FATIGUE DAMAGE ANALYSIS PROCEDURE FOR THERMAL L INERS SUBJECTED TO RANDOM THERMAL FLUCTUATIONS

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SUMMARY

Of particular concern in this paper is the structural analysis of thermal shock liners in liquid sodium systems subjected to sinusoidal and random thermal fluctuations produced by mixing of hot and cold fluid streams.

The development of the response function, relating thermally induced surface stresses to both low and high frequency sinusoidal surface temperature fluctuations, enables one to develop a method of determining fatigue damage of the structural components by conventional random vibration analysis techniques. Having the response function for the thermal shock liners enables one to predict the fatigue damage due to thermal fluctuation having either wide-band or narrow-band spectral characteristics. Although results are presented for a particular analysis—the analytical procedures can be easily adapted to consider other thermal environments.

A large portion of this report is focused on the results and implications suggested by the response function. The equation has the form:

\[
O = \frac{\left[\frac{1}{2} Q\right]}{\sqrt{1 + 2\left(\frac{Q}{Bo}\right) + 2\left(\frac{Q}{Bo}\right)^2}} \times \sqrt{\left[e^{-Q}(\sin(-Q) - \cos(-Q) + 1)\right]^2 + \left[e^{-Q}(\sin(-Q) + \cos(-Q)) + 2Q - 1\right]^2}
\]

Where \( Q = \frac{n h^2}{\tau \alpha} \) and \( Bo \) is the Biot number, \( h \) is the structural thickness, \( \alpha \) is the thermal diffusivity, and \( \tau \) is the period of the sinusoidal oscillation. A graphical presentation of these data is presented to assist in understanding certain conclusions reached in the paper. The graphical representation for example is quite explicit in pointing out sensitivity of the results, to the fluid film coefficient, and also the attenuation characteristics of the high frequency components. The plot is helpful to the analyst inasmuch as it allows him to visually pick out those frequency components that will potentially be a problem.

Unlike the stresses produced by a dynamically responding structure to random input, where the resulting stresses are in general developed through the entire wall of the structure, the randomly induced thermal stresses are very local in nature. In general the significant stress penetration is limited to a depth of \( 2.6 \sqrt{\tau \alpha} \), where \( \tau \) is the period of the spectral weighted frequency and \( \alpha \) is the thermal diffusivity.

Since these stresses are local in nature, it is a natural progression for one to proceed and evaluate fatigue damage due to these random loadings. The development of a general damage rate function for both wide and narrow band input is developed, based on Miner's damage criterion. Further simplifications of the damage rate function, in particular with regards to the material constant relating cycles to failure, gives a damage function which is much easier to use by the analyst. Still further manipulation of the data permits the development of a function (and graphical representation) in which the time to failure for wide band input can be determined as a function of spectral data, thickness, thermal diffusivity, the Biot number, and slope of the material S-N curve.

In summary, this paper presents simplified procedures for the analyst, by which the criticality of thermal fluctuation can be easily ascertained in liquid metal components such as mixing tees.
1. Introduction

The development of reactor and test facilities used to test large and small scale reactor components which will be used in future LMFBR's (Liquid Metal Fast Breeder Reactors), along with the rigorous design requirements imposed by both "nuclear" and "construction" codes, have created challenging problems for engineers working in these areas. Of particular concern to the engineer is the failure of components due to fatigue or stress cycling. Stress cycling in reactor and test facilities can come about because of "planned" events, and/or "unplanned" events. Cycling due to planned events is in general related to the operation "histogram", in which case stresses vary as the operating temperatures or pressures vary, where typically hours, even days may pass between cyclic events. Unexpected or "unplanned" stress cycling are those circumstances or situations that are not planned but in all probability may occur in a given facility. Stress cycling in this category may be a result of seismic loadings, hydrodynamic instabilities, or mechanically induced vibrations.

