APPLICATION OF THE BOUNDARY-INTEGRAL-EQUATION METHOD TO THE THREE-DIMENSIONAL THERMOELASTIC PROBLEM

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SUMMARY

We show how the three-dimensional steady state thermoelastic problem can be solved from data on the boundary alone, using the boundary integral equation method. The numerical procedure used to discretise the integral equation is presented, and two-dimensional examples are given to prove the feasibility and the accuracy of the method.

The formulation of the steady state thermoelastic boundary integral equation is accomplished by the use of an equivalent body force approach; a thermal field is equivalent to a body force proportional to temperature gradients, and tractions on the boundary proportional to temperature. But the volume integral induced by this formulation does not fit the interest of the integral equations. It is shown how this volume integral can be transformed in two surface integrals, in the case of a steady state thermal field.

It is therefore possible to solve three-dimensional steady state thermoelastic problems from data of temperature, fluxes, tractions, and displacements on the boundary alone.

The numerical discretisation of these integral equations is based on a technique similar to that of finite elements. The boundary alone needs to be discretised. Two-dimensional examples are given to prove the accuracy and the feasibility.

In the case of three-dimensional problems, the surface is represented by eight nodes quadrilateral elements and six nodes triangular elements. The unknown (temperature, flux, displacement, traction) may be considered to vary linearly, quadratically, or cubically, with respect to the intrinsic coordinates of each element. The integration is performed numerically using Gaussian quadrature formulas, for which the number of integration points is chosen automatically by the program, so that the upper bound of error in integration is minimized. In order to obtain a banded form matrix, and also to be able to study elongated structures, the body is divided into subregions, for each of which the integral equations are written.

The thermoelastic stress and deformation field are obtained from two successive calculations on the same mesh. The first gives the thermal field. The second, taking account of the thermal field, calculates the thermoelastic displacement and stress field.

This type of approach is especially suited to complicated three-dimensional thick structures for which finite element procedures are very expensive. The data are simple to generate, for data on the boundary alone have to be given. Much time can be saved by the user of the program in data generations and data checks.

The boundary integral equation formulation for elastic problems has already been used to study components of nuclear reactors and its efficiency has been proved. The three-dimensional thermoelastic program should be of great interest for the study of thick walled reactor vessels (P.W.R., B.W.R., or concrete reactors) where the thermal stresses are important (e.g.; nozzles).
Introduction

Boundary integral equations are a very useful and efficient tool to solve two and three dimensional elastic problems. Computer codes have been written and have proved to be efficient [1]. The boundary integral equation method is based on the numerical solution of a set of integral constraint equations which relate boundary tractions to boundary displacements. Thus the solution of a boundary equation reduces the dimensionality by one.

However this advantage is lost when there are body forces. The main object of this paper is to show that the thermoelastic problem can be solved, for body forces resulting of a steady state temperature gradient field, using only boundary data.

A method for the numerical formulation of the three dimensional problem is presented.

Examples are given for plane stress and plane strain cases.

1. Boundary integral equations : basic formulation

1.1. Three dimensional cases

1.1.1. Let us consider a closed domain V with boundary S. The steady state temperature \( T(x) \) at point \( x \) within the domain is obtained from equation (see ref. [2])

\[
\frac{\partial^2 T(x)}{\partial x^2} + \frac{\partial^2 T(x)}{\partial y^2} = \frac{\partial T(x)}{\partial t}
\]

(1)

1.1.2. The displacement \( u(x, y) \) at \( x \) is given by equation (see ref. [3])

\[
\begin{aligned}
\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} &= -\frac{1}{2} \int_V \left[ \nabla \cdot (\nabla \cdot (x, y)) \right] \nabla \cdot (x, y) dS_x \\
&+ \int_S T(x, y) \cdot n dS_y
\end{aligned}
\]

(2)

where

\[
\frac{\partial u(x, y)}{\partial x} = \int_0^\infty \int_0^{2\pi} \left[ \frac{\partial^2 u(x, y)}{\partial t^2} + \frac{\partial u(x, y)}{\partial t} \right] \frac{d\theta}{2\pi}
\]

