these surfaces were for aluminum specimens and a smaller number were for copper and brass. Figure 5 from [9] gives a sequence of two prestressings in combined tension and torsion for a pure aluminum specimen. The initial yield surface has been obtained first, then the specimen was prestressed to the point A and the first subsequent yield surface was obtained. Subsequently, the stress point starting from inside the first subsequent yield surface proceeded to prestress the same specimen to the point B and then the second subsequent yield surface was obtained. All prestressings were done at room temperature. The isothermals at four temperatures were obtained for the initial yield surface and for the first subsequent yield surface. For the second subsequent yield surface isothermals at only three temperatures were obtained. From the above test we may draw a number of important conclusions verified by a number of additional experiments. We observe that the yield surface does not pass through the prestressing point. Whether the yield surface will pass very near the prestress point or not depends on the time elapsed while the specimen is held at the prestressing point. If the specimen after being prestressed is allowed to remain at the prestressing point for considerable time the yield surface will pass very near the prestressing point. If, on the other hand, after prestressing the stress is immediately reduced to a value which lies within the corresponding yield surface, the yield surface will be at a considerable distance from the prestressing point.

This phenomena can be explained on the basis of our previous discussion concerning Fig. 4. In combined stresses the axis σ is replaced by the stress space and the points D and E do not coincide. Hence the prestressing point which corresponds to point E lies outside the yield surface which corresponds to that prestressing point. Our experiments have shown that for commercially pure aluminum the equilibrium yield surface does not pass through the prestressing point. The yield surface does not change even if several days pass, while the stress point lies within the elastic region; the yield surface is frozen. On the other hand by waiting at the prestressing point before unloading we succeed to have the yield surface move gradually towards the prestressing point and finally pass through the prestressing point as predicted above. It follows that hardening is due to both the prestressing and to the amount of plastic strain developed during prestressing.

Another example given here is that of yield surface for OFHC copper. Figs. 6 and 7 show the initial yield surface, and the first and second subsequent yield surfaces for a copper specimen loaded in combined tension, torsion, and internal pressure [10]. The prestressing is in the tension-torsion plane but the figures represent intersections of the three-dimensional yield surfaces with tension-internal pressure planes. We observe that the motion of the yield surface for copper due to prestressing follows the same hardening law which was presented for aluminum by this author [11,12].

4. The Loading Surface

Experiments with pure aluminum have shown that there exists a loading surface as described previously. Fig. 8 gives the essential features as found by our experiments [13]. Let the stress path be given by the line OA . Then the loading surface passes through A and is generated from the initial yield surface by means of isotropic expansion. The loading surface is denoted in Fig. 8 by the line I .

If the stress point had remained at A until all plastic and creep strains had time to develop, then the yield surface II , which is the boundary of the elastic region, will pass through the same point A but it will be much smaller than the loading surface, will be enclosed by I , and will be tangential to I at A .

Now suppose we continue the stress path from A to B within I . Then the loading surface I remains unchanged while the yield surface will move and pass through B , if of course again all plastic and creep strains had time to develop while the stress point was stationary at B at the end of the path. The new yield surface II' will be tangential to I if B is on I or very near to I . On the other hand if the point B is sufficiently far from I , as is the case with the point C at the end of the path AC then the new yield surface II" passing through C will be completely inside I and will not be tangent to I .

Suppose that we continue our stress path from B to D (or from C to D) where D is outside I . Then the loading surface I changes to the new loading surface I' which passes through D . If again, while the stress point is stationary at D , all plastic and creep strains had time to develop, the new yield surface II"' will pass through D , will be tangential to I' at D and will lie completely inside I' . During the motion of the stress point which generates a new loading surface, I → I' , the plastic and the creep strain rates generated are much larger than those strain rates generated by a motion of the stress point which keeps the loading surface unchanged.

Fig. 9 gives one example from a particular experiment in combined tension and torsion for pure aluminum. In this Figure for an experiment the seventh subsequent yield surface VII and the associated loading surface are shown. Then prestressing PR generates subsequent yield surface VIII which is tangential to the loading surface. Then prestressing ST generates subsequent yield surface IX which is again tangential to the loading surface. Finally prestressing TU generates a new loading surface and a yield surface X which is tangential to the new loading surface. We see that the model illustrated by Fig. 8 is correct. In addition, in Fig. 9 we see the plastic strain increment vectors. We observe that they are normal to the respective yield surfaces. We found also from other experiments that as the yield surface tends to become tangential to the loading surface the plastic strain increment vector also tends to become normal to the loading surface, if not immediately, then at some later stage.

