CREEP ANALYSIS BY THE PATH FUNCTION METHOD

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SUMMARY

The finite element method has become a common analysis procedure for the creep analysis of structures. As early as 1966 Greenbaum used a finite element program based on an intitial strain approach to solve for stresses and displacements in axisymmetric problems. Since then several other algorithms have been developed along lines which are similar to Greenbaum's in their treatment of the creep component of strain. Other related approaches are presented in the survey by Gallagher, ORNL 4756 (1973). The most recent programs are designed to handle a general class of material properties and are able to calculate elastic, plastic, and creep components of strain under general loading histories.

The constant stress approach is too crude a model to accurately represent the actual behavior of the stress for large time steps. The true path of a point in the effective stress-effective strain ($\sigma^e - \varepsilon^c$) plane is often one in which the slope is rapidly changing. Thus the stress level quickly moves away from the initial stress level and then gradually approaches the final one. The result is that the assumed constant stress level quickly becomes inaccurate.

What is required is a better method of approximation of the true path in the $\sigma^e - \varepsilon^c$ space. The method described here is called the path function approach because it employs an assumed function to estimate the motion of points in the $\sigma^e - \varepsilon^c$ space.

Consider a material at point A, at time t_A, in the $\sigma^e - \varepsilon^c$ plane. Assuming no sudden changes in applied loads over the time interval from t_A to t_B the true material will in general undergo a smooth change to arrive at a point B. Unless the material path from A to B has a small slope, the constant stress approximation (path ACD) path is relatively crude.—As an alternative, assume that the approximate path follows a straight line from A to a point E. The slope of line AE is that of the true path at point A. At point E, the material is relaxed elastically to reach a point F . It is logical to assume that in general path AEF would lie closer to the true path AB than does path ACD.

Assume that a general class of path functions $\sigma^e = P(\varepsilon^e)$ exist which can approximate the true material behavior over a specified time increment. It will be shown that a linear path function leads to an efficient solution algorithm, but it is not necessary to limit the choice of $P(\varepsilon^c)$.

For a path function analysis it is necessary to form a differential equation relating effective creep strain rate to stress and time. This relation is derived by substituting the assumed path function into the parametric creep law and establishing the relation for $\mathrm{d}t/\mathrm{d}\varepsilon^c$.

For example, if $\varepsilon_c = K\sigma_e^n t_e$ and $P(\varepsilon_c) = A + B\varepsilon_c$ the creep strain at the end of a time step is

$$\varepsilon_c = ([KBm\Delta t + (A + B\varepsilon_1)^m]^{1/m} - A)/B$$

where $m = 1 - n$ and ε_1 is the creep strain at the beginning of the increment, Δt.

The linear path function approach has been applied to a number of problems. The specific example of the relaxation of an axisymmetric cylinder will be presented in detail. Experience to date suggests that the path function analysis is usually four to five times faster than the constant stress procedure and should be investigated further.

1. Introduction

The finite element method has become a common analysis procedure for the creep analysis of structures. As early as 1966 Greenbaum [1],[2] used a finite element program based on an initial strain approach to solve for stresses and displacements in axisymmetric problems. Since then several other algorithms have been developed along lines which are similar to Greenbaum's in their treatment of the creep component of strain [3],[4],[5]. Other related approaches are presented in the survey by Gallagher [6]. The most recent programs are designed to handle a general class of material properties and are able to calculate elastic, plastic, and creep components of strain under general loading histories [7].

The common approach is to use the instantaneous material properties to solve for the initial stress-strain state at time t equal to zero. Based on this "elasticity" solution, some parametric creep equation is applied to predict an effective creep strain increment. A common form appears to be the power law

$$\varepsilon_c = K\sigma_e^n t_e^m \qquad (1)$$

where ε_c is the effective creep strain, σ_e is the effective stress, t_e is the parametric time, and K, n, and m are experimental material constants. A hardening rule is required to relate the material behavior under a general stress state to the uniaxial data described by equation (1). Once a choice of hardening law has been selected, the parametric equation is utilized with the Prandtl-Reuss equations to find the creep strain components. The creep strain increments are treated as "initial strains" and used to modify the force vector. Another elasticity solution can be obtained using the modified force vector and the original stiffness matrix. This two-phase process of dividing up the structural behavior into instantaneous effects and "creep" effects is known as the initial strain approach to the solution of time dependent material property problems.