(3)

\[
\frac{\partial^2 T(x, y)}{\partial x^2} + \frac{\partial^2 T(x, y)}{\partial y^2} = \frac{\partial T(x, y)}{\partial t}
\]

(4)

\( x \) is the distance between \( \tau(x_1, y_1, z_1) \) and \( y(y_1, y_2, y_3) \). In eq. (3) \( \frac{\partial T(x, y)}{\partial t} \) is the heat flux; In eq. (4) \( \oint \frac{\partial T(x, y)}{\partial t} \) is the body force, \( \delta_{ij} \) is the Kronecker delta. \( V_y \) the elementary volume integral, \( t(t_1, t_2, t_3) \) the traction on the boundary, \( n_n \) the outward normal at point \( y \) is the Young's Modulus, \( v \) is the Poisson's ratio. Let us quote that the Einstein convention of summation on the repeated indices is used.

If there are no body forces \( \oint \frac{\partial T(x, y)}{\partial t} = 0 \) it is obvious when \( \frac{\partial^2 u(x, y)}{\partial t^2} + \frac{\partial u(x, y)}{\partial t} \) are known on the surface \( S \), formula (1) and (2) give \( T \) and \( u \) for any point \( V \) within the domain.

Furthermore for \( x \) on the boundary eq. (2) relates then traction and displacement on the boundary alone.

When the body force are non zero (e.g., in case of temperature gradients) the volume integral in eq. (2) does not fit with the efficiency of the boundary integral equation; but we will show that this difficulty can be avoided for a thermoelastic field.

1.1.3. Thermoelastic problem : body force formulation

It is well known that a thermal field is equivalent to a body force \( \mathbf{f}_b(\mathbf{q}_t, \mathbf{q}_s, \mathbf{q}_s) \) and tractions \( F(F, F_2, F_3) \) on the boundary. In the three dimensional case we have:
where $\kappa$ is the thermal expansion coefficient.

Assuming, for the sake of simplicity, that all body forces are zero except those induced by the thermal field, we can deduce from eqs. (2), (5) and (6) the boundary integral equation for the thermoelastic problem:

$$
\kappa \delta_{y} = -K \frac{\partial T}{\partial x_{y}}
$$

where

$$
G = 1 + 4\pi
$$

Eq. (7) can also be written, integrating by parts the last two terms

$$
\kappa \delta_{y} \left( T_{y}(x, y) \right) \int_{y} T_{y}(x, y) \, dy + \int_{y} \left( \frac{\partial}{\partial y} \right)_{x}^{2} \left[ \frac{\partial u_{y}(x, y)}{\partial y} \right] \, dy
$$

Substitution of the displacement kernel (eq. (3)) in the volume integral of eq. (8) leads to:

$$
\kappa \int_{y} \left( \frac{A_{y}}{G} \left( \frac{\partial u_{y}(x, y)}{\partial y} \right) \right) \, dy = \frac{\kappa}{4\pi} \int_{y} \left( \frac{A_{y}}{G} \left( \frac{\partial u_{y}(x, y)}{\partial y} \right) \right) \, dy
$$

1.1.4. Transformation of the volume integral

The internal temperature obeys to Laplace's equation, therefore it should be possible to transform the volume integral (eq. (9)) in surface integrals.

Using the following property of Laplacian operator ($\nabla^{2}$)

$$
\nabla^{2} \left( k \right) = 1/k
$$

we obtain (since $\nabla^{2} T = 0$)

$$
\int_{y} \left( \frac{A_{y}}{G} \left( \frac{\partial u_{y}(x, y)}{\partial y} \right) \right) \, dy = \frac{1}{2} \int_{y} \left[ \nabla^{2} \left( \frac{A_{y}}{G} \right) \right] \, dy
$$

by the Green theorem eq. (11) can be transformed in eq. (12)

$$
\int_{y} \left( \frac{A_{y}}{G} \right) \, dy = \frac{1}{2} \int_{y} \left[ \frac{4\pi}{G} \left( \frac{\partial u_{y}(x, y)}{\partial y} \right) \right] \, dy
$$

