A NEW FINITE ELEMENT FOR PLATE BENDING ANALYSIS

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SUMMARY

An enormous amount of effort has been devoted to the development of finite elements for the bending of plates. Many users of finite element computer programs find a 'basic' four-node quadrilateral element particularly appealing due to its simplicity. It is our feeling that this appeal will become even greater when nonlinear applications are undertaken. In the nonlinear regime—and especially in nonlinear dynamics—computational cost is the prime concern. Due to frequent reformulations of tangent stiffnesses, complicated element routines can lead to exorbitant computational expenditures and may actually preclude nonlinear analysis. A simpler element of competitive accuracy becomes quite desirable under such circumstances. Other factors in nonlinear analysis support this viewpoint. For example, the accuracy level attainable in nonlinear problems is often severely limited due to the uncertainty of nonlinear material characterizations. Thus it makes little sense to engender significant computational cost for complicated bending elements which are only marginally more accurate than simpler elements, since the confidence level of the overall analysis may be affected only negligibly. Unfortunately, heretofore, no really simple alternative has existed.

In this paper we attempt to remedy this situation. We develop what we believe is the *simplest effective plate bending element yet proposed.* The element is a four-node quadrilateral with the basic three degrees-of-freedom per node, and is based upon the Mindlin plate theory. The element shape functions are bilinear for transverse displacement and rotations. The shear 'locking' associated with such low-order functions in application to thin plates is eleviated by splitting the shear and bending energies and using one-point quadrature on the shear term. One-point quadrature has a decisively positive effect on the accuracy of the element; two-by-two quadrature leading to worthless numerical results. *The upshot of all this is that by appropriately underintegrating troublesome terms, good bending behavior can be attained by the simplest shape functions.* The simplicity of the element lends itself to concise and efficient computer implementation. The element is also surprisingly accurate.

The main results of the paper are summarized as follows:

- The effectiveness of the element in thin plate bending is demonstrated by numerical examples. Comparison is made with many well-known elements, and the present element is shown to be competitively accurate.
- A simple computing strategy for dealing with the numerically sensitive case of extremely thin plates is presented.
- We also consider applications to thick plates. It is shown that the element is still effective for moderately thick plates. However, for very thick plates, in which the thickness of individual elements exceeds their characteristic lengths, a sligth modification of the shear quadrature need be employed.

1. Model Problem: The Linear Beam Element

To gain an analytical understanding of why underintegration of the shear term is necessary in developing bending elements based on Mindlin plate theory, it is useful to consider a simple "model problem." The equations of a rectangular cross-section beam, including shear deformation effects, emanate from the following expression for strain energy:

$$U(w,\theta) = \frac{1}{2}\frac{Et^3}{12}\left[\int_0^L \left(\frac{d\theta}{dx}\right)^2 dx + \kappa \frac{12G}{Et^2}\int_0^L \left(\frac{dw}{dx} - \theta\right)^2 dx\right] , \qquad (1)$$

where w is the transverse displacement of the center-line, θ is the rotation of the cross-section, E is Young's modulus, G is the shear modulus, κ is the shear correction factor (throughout we employ $\kappa = 5/6$), t is the depth, L is the length and x is the axial coordinate. The first term on the right-hand side of (1) is the bending energy and the second is the shear energy. With independent expansions for w and θ, (1) can be employed to derive beam element stiffness matrices. The case we are interested in is when both w and θ are assumed to behave linearly over an element. This leads to a four-degree-of-freedom element in which displacement and rotation are the nodal degrees-of-freedom. By virtue of the fact $d\theta/dx$ is constant within this element, the bending energy may be exactly evaluated by one-point Gaussian quadrature. On the other hand, two-point Gaussian quadrature is required to exactly integrate the shear energy term due to the explicit presence of θ, which is linear within the element. Employing one-point quadrature on the shear energy term "underintegrates" the element and it is our prime concern here to ascertain the effect of this procedure. (See also Gallagher [1], pp. 364-367.)