2. The Path Function Concept

In the constant stress method [6] σ_e is held constant during the time increment Δt, at the end of which an elastic stress redistribution is assumed to take place instantaneously. While iteration schemes can improve the accuracy of the constant stress approach, the constant stress approach is really too crude a model to accurately represent the actual stress behavior for large time steps. The true path of a point in the effective stress-effective strain ($\sigma^e - \varepsilon^c$) plane is often one in which the slope is rapidly changing. Thus the stress level quickly moves away from the initial stress level and then gradually approaches the final one. The result is that the assumed constant stress level quickly becomes inaccurate.

What is required is a better method of approximation of the true path in the $\sigma^e - \varepsilon^c$ space. The method described here is called the path function approach because it employs an assumed function to estimate the motion of points in the $\sigma^e - \varepsilon^c$ space.

Consider at time t_A a material point A in the $\sigma^e - \varepsilon^c$ plane as indicated in Fig. 1. If there are no sudden changes in applied loads over the time interval from t_A to t_B the true material will in general undergo a smooth change to arrive at point B. Unless the material path from A to B has a small slope, the constant stress approximation (path ACD) is relatively poor. As an alternative, choose as an approximate path the straight line from A to E and let the slope of line AE be that of the true path at point A. At point E, the material is relaxed elastically to reach point F. It is logical to assume that in general path AEF would lie closer to the true path AB than does path ACD. In general a class of path functions

$$\sigma^e = P(\varepsilon^e) \tag{2}$$

exist which can approximate the true material behavior over a specified time increment. It will be shown that a linear path function leads to an efficient solution algorithm, although it is not necessary to limit the choice of $P(\varepsilon^c)$.

For a path function analysis it is necessary to form a differential equation relating effective creep strain rate to stress and time. To derive this equation for strain hardening one begins with the parametric equation and takes its partial derivative w.r.t. ε^c. For the power law, equation (1), the result is

$$\frac{\partial t^e}{\partial \varepsilon^c} = \frac{(\varepsilon^c)^{(q-1)}}{m\left[K(\sigma^e)^n\right]^q} \tag{3}$$

where $q = 1/m$. Since the strain hardening criterion implies that

$$\frac{dt}{d\varepsilon^c} = \frac{\partial t^e}{\partial \varepsilon^c} \tag{4}$$

substituting the assumed path function (equation (2)) yields

$$\frac{dt}{d\varepsilon^c} = \frac{(\varepsilon^c)^{(q-1)}}{m\left[KP^n\right]^q} \tag{5}$$

This equation is the basis for the path function method as applied in the remainder of this analysis. It is an ordinary differential equation relating incremental creep strain to incremental real time. For a suitable choice of the function $P(\varepsilon^c)$, this expression leads to an explicit function for ε^c as a function to time t. In the case of time hardening the corresponding equation is

$$t^{m-1}dt = \frac{d\varepsilon^c}{mKP^n} \tag{6}$$

3. Linear Path Function

The explicit integration of equation (5) or (6) is possible for certain special cases. For example, with steady state creep ($m = 1$) and the linear form

$$P(\varepsilon_c) = A + B\varepsilon_c \tag{7}$$

chosen for the path function, equation (5) or (6) can be integrated in closed form and inverted to yield an explicit expression for ε^c as a function of time t. Substituting equation (7) into equation (5), with $m = 1$, yields

$$dt = \frac{1}{K}\frac{1}{(A + B\varepsilon_c)^n}\,d\varepsilon_c \tag{8}$$

so that

$$t - t_1 = \frac{1}{KB} \left. \frac{(A + B\varepsilon_c)^{1-n}}{1-n} \right|_{\varepsilon_1}^{\varepsilon} \qquad (9)$$

where t_1 = time at point A, ε_1 = effective creep strain at point A, t = time at arbitrary point C, and ε = effective creep strain at point C. This relation can be inverted to obtain

$$\varepsilon_c = \frac{1}{B}\left[KB(1-n)(t-t_1) + (A+B\varepsilon_1)^{1-n}\right]^{\frac{1}{1-n}} - \frac{A}{B} . \qquad (10)$$