Coming back to eq. (8), we obtain a formulation for the thermoelastic boundary value problem in terms of surface integrals alone:

$$
\kappa \delta_{y} \left( T_{y}(x, y) \right) \int_{y} T_{y}(x, y) \, dy = \frac{2\pi}{G} \int_{y} \left( \frac{\partial u_{y}(x, y)}{\partial y} \right) \, dy
$$

1.2. Two dimensional case

In this case the same procedure as in §1.1. can be used. We use the following property of the 2D Laplace's operator:

$$
\nabla^{2} \left( \frac{1}{\sqrt{a^{2} + b^{2}} - d} \right) = \nabla^{2} \left( \frac{1}{\sqrt{a^{2} + b^{2}}} \right) + d^{2} \nabla^{2} \left( \frac{1}{\sqrt{a^{2} + b^{2}} + 2d \sqrt{a^{2} + b^{2}}} \right) = 4 \delta_{y} \frac{1}{G}
$$

The integral equation is then:

$$
\kappa \delta_{y} \left( T_{y}(x, y) \right) \int_{y} T_{y}(x, y) \, dy = \int_{y} \left[ \frac{4\pi}{G} \left( \frac{\partial u_{y}(x, y)}{\partial y} \right) \right] \, dy
$$

where

$$
L_{y}(x, y) = \frac{4\pi}{G} \left( \frac{1}{\sqrt{a^{2} + b^{2}} - d} \right) \left( \frac{1}{\sqrt{a^{2} + b^{2}} + 2d \sqrt{a^{2} + b^{2}}} \right)
$$

$$
E' = \frac{E}{1 - \nu^{2}}
$$

for plane stress

$$
E' = \frac{E}{1 - \nu^{2}}
$$

for plane strain
Eq. (13) and (15) show that the thermoelastic stress and displacement field on the boundary can be calculated using temperatures and thermal fluxes on the boundary alone.

1.3. Stresses
The stress field at internal point \( x \) is obtained using Hooke's law i.e.:

\[
\sigma_{ij}(x) = \frac{E}{(1 - \nu^2)(1 + \nu)} \left[ \frac{\partial \varepsilon_{ii}(x)}{\partial x_j} \right] + \frac{E}{2(1 + \nu)} \left( \frac{\partial \varepsilon_{ij}(x)}{\partial x_j} \right) - K T(x) \tag{19}
\]

- plane stress: \( \varepsilon = \varepsilon_y, \nu = 0, K = \frac{E}{1 - \nu} \)
- plane strain and three dimensional case: \( \varepsilon = \varepsilon_y, \nu = \frac{1}{2}, K = \frac{E}{2(1 - \nu)} \)

2. Numerical discretisation
The solution of the thermoelastic problem requires two successive calculations with the same mesh. The first one gives the thermal field; the second, using this thermal field, leads to the thermoelastic displacement and stress field.

Assuming that at each node either temperature or flux is specified we obtain a linear system of equations relating unknowns (flux or temperature) on the boundary alone. After solving this system the same procedure can be applied to eqs. (13) or (15) to get displacement and stresses on the boundary. The corresponding values within the domain can be deduced from eqs. (13) or (15) and (19).

2.1. Three dimensional case
2.1.1. Discretisation of geometry and functions
The surface is modelled by \( p \) boundary elements each of which has either eight or six nodes (figure 1).
The cartesian coordinate \( x_L \) of an arbitrary point of an element is defined in terms of nodal coordinates \( x^i \) and shape functions \( N^k(y) \)

\[
x_L(y) = N^k(y) x^k \tag{20}
\]
The functions \( N^k(y) \) can have a linear quadratic or cubic variation over each element. The value \( \phi \) of a function at an arbitrary point on an element is defined in terms of its nodal values \( \phi^k \) and shape functions \( M^k(y) \)

\[
\phi(y) = M^k(y) \phi^k \tag{21}
\]

2.1.2. Subregions
The elastic body is considered to be consist of several subregions \( R^{(k)} \) the elastic properties of which may differ (figure 2). The integral equations (1) and (13) are written for each subregion \( R^{(k)} \). Compatibility equations (i.e. continuity of displacements temperature and flux) have to be added to these equations.