When considering planned cyclic events, the analyst in general must also consider creep effects on the material due to isothermal hold times between cycles. These high temperature creep effects complicates the analysis considerably. However, the analyst is also concerned with equally challenging problems when considering the higher frequency "unplanned" cyclic occurrences. Unlike "planned" events, where the loading is explicitly defined, the analyst has to consider the probability and randomness of "unplanned" events.

Mixing-tees in liquid sodium facilities are particularly susceptible to stress fluctuations which fall into the "unplanned" category of stress cycling. Specifically, thermal fluctuations produced by the mixing of hot and cold sodium streams which are created by the hydrodynamic effects of the fluid. Recent studies and tests [1] undertaking investigations of this phenomenon have given credibility to the potential severity of this problem.

The rigorous design requirements of liquid sodium reactor systems coupled with a more thorough understanding of this phenomenon, developed through recent studies and investigation, has created a need in the engineering community for a solution to this problem. The material presented in this report represents the author's approach to solving this particular problem.

This paper does not consider the sources of the thermal fluctuations but instead focuses on the techniques for evaluating accumulative damage due to the stresses created by these fluctuations (see Figure 1).

2. General Discussion

The results obtained from studies on mix-tee flow characteristics [1] indicate that both wide band (pink noise) and narrow band random thermal fluctuations are created during the same mixing process at different locations. The actual physical locations where wide and narrow band fluctuations occur depend on parameters such as branch-run flow ratios, nozzle designs, and fluid properties, however, these studies do show that regardless of whether these fluctuations are wide or narrow band ($\Delta T_{\text{FLUCT}}/\Delta T_{\text{IN}}$) is about 90-100% at the location where the hot and cold streams meet. Test results also suggest that the narrow band fluctuations are to be expected if unstable or "flip-flop" flow conditions exist; that is, if vortex streets are generated or hot and cold slugs of fluid oscillate in local areas, and white or pink noise spectrum can be expected in most
other situations, particularly in those situations where the Reynolds number is large. Since good mix-tee design precludes those conditions that are conducive to producing narrow band oscillations, and since white or pink noise fluctuations are exhibited in nearly all other situations, it is believed that the damage analysis should be based on the probability characteristics of the white or pink noise phenomena. Also, since test data indicates that the peak fluctuations values generally have the same magnitude, regardless of whether narrow or wide band fluctuations are created, it seems reasonable to expect a conservative prediction of fatigue damage can be obtained if a presumption of wide band characteristics is made. As a result of this presumption, the probability density distribution will be Gaussian for the input as well as the output.

Further reflection, however, on the probability or statistical characteristics of the thermal fluctuations indicates that a Gaussian probability is not a rigorous representation of the probability distribution, primarily because of the physical constraints of the system. A Gaussian probability distribution suggests that there is a probability of finding a fluid temperature less than the cold stream temperature \( T_c \), or greater than the hot stream temperature \( T_h \), even though the probability is small. In actuality the fluid temperatures are limited by the hot and cold fluid stream temperatures, that is \( T_c \leq T_{FLUID} \leq T_h \). However, since it appears reasonable to assume a Gaussian probability distribution, the assumption will be retained with the knowledge that some conservatism will be retained in the analysis by considering temperature \( T_p \) outside of the limits (see Figure 2).

Figure 1 shows a typical temperature time history of the fluid at a point in the system, typically on the wall of the liner, as it oscillates randomly between \( T_h \) and \( T_c \), the hot and cold temperatures of the two fluid streams respectively. \( T_m \) is the mean fluid temperature. Figure 3 shows the stress response to the input shown in Figure 1.

The symmetric probability distribution curve shown in Figure 2 would be expected in the case where the main flow stream boundary layer has been significantly effected by the branch flow and the branch run flow ratio is near unity. In other cases the actual probability distribution would be represented by a skewed distribution as shown in Figure 3. However, since \( Q_b/Q_l \), where \( Q_b \) and \( Q_l \) are the branch and run flow rates respectively, is usually near unity and \( Q_b \) rates are usually high to produce mixing, the probability distribution curve should tend to be symmetric as shown in Figure 2. It is realized that some skewness may exist in the probability distribution curve, however since a rigorous solution cannot be obtained without this data, it is reasonable to presume that a Gaussian distribution can adequately represent the test data.