Thus for a given time step (within limits prescribed below) the creep strain, ε_c, can be directly obtained. A limitation in Δt occurs for $B > 0$ and $n > 1$. The value of ε_c becomes infinite as the term in brackets in equation (10) goes to zero. That is, the expression

$$KB(1-n)(t-t_1) + (A+B\varepsilon_1)^{1-n} = 0 \qquad (11)$$

defines a limit on the maximum time step for the linear path function. Define $\Delta t = t - t_1$. Then

$$\Delta t_{max} = - \frac{(A+B\varepsilon_1)^{1-n}}{KB(1-n)} \qquad (12)$$

is an upper bound on time step for the current path function and parametric equation.

There is, of course, also a stability limit on the size of time step as discussed by Cormeau [8]. For the constant stress method this limit is given by

$$t_c = \frac{4}{3} \frac{(1+\nu)}{EKt^{m-1}} \frac{1}{n(\sigma^e)^{n-1}} \qquad (13)$$

Experience has shown that the stability limit for the linear path function method is at least as large as that for the constant stress method.

To start the solution the loads are applied and the material deforms elastically to point A in Fig. 2. A very small constant stress increment is applied, points B and C, to establish the initial slope of the path. (The size of the constant stress increment in Fig. 2 is exaggerated for clarity.) Next the full time increment is used to predict the effective creep strain increment, point D. An elastic stress redistribution completes the first solution cycle arriving at the final point E.

The next time increment starts at point E using the final slope of the previous time increment as a good starting approximation for the next time increment. Applying the effective creep strain equation takes the solution to point F of Fig. 2. An elastic redistribution phase then completes the solution cycle at point G.

The solution obtained at the completion of any solution cycle may be improved by iteration. Referring to Fig. 3, if EG is the path predicted after elastic redistribution, the slope of path EG can be used in equation (10) to improve the estimate of creep strain increment obtained using path EH. This predicts point H for the creep increment and point I for the elastic redistribution. If point I is deemed acceptable then path EI is considered to be the final solution. If point I is not acceptable the iterative steps can be repeated until suitable convergence is achieved. A simple test on convergence can be established if desired. Alternatively, a maximum number of iterations could be allowed per time step. Experience indicates that good results can be obtained with a maximum of three iterations.

There are obviously alternatives to a linear path function. At this time, no other choices for $P(\varepsilon^c)$ are known which allow for a closed form solution of ε^c as a function of t. However, a two step numerical procedure could be adopted. First equation (5) would be integrated numerically for some choice of ε. The obtained value of t would then need to be compared to the actual value of t. An iterative loop could be established by making an improved guess on ε and repeating the cycle until suitable agreement between predicted time and actual time was achieved.

4. Examples

Two simple axisymmetric examples will be presented to illustrate typical results. The results were obtained from quadratic isoparametric elements, but are basically one-dimensional problems. The constants in equation (1) were assumed to be $K = 6.4 \times 10^{-18}$, $m = 1$, and $n = 4.4$.

The first problem is a "single element test" [9], of a relaxation condition. The nodal displacements were specified to be $u = c_1 r$ and $w = c_2 r + c_3 z$ in the radial and axial directions, respectively. Thus the constant strain components were $\varepsilon_r = c_1 = \varepsilon_\theta$, $\varepsilon_z = c_3$, and $\varepsilon_{rz} = c_2/2$. The modulus of elasticity and Poisson's ratio were taken to be $E = 20 \times 10^6$ psi and $\nu = 0.499$, respectively. The values of c_1, c_2 and c_3 were set to 0.001, 0.002, and 0.003. The calculated stresses agreed with the exact values for all the significant digits (six) printed by the code.