5. Generalization for Stress-Temperature Space

The concept of the equilibrium stress-strain curve can be generalized into an upper and a lower equilibrium stress-strain curve. As the plastic strain or stress increases the width of the yield surface decreases and as we shall see in Fig. 11 at a given plastic strain (or stress) the yield surface will be reduced to zero width, that is into a straight line of limited length.

Fig. 10 from [14] gives a schematic illustration of the generalization of the upper and lower equilibrium stress-strain curves for different temperatures. We see that as the temperature is raised the upper curve is lowered while the lower curve is raised. At some particular plastic strain which is a function of the temperature the two curves meet, which indicates that at that stage the yield surface for that temperature was reduced into a straight line. It is obvious that the lower the temperature is the larger will the plastic strain be at which the yield surface is reduced into a straight line.

The generalization of the yield surface in the stress-temperature space is shown in Fig. 11 from [14] in which the first subsequent yield surface of Fig. 5 is reproduced. In the lower part of this figure a temperature-stress diagram is drawn. We observe that there exists a maximum temperature θ_{max} for which the yield surface reduces to a straight line for that prestressing. We also can show that point P' is a center of the yield surface. This point has been called the thermodynamic reference stress [15]. The existence of this center follows from the second law of thermodynamics.

The loading surface can be generalized in the stress-temperature space in the same manner as the yield surface. It can be shown that the yield surface and the corresponding loading surface are tangential to each other at some stress-temperature line over a range of temperatures if they are tangential at a point for one temperature. The relative behavior of the loading surface versus the yield surface in the stress-temperature space is similar to the one described above for one temperature.

A study of the relation between the motion of the thermodynamic reference stress, the deformation of the yield surface, and the motion of the loading surface gives an insight on the hardening law of aluminum that is more fundamental than it was possible up to now.

6. Acknowledgment

This work was sponsored by a grant to Yale University by the National Science Foundation.

References

[1] PHILLIPS, A., SIERAKOWSKI, R. L., "On the Concept of the Yield Surface," Acta Mech. 1, 29-35 (1965).

[2] MELAN, E., "Zur Plastizität des Raümlichen Kontinuums," Ingenieur-Archiv 9, 116-126 (1938).

[3] MCLAUGHLIN, JR., P. V., "Properties of Work-Hardening Materials with a Limit Surface," Journal of Applied Mechanics 40, 803-807 (1973).

[4] MROZ, Z., "On the Description of Anisotropic Workhardening," Journal Mech. Phys. Sol. 15, 163-175 (1967).

[5] JUSTUSSON, J. W., PHILLIPS, A., "Stability and Convexity in Plasticity," Acta Mech. 2, 251-267 (1966).

[6] EISENBERG, M. A., PHILLIPS, A., "A Theory of Plasticity with Noncoincident Yield and Loading Surfaces," Acta Mech. 11, 247-260 (1971).

[7] DAFALIAS, Y. F., POPOV, E. P., "A Model of Nonlinear Hardening Materials for Complex Loading," Acta Mech. 21, 173-192 (1975).

[8] KRIEG, R. D., "A Practical Two Surface Plasticity Theory," Journal of Applied Mechanics, 42, 641-646 (1975).

[9] PHILLIPS, A., TANG, J. L., "The Effect of Loading Path on the Yield Surface at Elevated Temperatures," Int. Journal Solids and Struct. 8, 463-474 (1972).

[10] PHILLIPS, A., DAS, P. K., to be published.

[11] PHILLIPS, A., KASPER, R., "On the Foundations of Thermoplasticity - An Experimental Investigation," Journal of Applied Mechanics, 40, 891-896 (1973).

[12] PHILLIPS, A., WENG, G. J., "An Analytical Study of an Experimentally Verified Hardening Law," Journal of Applied Mechanics, 42, 375-378 (1975).

[13] PHILLIPS, A., MOON, H., "An Experimental Investigation Concerning Yield Surfaces and Loading Surfaces," Acta Mech. (in press).

[14] EISENBERG, M. A., LEE, C. W., PHILLIPS, A., "Observations on the Theoretical and Experimental Foundations of Thermoplasticity." (submitted for publication).

[15] PHILLIPS, A., EISENBERG, M., "Observations on Certain Inequality Conditions in Plasticity," Int. J. Nonlinear Mechanics, 1, 247-256 (1966).