2.1.3. Discretisation of integral equations (1) and (13)
The temperature field is obtained over each subregion for each node \( x^a \in S^{(k)} \) by the equation:

\[
dT(x^a) = \sum_{l=1}^{8} \sum_{e=1}^{4} T^e(x^a) \int_{S_l} \frac{1}{2} \frac{\partial \psi(x)}{\partial n_l} M^e(\overline{r}) J(\overline{r}) dS_l \tag{22}
\]
Once the temperature field is known, the integral equation (13) is first rewritten in terms of the components of \( \dot{u} \) and \( \int \) in the fixed and free directions at point \( x \).

\[
C_{ij}^{(12)} \int_{S_i} T_j(x,y) \dot{u}_j(y) \, dS_j = \int_{S_i} \sum_{k} u_k(x) \frac{\partial T_j}{\partial n_k}(y) \, dS_j + B_j^{(12)}
\]

(23)

where

\[
\dot{u}_j^k = \dot{u}_j \frac{\partial u_k}{\partial x_j}, \quad \dot{u}_j^k = \dot{u}_j \frac{\partial u_k}{\partial x_j}, \quad \dot{u}_j^k = \dot{u}_j \frac{\partial u_k}{\partial x_j}, \quad \dot{u}_j^k = \dot{u}_j \frac{\partial u_k}{\partial x_j}
\]

\[
\begin{align*}
&\lambda_{ij} = \lambda_{ij} \frac{\partial \lambda_{ij}}{\partial x_j} \\
&\lambda_{ij} = \lambda_{ij} \frac{\partial \lambda_{ij}}{\partial x_j} \\
&\lambda_{ij} = \lambda_{ij} \frac{\partial \lambda_{ij}}{\partial x_j} \\
&\lambda_{ij} = \lambda_{ij} \frac{\partial \lambda_{ij}}{\partial x_j}
\end{align*}
\]

(24)

\( \beta_i \) are the direction of cosines of the unknowns at \( x \).

Equation (23) is then written for each node \( x^k \in S^k \)

\[
\begin{align*}
C_{ij}^{(12)} \dot{u}_j^k(x^k) + \sum_{m} \sum_{n} u_m(x^k) \int_{S_i} T_n(x^k, y^m) M^m(y) J^m(y) \, dS_j^m & = \sum_{m} \sum_{n} \dot{u}_j^m(x^k) \int_{S_i} T_n(x^k, y^m) M^m(y) J^m(y) \, dS_j^m \\
& \quad + \sum_{m} \sum_{n} \dot{u}_j^m(x^k) \int_{S_i} T_n(x^k, y^m) M^m(y) J^m(y) \, dS_j^m
\end{align*}
\]

(25)

where

\( d_i \) is the \( i \)th node of the \( b \)th boundary element

\( M^m \) is the shape function associated with the variation of unknown \( \lambda^k \)

\( n_m \) is the number of boundary element of subregion

\( \beta_i \) is the number of nodes on each element

\( J^m \) is the Jacobian.

The integrals of kernel shape functions products appearing in eq.(25) are evaluated using gaussian quadrature formula the integration scheme being chosen according to the rapidity of variation of the integrand.

2.1.4. Procedure of integration

The upper bound of error in gaussian integration of function \( f \) is given by the formula:

\[
\left| \int_{S_i} f(x) \, dS_i - \sum_{i=1}^{n_i} \int_{S_i} f(x) \, dS_i \right| \leq 2 \sum_{i=1}^{n_i} \epsilon_i \, M_i
\]

(26)

where

\( M_i \) = \frac{n_i}{\int_{S_i} f(x) \, dS_i}

\( \theta_i = \frac{G}{\int_{S_i} f(x) \, dS_i} \)

\( n_i \) number of integration point in direction \( i \)

It would be very impractical to calculate the proceeding upper bound because the integrands \( \int_{S_i} f(x) \, dS_i \) are very complicated. Thus the function \( f(x) \) is taken to be representative of the integrands. Even the derivative of \( f(x) \) with respect to the intrinsic coordinate is complicated. Further simplifications are made.