It can be shown that this relaxation problem has an exact linear path in the $\sigma^e - \varepsilon^c$ plane with a theoretical slope of

$$\frac{d\sigma^e}{d\varepsilon^c} = - \frac{3E}{2(1+\nu)} \; . \qquad (14)$$

Figure 4 shows the results of three runs made with time increments of 0.1 hours, 2.0 hours, and 10,000 hours, respectively. The first run included twenty time increments for a total of 2.0 hours total elapsed time while runs two and three used only one time increment each. The values obtained at the centroid of the element are shown in Fig. 4. The finite element solutions fall on a linear path with a slope of -2.001E7 which agrees with the theoretical value. The stress-strain values at $t = 2$ hours agree exactly for runs one and two.

This single element test is one clear example in which the linear path function method is superior to the constant stress method. Since the Cormeau stability criteria limits the time increment size that can be used with the constant stress approach, if the final time t is large, many solution cycles would be required for a solution by that procedure.

The second example is the thick wall cylinder creep model considered by Greenbaum [1], [2]. The internal radius was 0.16 in. and the outer radius was 0.25 in. The material properties were the same as above and the internal surface was subjected to a step pressure of $p = 365$ psi at $t = 0$. The model consisted of 12 equal spaced elements in the radial direction. Since this is a creep problem in the classical sense, it is a favorable one for the constant stress method. This follows from the fact that the true path quickly settles down to a constant stress level. Solutions were obtained using time increments of 0.3 hour, 0.2 hour, 0.1 hour, 0.05 hour, and 0.01 hour for both the constant stress method and the linear path methods. As shown in Fig. 5, the solution for the axial stresses varies considerably as the time increments are reduced from 0.3 hour to 0.01 hour but both methods approach the same solution as the time increments are refined. Similar results are shown in Fig. 6 for the tangential stresses.

A comparison of the finite element solution and the limiting steady state creep stress solution [10] is shown in Fig. 7. Disagreement with the initial elastic stress state was 0.09 percent maximum. After two hours elapsed time, the creep solution differs from the theoretical steady state solution by a maximum of 0.6 percent. By four hours the difference had dropped to within 0.1 percent. Fig. 8 shows the convergence of the effective stress to the steady state solution at the centroids of elements number one and twelve.

To determine the effectiveness of the iteration routine, three runs were made at time increments of 0.3 hour, 0.1 hour, and 0.01 hour, respectively. The iteration was continued for each time increment until the maximum difference in effective stress from one iteration to the next was less than 1.0 psi at every integration point.

For the 0.3 hour time increment run, the number of iterations required to satisfy the above convergence criteria for each time increment was: 11, 10, 7, 5, 4, 3, 2, 2, 2, and 1 iteration per time increment thereafter. Thus for the total elapsed time of 2.7 hours a total of 48 iterations including the initial solution were required. For the 0.1 hour time increment, the number of iterations per time increment was: 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 2, 2, and 1 thereafter. The total number of iterations to reach the time of 2.7 hours was 47 iterations. For the 0.01 hour run only 1 iteration per time increment was necessary, but a total of 272 iterations were required for a time of 2.7 hours.

A comparison of results from the 0.1 hour and 0.3 hour increment runs is shown in Fig. 9. The results from an 0.01 hour run agreed with that of 0.1 hour run within 1.0 psi maximum for all values of calculated stresses.

The above results indicate that there exists an optimum time increment which minimizes the total number of iterations required to reach a given total elapsed time. Moreover, if the results above are typical, this optimum time increment will yield this solution at little or no expense to the accuracy of the solution. In fact even the 0.3 hour iterated solution appears to be more accurate than an 0.05 hour constant stress solution.