A series of test computations were performed to determine the behavior of the element. A cantilever beam subjected to an end load (see Fig. 1) was analyzed for various discretizations. The first example is for a relatively deep beam. The data are:

$$E = 1000 \qquad G = 375 \qquad t = 1 \qquad L = 4$$

Tip displacement results for several discretizations are presented in Table I. As is evident, the one-point quadrature results are vastly superior to the two-point results. A more severe test for linear elements is bending governed by Bernoulli-Euler theory. In this case shear strains are to be equal to zero. Such a situation can be brought about in the present theory if depth is taken very small compared with element length. Alternatively, a very large fictitious value of G can be specified. In the second example we attempt to ascertain the behavior of the linear element when the assumptions of the Bernoulli-Euler theory apply. The data of the previous example are employed with the exception of G which is set here to 375×10^5. Results are listed in Table II. The one-point quadrature results are quite accurate whereas the two-point results are in error by approximately three orders of magnitude. Early attempts at developing bending elements with simple shape functions were abandoned because of results like those for the two-point quadrature presented here.

We shall now proceed to give a heuristic argument why two-point quadrature causes such an overly stiff element. Consider a cantilever beam discretized into N elements. In the assembled stiffness matrix there are 2N degrees-of-freedom -- two degrees-of-freedom per element. The shear contribution to the stiffness represents a constraint on the shear strains. If one-point quadrature is employed, one constraint is imposed upon the element, whereas if two-point quadrature is employed, two constraints are imposed upon the element. In the latter case the number of constraints per element equals the number of degrees-of-freedom per element, and the result is that the mesh "locks."

This can be seen more precisely by looking at the stiffness contributions of the bending and shear terms. We assume the nodal degrees of freedom are ordered as follows: w_1, θ_1, w_2, θ_2; and h is the element length. The stiffnesses are:

$$k_b = \frac{Et^3}{12h}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{2a}$$

$$k_s^{(1)} = \frac{\kappa Gt}{h}\begin{bmatrix} 1 & h/2 & -1 & h/2 \\ h/2 & h^2/4 & -h/2 & h^2/4 \\ -1 & -h/2 & 1 & -h/2 \\ h/2 & h^2/4 & -h/2 & h^2/4 \end{bmatrix} \tag{2b}$$

$$k_s^{(2)} = \frac{\kappa Gt}{h}\begin{bmatrix} 1 & h/2 & -1 & h/2 \\ h/2 & h^2/3 & -h/2 & h^2/6 \\ -1 & -h/2 & 1 & -h/2 \\ h/2 & h^2/6 & -h/2 & h^2/3 \end{bmatrix} \tag{2c}$$

where k_b is the bending stiffness, and $k_s^{(1)}$ and $k_s^{(2)}$ are the one-point and two-point quadrature shear stiffnesses, respectively. It is easily verified that the rank of $k_s^{(1)}$ is one and the rank of $k_s^{(2)}$ is two. In the latter case, k_b is completely dominated by the shear stiffness, as the following simple example illustrates.

Consider the case of a one-element cantilever beam, subjected to an end load P and moment M.

(i) <u>One-point quadrature</u>

Combining (2a) and (2b), eliminating appropriate rows and columns, and solving for the tip displacement and rotation yields

$$w = (h^2/4\alpha + \beta^{-1})P + hM/2\alpha \ , \tag{3a}$$

$$\theta = (hP/2 + M)/\alpha \ , \tag{3b}$$

where

$$\alpha = Et^3/12h \ , \tag{3c}$$

$$\beta = \kappa Gt/h \ . \tag{3d}$$

In the thin beam limit (i.e., when $\beta \gg \alpha$), (3a) becomes

$$w = h(hP/2 + M)/2\alpha \ , \tag{4}$$

and (3b) remains unchanged. Thus we are left solely with the deformation due to bending as is right.

(ii) Two-point quadrature

Carrying out the same steps as in case i, with (2c) in place of (2b), yields

$$w = \left(\frac{\alpha + h^2\beta/3}{\beta\gamma}\right) P + hM/2\gamma \ , \tag{5a}$$

$$\theta = (hP/2 + M)/\gamma \ , \tag{5b}$$

where

$$\gamma = \alpha + h^2\beta/12 \ . \tag{5c}$$

In the thin beam limit (5a) and (5b) become

$$w = (4P + 6M/h)/\beta \ , \tag{6a}$$

$$\theta = 6(hP + 2M)/h^2\beta \ , \tag{6b}$$

respectively. In this case only deformations due to shear are in evidence and (6a) and (6b) are $0(t^{-2})$ in error.

In passing we note that there are some circumstances in which the present element may have some practical value. For example, an axisymmetric shell version might be useful for shell covered solids in which bilinear elements are used to model the solid. The fact that only one quadrature point is involved may lead to more economical computations in nonlinear analysis.