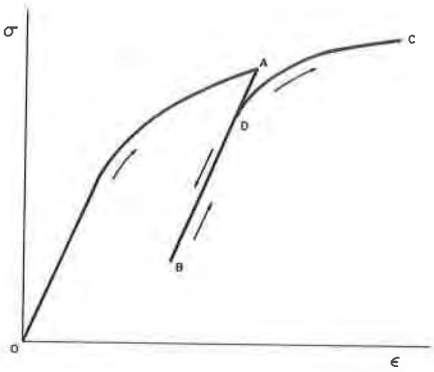


Fig. 1 The Stress-Strain Diagram

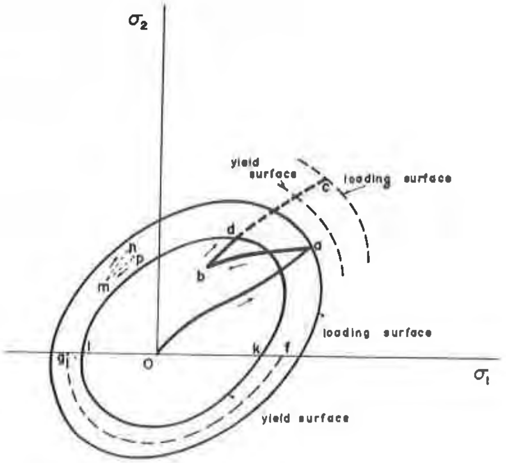


Fig. 2 The Yield Surface and the Loading Surface

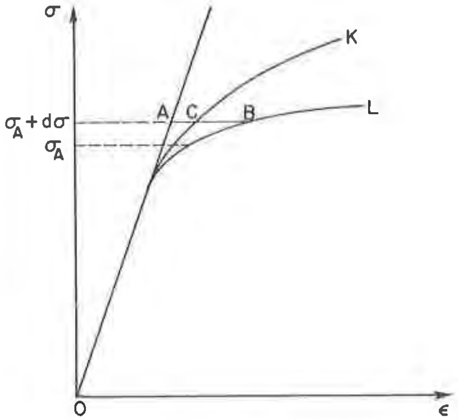


Fig. 3 The Equilibrium Stress-Strain Curve

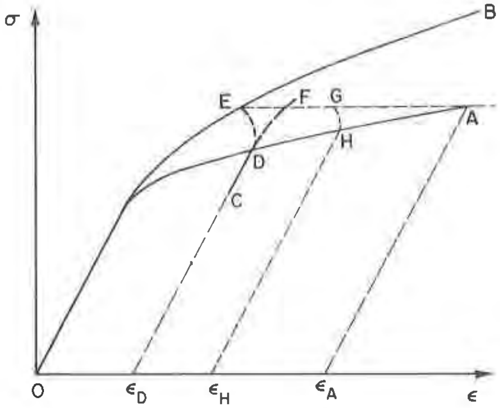


Fig. 4 Creep Strains

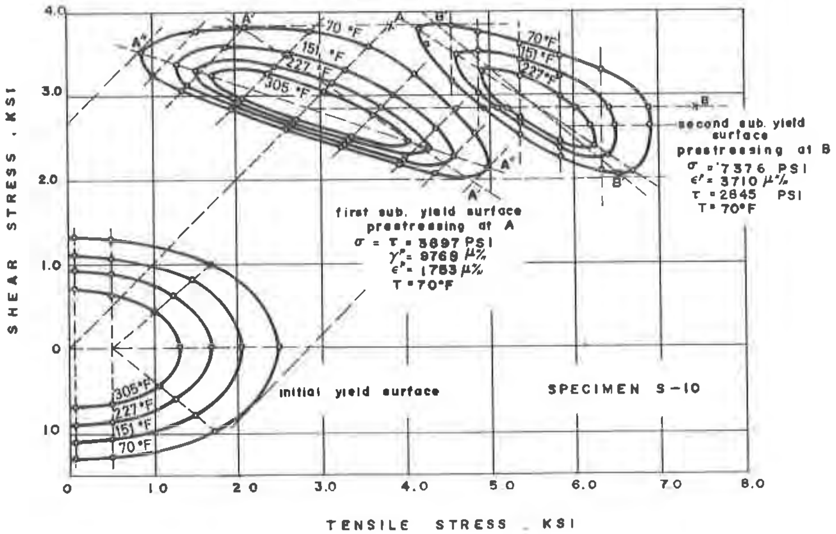
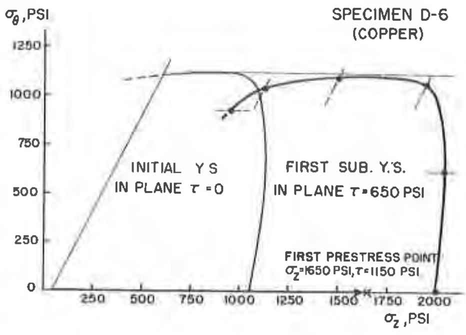
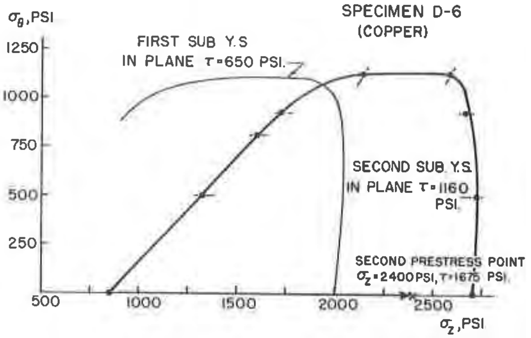