5. Conclusions

The previous examples demonstrate the utility and efficiency of the linear path function approach for creep analysis.

A more practical relaxation problem [11] clearly shows the superior performance of the linear path function approach over that of the constant stress approach. In that problem the 0.3 hour time increment linear path function solution clearly gave more accurate results than the 0.1 hour increment constant stress solution. The fact that the linear path iteration rapidly approaches the true path is also significant.

Results from the creep problem indicate that comparable accuracy can be achieved with the linear path function approach using time increments that are four to five times larger than that of the constant stress approach. Also the additional program complexity does not add substantially to the total time required for a typical solution cycle. For example, the total execution time for 200 steps by the linear path approach was only five percent higher than that for the constant stress approach. So it is likely that the utilization of the linear path method should realize substantial time savings in obtaining a solution to a given accuracy.

It is important to note that the creep problem presents the constant stress approach at its best. Yet even in this case, the results indicate a reduction in time, by a factor of four or five, can be obtained by the path function procedure.

6. References

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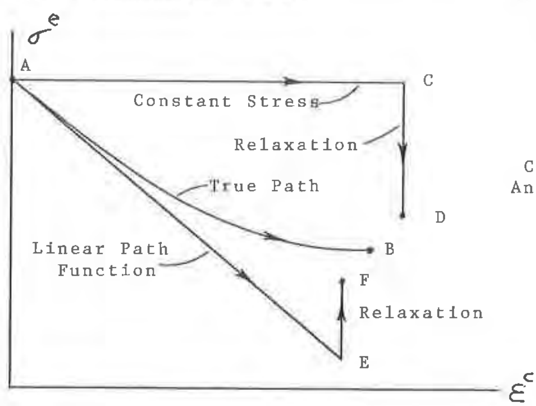