2. Bilinear Plate Bending Element

According to Mindlin plate theory, the strain energy for an isotropic, linear elastic plate, including shear deformation, is

$$U(w,\theta_1,\theta_2) = \frac{Et^3}{24(1-\nu^2)} \left\{ \iint_A \left[\left(\frac{\partial\theta_1}{\partial x_1}\right)^2 + 2\nu \frac{\partial\theta_1}{\partial x_1}\frac{\partial\theta_2}{\partial x_2} + \left(\frac{\partial\theta_2}{\partial x_2}\right)^2 + \frac{(1-\nu)}{2}\left(\frac{\partial\theta_1}{\partial x_2} + \frac{\partial\theta_2}{\partial x_1}\right)^2 \right] dx_1\, dx_2 \right.$$
$$\left. + \frac{6\kappa(1-\nu)}{t^2} \iint_A \left[\left(\frac{\partial w}{\partial x_1} - \theta_1\right)^2 + \left(\frac{\partial w}{\partial x_2} - \theta_2\right)^2 \right] dx_1\, dx_2 \right\} \ , \tag{7}$$

where x_1 and x_2 are cartesian coordinates, w is the transverse displacement, θ_1 and θ_2 are the rotations about the x_1 and x_2 axes, respectively, E is Young's modulus, ν is Poisson's ratio, κ is the shear correction factor, t is the plate thickness and A is its area. The first integral in (7) represents the bending energy and the second represents the shear energy. We consider a four-node quadrilateral element and assume the displacement and rotations are expanded in independent bilinear shape functions. The isoparametric concept is employed (see Zienkiewicz [2]). This results in three degrees-of-freedom -- one displacement and two rotations -- at each of the corners.

For very thick plates two-by-two Gaussian quadrature leads to acceptable results, however, for thin plates it causes "locking" as indicated for the beam in the previous section. In this case we use two-by-two Gaussian quadrature on the bending energy term and one-point Gaussian quadrature on the shear energy term. This results in two constraints per element. In large meshes there are approximately three equations per element, thus there is no danger of the mesh "locking." As is apparent, the proposed element is extremely simple, and easily and concisely coded. We are certain that the element routines are faster than any other plate bending element yet proposed. In the next section we will show that the element is also surprisingly accurate.

3. Numerical Examples: Thin Plates

In this section we present several numerical examples which have become more or less standard ones for evaluating thin plate elements. All computations were performed on a CDC 6400 computer in single precision. (A single precision word consists of 60 bits on the CDC 6400.)

Square Plate

The data for this example consists of the following (see Fig. 2):

$$E = 10.92 \times 10^5 \qquad \nu = .3 \qquad t = .1 \qquad L = 10$$

Both simply supported and clamped boundary conditions were considered as well as concentrated and uniformly distributed loadings. Results are presented in Tables III and IV for κ values of 1000 and 5/6. The former value is set to artificially maintain the Poisson-Kirchhoff constraint. Due to the fact that the plate is rather thin ($L/t = 100$), there is little difference in the results for the two values of κ. In fact, the bending moments are identical. In practical situations there seems no point in exceeding the "natural" value of $\kappa = 5/6$.

The simply supported concentrated load case has, it seems, taken on the role of the preeminent comparison problem for bending elements. In Fig. 3 the present element, with $\kappa = 1000$, is compared with data taken from Gallagher [1]. The good convergence of the element is evident.

Clamped Circular Plate

The data for this example is the same as for the previous problem except (see Fig. 4):

$$R = 5 \qquad t = .1$$

Results are presented in Table V for concentrated and uniform loadings, and κ values of 1000 and 5/6.

Again, due to the thinness of the plate, there is little difference in the displacement results for the two values of κ, and the moment results are identical.

4. Numerical Sensitivity due to Extreme Thinness and Application to Thick Plates

Our numerical experience with the present element in extremely thin and moderately thick plate situations has been good. A computing strategy for dealing with numerical sensitivity on computers with short word length, modifications necessary for very thick plates, and further numerical studies are contained in [3].