3. Discussion of Linear Response Analysis

The actual physical phenomena associated with the mixing of two fluid stream is quite complex - much too complex to explicitly define in terms of a mathematical expression or function. Our best attempt at expressing this data in a form that is useful to the analysis is to determine the statistical properties of this information and apply appropriate analysis techniques - in this case random analysis. The same type of complex problems are associated with calculating the thermal response of the structure or thermal shock liner and the stresses resulting from this thermal response. However, in order to calculate the resulting thermal stress fluctuations we cannot rely on statistical means for calculating stresses but must set forth a set of rules or constraints from which
stresses can be calculated.

It is proposed that it is reasonable to assume that the thermal liner can be considered to be a slab without significantly altering the results or conclusions. So basically we are confronted with the problem of determining stresses for a plate or slab subjected to fluid thermal fluctuation on one surface. In general, problems of this type fall into a category of analysis called plane strain problems. In the case of the thermal stress problem for slabs or cylinders with radial thermal gradients, the resulting stress in most cases can be found to be proportional to \((T(X) - \overline{T})\); where \(T(X)\) is the temperature at a point and \(\overline{T}\) is the average temperature across the wall at that point. However, the problem arises because \(T(X)\) & \(\overline{T}\) are not known as a function of time, but rather as statistical quantities. Since both quantities are not known as explicit quantities it would seem reasonable to assume that the resulting thermal stresses are proportional to the statistical values of the difference \((T(X) - \overline{T})\).

The quantity \(T(X)\) is of course related directly to the thermal fluctuations on the surface \(T(0)\), but \(\overline{T}\) of the structure is much more difficult to define. It too is a statistical quantity but there is not experimental data that will define this value. So it is proposed that \(\overline{T}\) be based on the average temperature of the responding structure at the point where \(T_F\) is defined. This is probably a reasonable assumption, since it is believed that, in contrast to \(T_F\), \(\overline{T}\) is independent of location in the area of mixing. Since the thermal fluctuations presumably vary between \(T_H\) and \(T_C\) at all locations in proximity to the location where \(T_F\) is measured, \(\overline{T}\), where \(T_F\) is measured should be representative of the average temperature of the structure.

4. Development of Liner Response Equation

The problem of periodic temperature changes in engineering systems has been known for quite a while. A common feature of these natural or artificially occurring periodic changes is that they are often not sinusoidal, but rather complex, often asymmetric, and even stepped. However, since many of these complex functions can be represented and analyzed by Fourier harmonic analysis techniques, the harmonic temperature problem has been studied in considerable detail, the results of which will be of considerable use in the development of the thermal response function.

For reasons of simplicity the results that will be utilized are those that have been derived for an infinitely thick plate. At first, this may seem to be quite restrictive. However, it will be shown that the equations for an infinitely thick plate can be applied to rather thin walls. If necessary, this analysis can be modified to consider temperature variations in a finite plate, but again, for reasons of simplicity, this approach was not considered.

Inasmuch as our primary concern is the analysis of thermal shock liners, the thermal response equation for a slab has been used in this analysis. In general, thermal shock liners in mixing tees or the tcc walls themselves usually have large radius/thickness ratios, therefore the slab analysis can replace the cylinder analysis.