Fig. 1

Comparison of Constant Stress And Linear Path Approximations

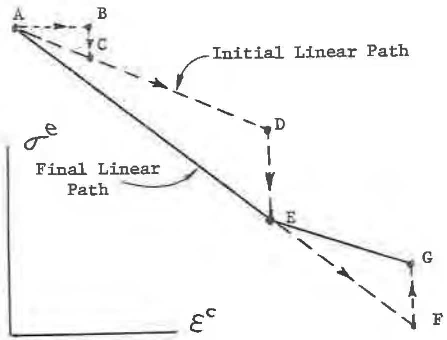

Fig. 2 The First Two Linear Increments

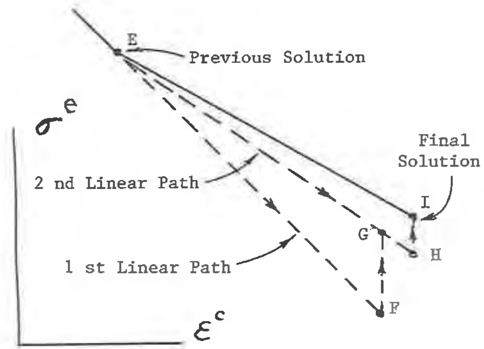

Fig. 3 The Iteration Algorithm

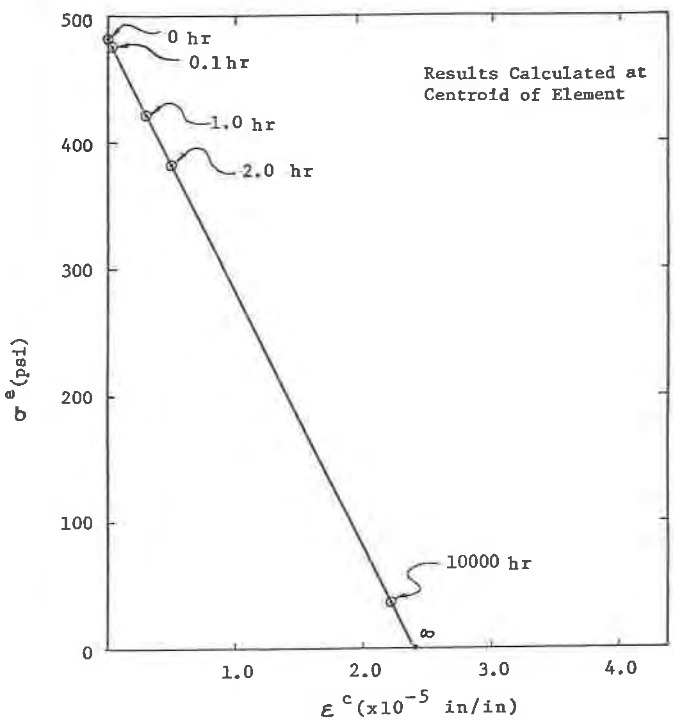