5. Generalizations

Since submission of the summary on the present work it was given the status of an invited talk and thus it was decided to expand the scope of the presentation. Topics to be presented are:

- Evaluation of standard families of higher-order elements derived from Mindlin plate theory.
- The relationship of the present elements with elements derived from mixed variational formulations and "discrete Kirchhoff" elements.
- Use of the present elements in transient analysis, lumped mass matrices and critical time step estimates.

Further details on some of the above topics can be found in [4].

6. References

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Table I

Normalized Tip Displacement for a Thick Cantilever Beam

Number of Elements	1-point Gauss	2-point Gauss
1	.762	$.416 \times 10^{-1}$
2	.940	.445
4	.985	.762
8	.996	.927
16	.999	.981

Table II

Normalized Tip Displacement for a Thin Cantilever Beam

Number of Elements	1-point Gauss	2-point Gauss
1	.750	$.200 \times 10^{-4}$
2	.938	$.800 \times 10^{-4}$
4	.984	$.320 \times 10^{-3}$
8	.996	$.128 \times 10^{-3}$
16	.999	$.512 \times 10^{-3}$

Table III

Normalized Center Displacement and Bending Moment for a Simply Supported Square Plate

Number of Elements	Displacement--Concentrated Load	Displacement--Uniform Load	Moment--Uniform Load
4	.9922	.9770	.851
16	.9948	.9947	.963
64	.9982	.9987	.991

(a) $\kappa = 1000$

Number of Elements	Displacement--Concentrated Load	Displacement--Uniform Load	Moment--Uniform Load
4	.9957	.9782	.851
16	.9991	.9960	.963
64	1.0034	.9997	.991

(b) $\kappa = 5/6$

Table IV

Normalized Center Displacement and Bending Moment for a Clamped Square Plate

Number of Elements	Displacement--Concentrated Load	Displacement--Uniform Load	Moment--Uniform Load
4	.8652	.9535	.822
16	.9650	.9850	.955
64	.9920	.9937	.986

(a) $\varkappa = 1000$

Number of Elements	Displacement--Concentrated Load	Displacement--Uniform Load	Moment--Uniform Load
4	.8720	.9575	.822
16	.9748	.9890	.955
64	1.0034	.9976	.986

(b) $\varkappa = 5/6$

Table V

Normalized Center Displacement and Bending Moment for a Clamped Circular Plate

Number of Elements	Displacement--Concentrated Load	Displacement--Uniform Load	Moment--Uniform Load
3	.9197	.8587	.827
12	.9579	.9535	.957
48	.9883	.9888	.990

(a) $\varkappa = 1000$

Number of Elements	Displacement--Concentrated Load	Displacement--Uniform Load	Moment--Uniform Load
3	.9267	.8621	.827
12	.9674	.9570	.957
48	1.0005	.9925	.990

(b) $\varkappa = 5/6$

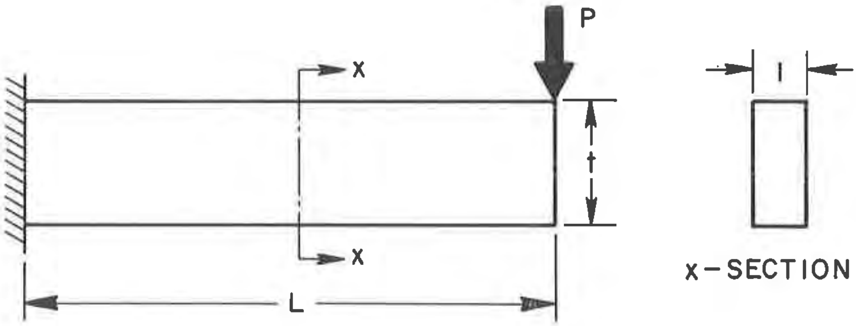

Figure 1 Cantilever beam subjected to end load.