Jakob \([2]\) has developed an expression for the response temperature of a slab subjected to harmonic temperature fluctuations on the surface. The expression for the temperature is a function of depth \((x)\), time \((t)\), thermal conductivity \((k)\), film coefficient \((h_c)\), fluid oscillation frequency \((\omega)\), temperature range \(\theta_m\), and \((\alpha)\), thermal diffusivity.
\[ \dot{\theta}(x, t) = \Theta \eta e^{-mx} \sin(\omega t - mx - \phi) \]  
(1)

\[ \eta = \sqrt{\frac{1}{1 + \left( \frac{m^2}{B^2} \right) \left( \frac{m^2}{L^2} \right)}} \]  
(2a)

\[ \phi = \tan^{-1} \left( \frac{m}{\sqrt{m^2 + b^2}} \right) \]  
(2b)

\[ b = \frac{h_c}{k} \]  
(2c)

\[ m = \sqrt{\frac{m}{2\gamma}} \]  
(2d)

Equation (1) represents the response equation for the structure. This particular equation allows one to determine the temperature while considering surface convection effects.

From Equation (1) we can calculate the reduction in amplitude of the temperature fluctuation as a function of the depth of penetration.

\[ \frac{\dot{\theta}(x, t)}{\dot{\theta}(0, t)} = \frac{1}{R_0} = e^{-mx} \]  
(3)

where \( R_0 \) is the amplitude ratio. Rearranging equation and solving for \( x \) we obtain:

\[ x = \frac{1}{m} \ln R_0 = \sqrt{\frac{\sigma c_c}{\pi}} \ln R_0 \]

\[ x = 1.3 \sqrt{\frac{\sigma c_c}{\pi}} \log R_0 \]  
(4)

where:

\[ c_c = \frac{2\gamma}{\alpha} \]

The depth at which the amplitude is damped to that of 1% of the surface is:

\[ x (R_0 = 100) = 2.6 \sqrt{\frac{\sigma c_c}{\pi}} \]  
(5)

A wall of this thickness, and even considerably thinner can be considered as infinitely thick. Consequently the equations of an infinitely thick plate can be applied to a relatively thin cylinder, such as a thermal shock liner.

Again, in order not to lose sight of our primary objective and be burdened with needless mathematics, it would be to our benefit to assume that the modulus of elasticity and thermal coefficient of expansion of the material are temperature independent. It is obvious that this is a self imposed restriction, that can be removed if desired.

Knowing the temperature profile at any instant of time permits one to determine the corresponding stress distribution. The stress distribution can be expressed as:

\[ \sigma(x) = C \left( T(x) - T_0 \right) \]  
(6)
Where: \( C = \frac{Eh}{S} \) (Restrained slab)  
\( C = \frac{Eh}{S} / (1 - \mu) \) (Plane Strain Cylinder)  
\( T(X) = \text{Temperature at a distance } X \text{ from the surface} \)  
\( h_0 = \text{Wall thickness} \)  
\( \overline{T} = \text{Average temperature of wall} = \frac{1}{h_0} \int_0^{h_0} T(x) \, dx \)

Using the wall response results given in Equation (1),

\[
\Theta(x,t) = \Theta_m \sin \left( \omega t - \frac{mx}{h_0} - \phi \right)
\]

where: \( \Theta_m = T_0 \sin \left( \omega t \right) \)

One can proceed to calculate the terms necessary to evaluate the wall stresses.

\[
\overline{T} = \frac{1}{h_0} \int_0^{h_0} T_0 e^{-\eta x} \sin(\omega t - \frac{mx}{h_0} - \phi) \, dx
\]

or:

\[
\overline{T} = -\frac{\eta T_0}{2 \eta h_0} \left[ e^{-\eta h_0} \left( \sin(\omega t - \phi)(\sin(-mh_0) + \cos(-mh_0)) \right. \right.
\]

\[
+ \cos(\omega t - \phi)(\sin(-mh_0) - \cos(-mh_0)) - \sin(\omega t - \phi) + \cos(\omega t - \phi) \right]
\]

The surface will have the largest stress because the temperature fluctuation is greatest at this point. The temperature fluctuation attenuates as "x" increases, hence \( T(X) \) approaches \( T \), therefore the term \( (T(X) - \overline{T}) \) decreases with increasing "x".