Fig. 4 Solution for Single Element Test

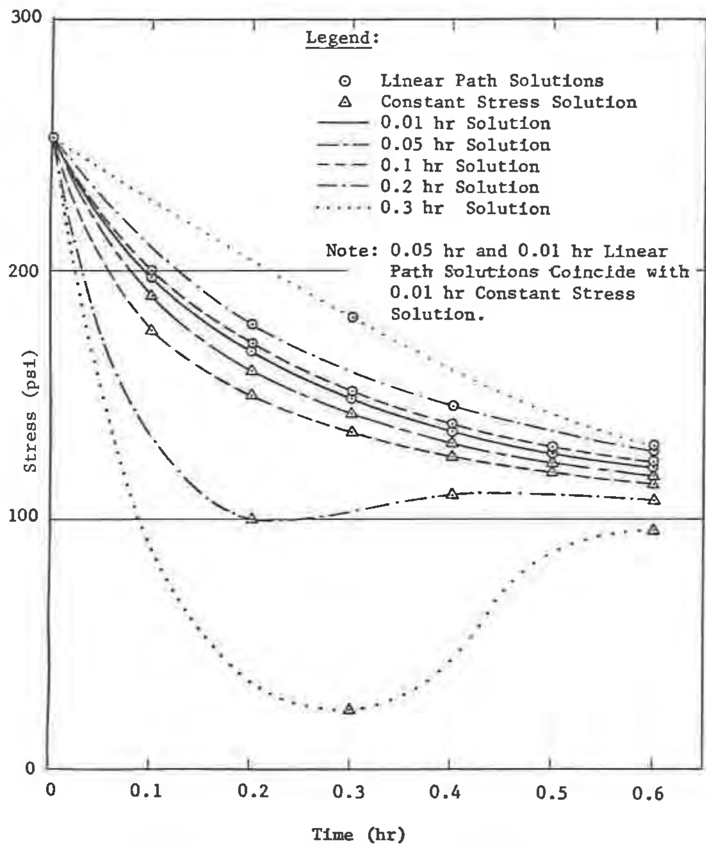

Fig. 5 Axial Stress in Element 1 for Creep Case

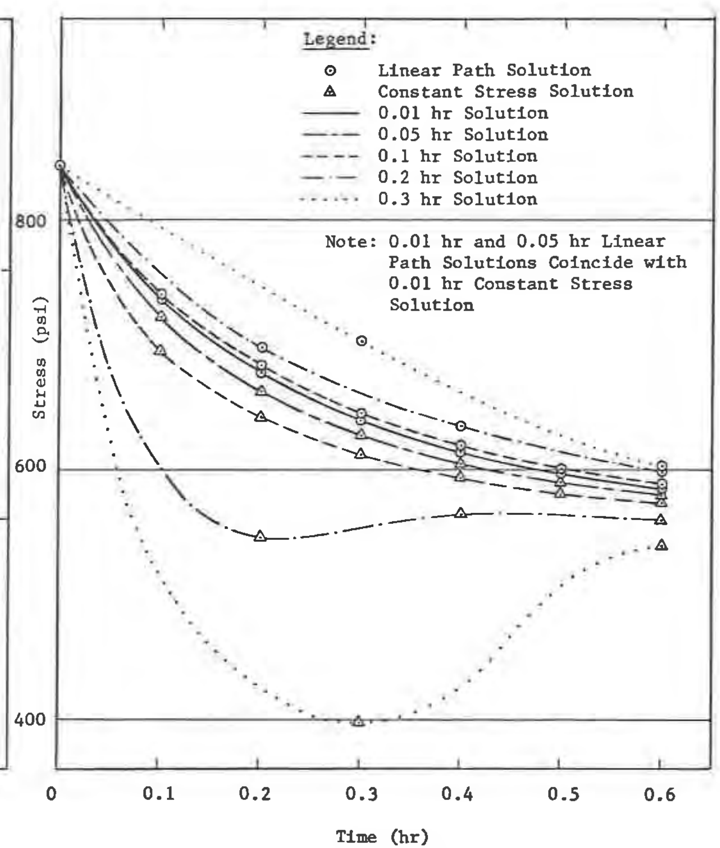

Fig. 6 Tangential Stress in Element 1 for the Creep Case

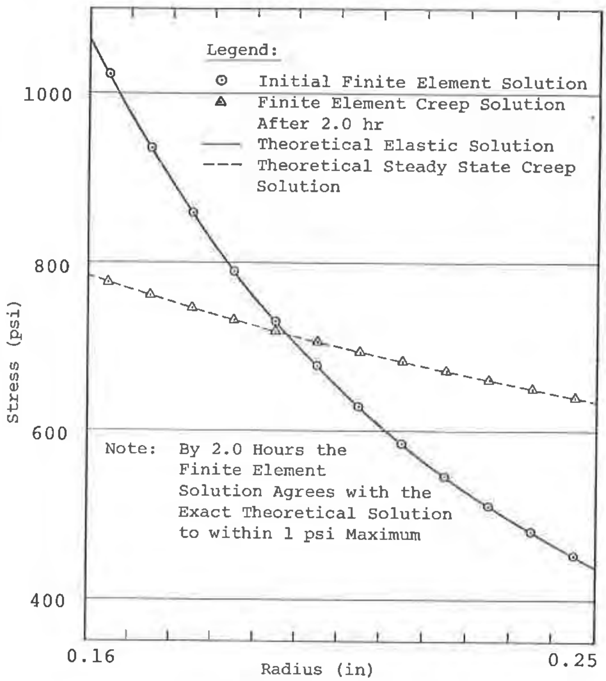