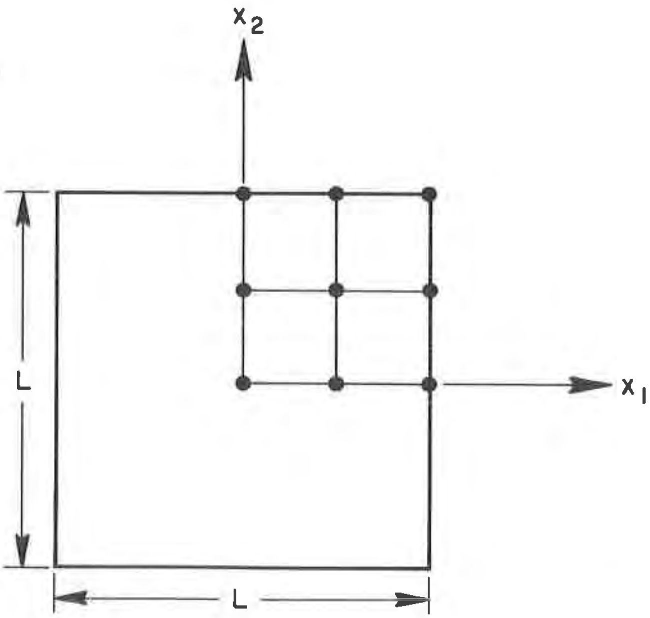

Figure 2 Square plate. Due to symmetry only one quadrant is discretized.

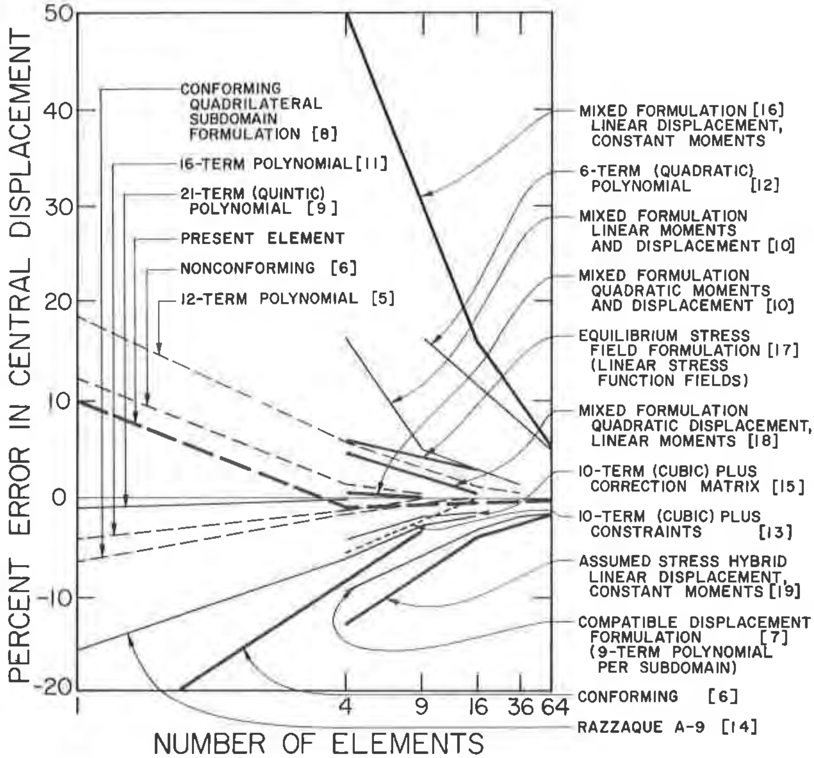

Figure 3 Simply supported square plate subjected to a concentrated load. Comparison of center displacement for various bending elements.

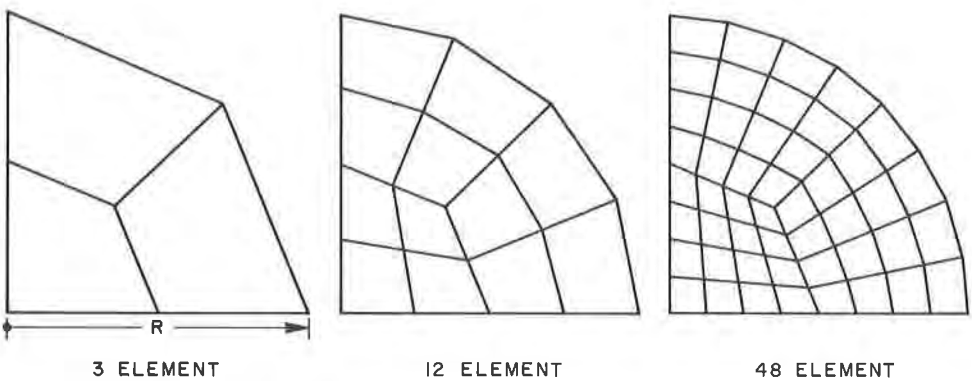

Figure 4 Finite element meshes for clamped circular plate. Due to symmetry only one quadrant is discretized.