\[
(T(X) - \overline{T}) = \eta T_0 \sin(\omega t - \phi) + \frac{\eta T_0}{2 \eta h_0} \left[ e^{-\eta h_0} \left( \sin(\omega t - \phi)(\sin(-mh_0) + \cos(-mh_0)) \right. \right.
\]

\[
+ \cos(\omega t - \phi)(\sin(-mh_0) - \cos(-mh_0)) - \sin(\omega t - \phi) + \cos(\omega t - \phi) \right]
\]

To find the time when the surface stress is maximum we must differentiate the expression with respect to time.

\[
\frac{dT}{dt} (T(X) - \overline{T}) = 0
\]

The solution of this equation indicates that the maximum stress will occur when:

\[
\tan(\omega t - \phi) = \frac{e^{-\eta h_0}(\sin(-mh_0) + \cos(-mh_0)) + 2 mh_0 - 1}{e^{-\eta h_0}(\sin(-mh_0) - \cos(-mh_0)) + 1}
\]

Substituting the results of Equation (12) into Equation (10) we obtain the expression for the maximum stress parameter \( (T(0) - \overline{T}) \).

\[
\frac{T(0) - \overline{T}}{\eta T_0} = \frac{1}{2 \eta h_0} \sqrt{\left[ e^{-\eta h_0}(\sin(-mh_0) + \cos(-mh_0)) + 1 \right]^2 + \left[ e^{-\eta h_0}(\sin(-mh_0) + \cos(-mh_0)) + 2 mh_0 - 1 \right]^2}
\]

5. Discussion of Response Equation Results

One particular observation as a result of the development of the response equation is the sensitivity of the thermal stress response to the BIOT number which of course is related to the film coefficient. The convective boundary layer is probably the single most influential parameter in regards to the magnitude of the cyclic stresses produced in the thermal liner or component wall - at least it is for the higher frequency thermal fluctuations \( f > \frac{\alpha}{h^2} \). However, at low frequency thermal fluctuations \( f < \frac{\alpha}{h^2} \) the response is not very sensitive to the film coefficient. As the thickness of the boundary layer
increases the thermal fluctuations on the wall surface attenuate or decrease, thus for a
given film coefficient there exists a frequency at which thermal oscillations create in-
significant thermal stresses on the surface of the liner. This frequency can be approxi-
mated by: \( f = 35 \sigma / \rho \sigma \). That is, for all cyclic thermal fluctuations having a fre-
quency greater than \( f_c \), insignificant thermal stresses are created on the surface of the
liner. However because of the complexity of the problems associated with fluid mixing, the
analyst may have difficulty determining the mean film coefficient \( h_c \) to be used in the
analysis.

At the other end of the spectrum when low frequency thermal oscillations are present
the film coefficient has little effect. In this situation the average temperature of the
structure follows very nearly the temperature of the fluid; therefore resulting in a
desirable low stressed condition. Hence, the buffering effect of the film coefficient is
inconsequential.

The function \( \frac{T - T_c}{T_F} \) in Figure 4 for values \( \varepsilon \leq 1.0 \) is in error because the liner or
wall is relatively too thin and the back wall sees a significant temperature fluctuation.

For example when \( \varepsilon = \ln(\frac{1}{2}) = .693 \) the back wall temperature will fluctuate with half
the magnitude of the front surface temperature fluctuation. Steps could be taken to correct
this portion of the curve to account for the actual boundary conditions but as a result
would complicate the analysis considerably. The curves as presented in Figure 4 actually
over estimate the value of the function \( \frac{T - T_c}{T_F} \) in the range where \( \varepsilon \leq 1 \), so it is believed
in the opinion of the author that the additional conservatism is acceptable in light of
other uncertainties in the analysis procedure. The effect of utilizing this approximation
for the low frequency components of temperature and stress fluctuations will have an even
more diminished influence when it comes to calculating the accumulated fatigue damage.