Fig. 7 Effective Stress Distribution

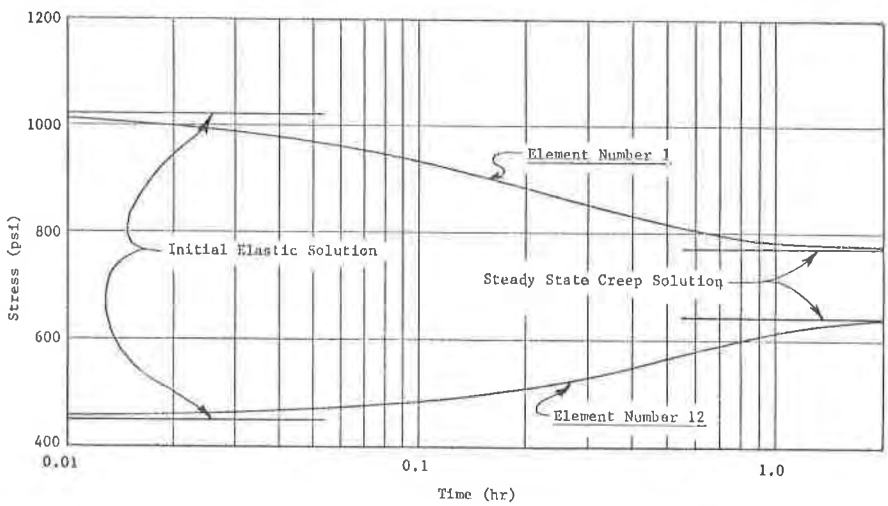

Fig. 8 Effective Stress History

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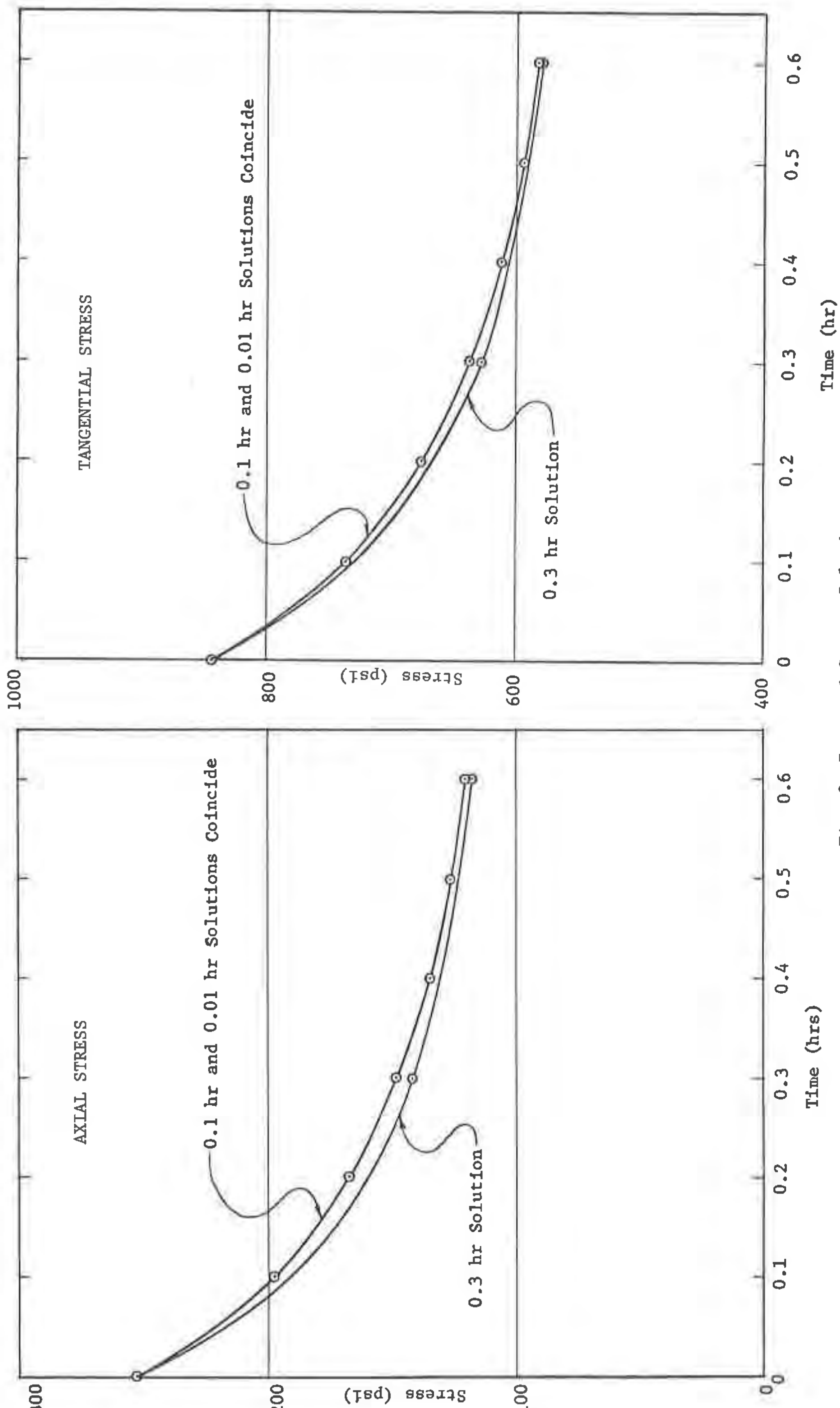

Fig. 9 It[illegible]ed Stress Solutions