Fig. 5 Two Prestressings for a Pure Aluminum Specimen



FIRST SUBSEQUENT YIELD SURFACE

Fig. 6 A First Subsequent Yield Surface for Copper



SECOND SUBSEQUENT YIELD SURFACE

Fig. 7 A Second Subsequent Yield Surface for Copper

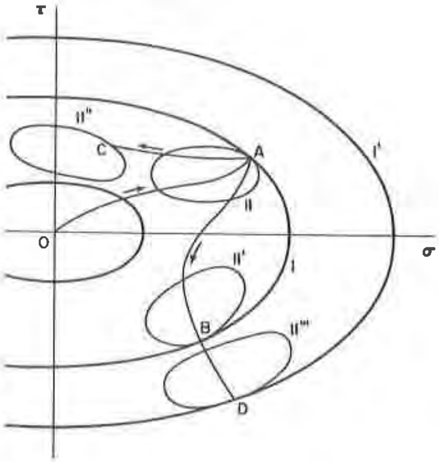


Fig. 8 Schematic Description of the Motions of the Yield and Loading Surface

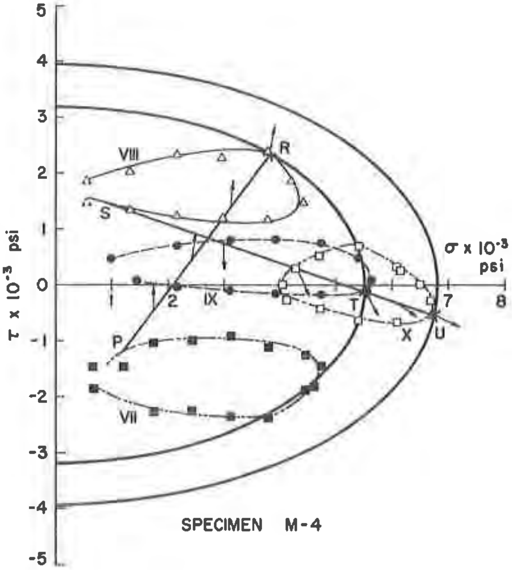


Fig. 9 Experimental Verification of the Motion Proposed in Fig. 8

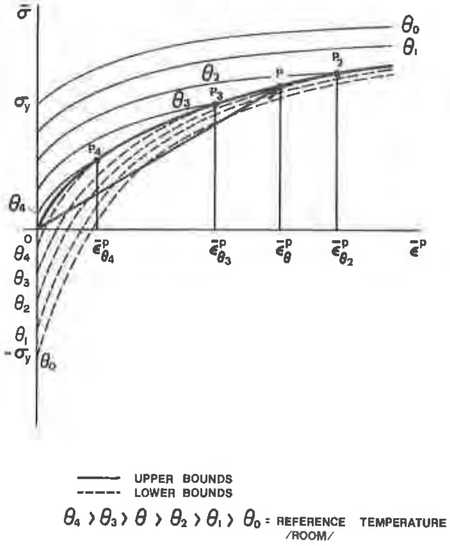


Fig. 10 The Equilibrium Stress-Strain Curves for Different Temperatures

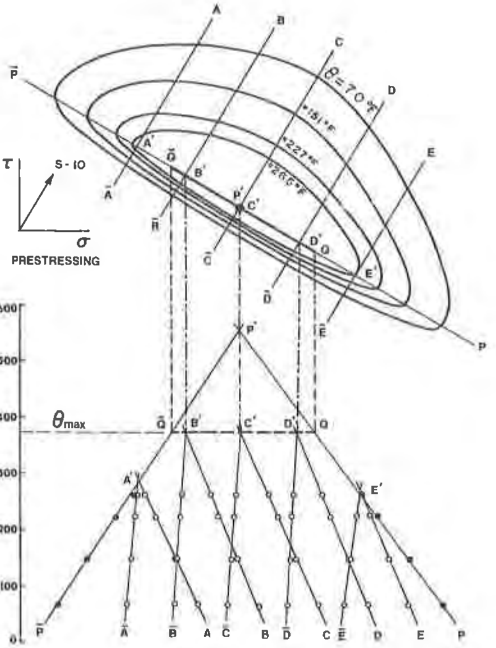


Fig. 11 The Yield Surface in the Stress-temperature Space