6. Discussion of Fatigue Analysis Procedure

When the stress history does not consist of repeated cycles of stress of constant
amplitude \( \pm \sigma \) leading to failure after \( N_f \) cycles, but a sequence where \( N_1 \) cycles at \( \pm \sigma_1 \),
followed by a sequence of \( N_2 \) cycles at \( \pm \sigma_2 \), and so on; and \( N_{f1} \) would be a sufficient
number of cycles to cause failure at \( \pm \sigma_1 \), and \( N_{f2} \) at \( \pm \sigma_2 \), and so on, failure is assumed
to occur when:

\[
\frac{N_1}{N_{f1}} + \frac{N_2}{N_{f2}} + \cdots + \frac{N_f}{N_{f}} = 1
\]  

(14)

This is known as Miner's criterion. Because of its simplicity and wide acceptance, this
criterion of failure will be used for the fatigue damage evaluation.

A number of interesting attributes of \( \sigma(t) \) have been deduced by Rice \([3]\) where \( x(t) \) is
a Gaussian random process. These properties which will be used in the development of the
fatigue expression in terms of the auto-correlation functions include:

\[
N_0 = \frac{1}{\pi} \left( \frac{R}{\rho x^2} \right)_{t=0}^{1/2}
\]  

(15)

and

\[
n_x = \frac{N_0}{\pi} \exp \left( -\frac{x^2}{2R(0)} \right)
\]

(16)

Where \( N_0 \) is the number of times that \( x(t) \) passes through zero per unit of time, and \( n_x \) is
the number of times that \( x(t) \) passes through a particular value of \( x \) with a positive slope
dx/dt, per unit of time. Strictly speaking \( n_x \) is the number of peaks in excess of troughs.
However, when \( x \), where \( x \) is stress, is large compared to the rms value, the probability of troughs goes to zero. Therefore, this expression is useful for analyzing fatigue damage since lower stress values (\( x \)) in general do not contribute significantly to fatigue damage.

As noted, Equation (16) indicates the probability of peak occurrences during the time interval \( t \) without considering minimum. Since we are considering a random process, we can expect the same number of minimum values, that is those that have a \(-dx/dt\) value. In other words for every peak \( i \), there exists a negative peak \( i' \). Therefore, in the development of the fatigue expression the full stress excursion must be considered as shown in Figure 5.

7. Development of Fatigue Damage Expression

We of course shall begin with the mathematical representation of Miner's Rule for the accumulation of fatigue damage of a structure.

\[
D = \sum \frac{n_i}{N_i} \tag{17}
\]

where: \( D \) = Accumulative fatigue damage; \( n_i \) = Number of cycles having a stress range \( \sigma_i \); and \( N_i \) = Number of cycles at which failure occurs with a stress range \( \sigma_i \); \( N_i = N_i (\sigma_i) \).

Next we utilize the linear approximation of the \((S,N)\) curve where:

\[
N_i = \frac{(2\sigma)^B}{\sigma_i^B} \tag{18}
\]

Substitute into Equation (3) gives:

\[
D = \sum \frac{n_i}{(2\sigma)^B} = \sum \frac{n_i \sigma_i^B}{\sigma_i^B} \tag{19}
\]

Utilizing the probability distribution characteristics for a Gaussian random process, the number of times \( n_i \) that \( \sigma(t) \) passed through a particular value of \( \sigma \) with a positive slope \( d\sigma/dt \), per unit time is on the average:

\[
n_i = \frac{1}{2\pi} \left[ -\frac{R''(\tau)}{R} \right] e^{-\sigma^2/2R} \tag{20}
\]

where the auto-correlation functions are:

\[
R = R(t) = \langle x(t), x(t + \tau) \rangle 
\]

\[
R'' = R''(\tau) = \langle x(t), x''(t + \tau) \rangle \tag{21}
\]

or in terms of the spectral density:

\[
R = \int_0^\infty S(f) df 
\]

\[
R'' = \int_0^\infty f^2 S(f) df \tag{22}
\]

From Equation (20) the number of peaks of \( \sigma \), per unit time, occurring between \( \sigma \) and \( \sigma + \Delta\sigma \) is then, on the average:

\[
n_i = \frac{1}{2\pi} \frac{\sigma}{R} \left[ -\frac{R''(\tau)}{R} \right] e^{-\sigma^2/2R} \Delta\sigma \tag{23}
\]