The Jacobian \( J \) is assumed to be constant over each element (which is a reasonable approximation if the elements are not badly distorted). We can assume that:

\[
\left| \frac{\partial J}{\partial n_i} \left( f(x) \right) \right| \leq \frac{(2n_i + 1)^2}{4n_i} \left( \frac{2n_i + 1}{2n_i} \right)^{\epsilon_i}
\]

(27)

Where \( R \) is the minimum distance between point \( x^k \) and any point of boundary element \( S^k_b \), supposing that integration is equally precise in each direction.

If we want to limit the upper bound of error by \( \epsilon_i \), taking account of eq.(26) and eq.(27)
we obtain:
\[
\left( \tilde{a}_m \right)_n \cdot 4 \cdot \tilde{c}_m \frac{\alpha_n}{\rho} \leq \tilde{c}_j \cdot \frac{\alpha_n}{\rho} = \left[ \frac{2 \tilde{c}_n}{\alpha_n} \right]_{n=1}^{2n}.
\]
(28)

The value of \( \tilde{c} \) has to be chosen in the light of experience.

The inequality (28) allows the calculation of \( \alpha_n \), number of integration points in each direction. The effect of this procedure is to concentrate the integration points around the singularity (fig.3).

These procedures have already been tested on the three dimensional elastic cases and have proved to be accurate and cheap [1].

2.2. Two dimensional case

2.2.1. Discretisation

The boundary \( \tilde{c} \) is modelised by a set of curved or straight elements each of which has three nodes. The same procedure as in §2.1.1. and 2.1.3. is used for the discretisation of integral equations in the 2D case.

2.2.2. Examples

In order to show the feasability and the accuracy of the method, two examples are given below and compared to theoretical solution.

- **Plastic strain example**: Beam with a longitudinal linear temperature variation the material has the following properties:
  - conductivity \( \lambda = 1 \text{W/m}^2\text{C} \)
  - density \( \rho = 7.8 \times 10^3 \text{Kg/m}^3 \)
  - specific heat \( C = 1 \text{J/K}^\circ \text{C} \)
  - Young's modulus \( E = 210000 \text{MPa} \)
  - Poisson's ratio \( \nu = 0.3 \)
  - thermal expansion coefficient \( \alpha = 12.10^{-6} \text{m/m}^\circ \text{C} \)

Modelisation is given on figure 4. Because of symmetry half of the structure is represented. Four boundary elements with parabolic variation of the unknown have been chosen. The boundary conditions are given on fig.5 for the thermal case and on fig.6 for the thermoelastic calculation. The first calculation gives the temperatures and flux which are found to approach theory by less than 1%.

In the second step these values are used for the computation of the thermoelastic displacement field. The calculated displacements of node 1 (fig.6) is 0.602 (theory 0.600) and the stresses are about \( 10^{-3} \text{MPa} \). Despite of the coarse discretisation used, the comparison is rather well.

- **plane strain example**: Cylinder

Two modelisations were used (fig.7) with the same material as in plane stress example. Boundary conditions are shown on fig.(8) for the thermal case and on fig.(9) for the thermoelastic problem. Figure (10) show the comparison between the theoretical values and the calculated ones for hoop stresses. Here again the results are quite good.
Conclusion

In this paper it has been shown how to write the boundary integral equations for the thermoelastic problem in terms of boundary integrals alone.

The feasibility and accuracy of the method has been proved on two dimensional examples. The generalisation to three dimensional thermoelastic problem presented in the paper should then be of great practical interest.

Acknowledgement

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References


Figure 4: Plane stress example: geometry and modelisation

Figure 5: Plane stress example: thermal boundary conditions

Figure 6: Plane stress example: thermoelastic boundary conditions

Figure 7: Plane strain example: geometry and modelisation

Figure 8: Plane strain example: thermal boundary conditions

Figure 9: Plane strain example: thermoelastic boundary conditions
Figure 10: Plane strain example: comparison theory - calculation; hoop stress