Since \( \Delta n_i \) represents the number of peaks having a positive \( d\sigma/dt \) slope the actual stress range will be \( 2\sigma \) rather than just \( \sigma \) as suggested by Equation (20), this equation should reflect this fact, therefore Equation (19) becomes:

\[
D = \sum \frac{n_i}{\sigma_i^B} (2\sigma)^B \tag{24}
\]
However, since we are considering the number of peak cycles, rather than just peaks, with magnitudes between \( \sigma \) and \( \sigma + \Delta \sigma \), the total number of cycles will be:

\[
n'_{1} = \frac{1}{2} n_{1} = \frac{1}{4 \pi} \frac{\sigma}{R} \left[ \frac{-R''}{R} \right]^{\frac{3}{2}} e^{-\frac{\sigma^{2}}{2R}} \Delta \sigma
\]  

(25)

Now substituting Equation (25) into Equation (24) we obtain:

\[
D = \sum \frac{(2\sigma)^{\beta}}{2(\sigma_{p})^{\beta}} \frac{1}{4 \pi} \frac{\sigma}{R} \left[ \frac{-R''}{R} \right]^{\frac{3}{2}} e^{-\frac{\sigma^{2}}{2R}} \Delta \sigma
\]

(26)

or writing in the integral form with the appropriate limits we obtain:

\[
D = \int_{0}^{\infty} \frac{(2\sigma)^{\beta}}{2(\sigma_{p})^{\beta}} \frac{1}{4 \pi} \frac{\sigma}{R} \left[ \frac{-R''}{R} \right]^{\frac{3}{2}} e^{-\frac{\sigma^{2}}{2R}} d\sigma
\]

(27)

Since the damage expression is expressed in terms of stress peaks having a positive \( d\sigma/dt \) slope, the limits of integration are confined to the positive stress space.

Rearranging the terms of Equation (27) we have:

\[
D = \frac{1}{4 \pi} \left( \frac{2}{\sigma_{p}} \right)^{\beta} \frac{1}{\beta} \left[ \frac{-R''}{R} \right]^{\frac{3}{2}} \int_{0}^{\infty} \sigma^{\beta} e^{-\frac{\sigma^{2}}{2R}} d\sigma + 1
\]

(28)

Evaluation of the integral leads to the following damage expression:

\[
D = \frac{1}{4 \pi} \left( \frac{2}{\sigma_{p}} \right)^{\beta} \frac{1}{\beta} \left[ \frac{-R''}{R} \right]^{\frac{3}{2}} \left( 2R \right)^{\beta/2} \Gamma \left( \frac{\beta + 1}{2} \right)
\]

(29)

Equation (29) now permits one to calculate the accumulative damage as a function of time when one knows: 1) The \((S,N)\) curve slope \((B)\); 2) \(\sigma_{p}\), the stress at one cycle failure from the \((S,N)\) curve; and 3) a PSD or response spectrum representation of the random thermal fluctuations. Where \(D\) is now a damage rate.

8. Calculation of Damage

The damage expression can be solved by using numerical techniques to solve for \(R\) and \(R''\) when a generalized PSD curve of the thermal fluctuations is considered. However, since the thermal fluctuations many times have white or pink noise spectral characteristics simplifications can be made to Equation (29). If we assume white noise characteristics of the thermal fluctuations, that is, a constant PSD level of \(S_0\) between 0 and \(f_0\) cycles, and zero for all other frequency values, Equation (29) can be written as:

\[
D = C_{1} C_{3} \left[ C_{2} \frac{E}{h} S_{0} \right]^{\frac{B}{2}} \left( \frac{1}{\sigma_{p}} \right)^{\frac{B}{2}} \left( \frac{\alpha}{m^{2}} \right)^{\frac{B + 2}{2}}
\]

(30)

where: \(B\) = slope of \((S,N)\) curve; \(\sigma_{p}\) = 1 cycle failure stress from \((S,N)\) curve; \(h\) = wall thickness; \(\alpha\) = thermal diffusivity of liner; \(\frac{E}{h} = E/\nu(1-\nu)\) where \(E\) is modulus of elasticity, \(\sigma\) is thermal coefficient of expansion, and \(\nu\) is poisson's ratio; \(S_0\) = PSD level of thermal fluctuations; \(C_{1} = C_{1}(f_0, N_p)\), \(C_{2} = C_{2}(f_0, N_p)\); \(C_{3} = C_{3}(\beta)\); \(N_p = \text{BIOT number} (h_a h/k)\) where \(h_a\) is the convective film coefficient, \(h\) is the wall thickness and \(k\) is the conductivity of the wall; \(D\) = damage rate per unit time. See Figures 6 & 7 and Table 1 for \(C_{1}, C_{2},\) and \(C_{3}\).

SAMPLE PROBLEM: Find the fatigue damage caused by thermal fluctuations having a PSD level of \(40,000 \text{F}^2/\text{Hz}\) between 0 and 30 Hz. The inner wall thickness is .1875 inches and is made of stainless steel. The \((S,N)\) curve for the material can be approximated by the following expression.
\( N_f = \left( \frac{\tau_f}{S} \right)^8 = \left( \frac{396,000}{5} \right)^5.0 \)

where \( N_f \) is the number of cycles to failure when the stress range is \( S \).

**PARAMETERS ARE:**
- \( \bar{\tau} = 10 \times 10^{-6} \) in/in./sec
- \( \tau_f = 396,000 \) psi
- \( h_c = 3500 \) Btu/hr/ft²/F
- \( k = 12.0 \) Btu/hr/ft²/F
- \( S_0 = 40,000 \) Btu/hr/ft²/F
- \( u = 0.3 \)
- \( \alpha = 0.0072 \) in²/sec
- \( E = 25 \times 10^9 \) psi

**CALCULATIONS ARE:**
\[
\begin{align*}
N_f &= \frac{h_c h_{eb}}{k} = \frac{3500(1.875/12)}{12} = 4.55 \text{ say } 5.0 \\
F_0 \frac{\alpha h^2}{\tau} &= \frac{30 (\pi)(1.875)^2}{0.0072} = 460 \\
C_1 &= C_1(f_0, N_f) = 200; \quad C_2 = C_2(f_0, N_f) = 13.5; \quad C_3 = C_3(b) = 270.8 \\
\alpha/\tau h^2 &= \frac{0.0072/(\pi)(1.875)^2}{0.06519} = 0.06519 \\
E \bar{\tau} &= 25(10^6)(10)(10^{-6})(1 - 0.3) = 357
\end{align*}
\]

**RESULTS ARE:**
\[
\begin{align*}
D &= 200(270.8) \left[ 13.5(357) \frac{40,000}{396,000} \right]^{2.50} \left( \frac{L}{396,000} \right)^{5.0} \left( \frac{0.06519}{3.5} \right) \\
D &= 2.02 \times 10^{-7} / \text{sec} \\
D &= 1 \text{ after } 1375 \text{ hours}
\end{align*}
\]

**REFERENCES**


**TABLE 1**

<table>
<thead>
<tr>
<th>C3 VERSUS B</th>
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<tr>
<td>B</td>
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<td>2.0</td>
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<td>2.5</td>
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</tbody>
</table>

- \( \alpha = 0.0072 \) in²/sec
- \( E = 25 \times 10^9 \) psi
- \( \alpha = 0.0072 \) in²/sec
- \( u = 0.3 \)
1  MIX-TEE THERMAL LINER CONFIGURATION

2  TEMPERATURE PROBABILITY DISTRIBUTION CURVE

3  SKEW PROBABILITY DISTRIBUTION

4  THERMAL RESPONSE FUNCTION
IDENTIFICATION OF STRESS EXCURSION