A COLLOCATION FINITE ELEMENT METHOD WITH PRIOR MATRIX CONDENSATION

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SUMMARY

For thin shells with general loading, sixteen degrees of freedom have been used for a previous finite element solution procedure using a Collocation method instead of the usual variational based procedures. Although the number of elements required was relatively small, nevertheless the final matrix for the simultaneous solution of all unknowns could become large for a complex compound structure.

The purpose of the present paper is to demonstrate a method of reducig the final matrix size, so allowing solution for large structures with comparatively small computer storage requirements while retaining the accuracy given by high order displacement functions.

The principle used is based on the fact that, of the shell equations to be satisfied at Collocation points, a number are equilibrium conditions which must be satisfied independently of the overall compatibility of forces and deflections for a complete structure.

In order to implement this principle the displacement functions can be rewritten with the independent conditions separated from the remainder. Then by writing the equations for external axial and lateral loads and the three possible surface loads, the calculations involving the independent conditions can be performed for each element before final matrix assembly. The final solution for the complete structure then involves a correspondingly smaller number of unknowns. This makes explicit this condition that, of the total number of equations to be satisfied, the majority (six out of eleven for axi-symmetric loading) arise from the requirements of equilibrium conditions which are independent of inter-element effects. It may be noted that when pressure loads are reduced to equivalent forces concentrated at the nodes, as is commonly the case in variational based solutions, four of these degrees of freedom are lost for axial loading and six for unsymmetrical loading.

Using the prior condensation method in a typical example the maximum number of equations for simultaneous solution can be reduced from 77 to 28, without any reduction in the effective degrees of freedom of each element. Furthermore, in principle, the number of degrees of freedom can be increased, if required. This will increase the number of equations in the prior condensation stage but not in the final element assembly stage and not, in any practical case, above the number in the final assembly stage.

1. Introduction

The method of solution proposed in the present paper is a development of that described previously [1] [2]. In those papers a finite element in the form of a thin ring was used and a Collocation method was applied to solve axi-symmetrical thin shell problems with symmetrical and unsymmetrical loads.

The displacement relations used were in the form of series in S, the co-ordinate of distance along a meridian. For symmetrical loads these power series assumed five terms for v, the displacement along the meridian, and six terms for w, the displacement perpendicular to the meridian. In addition, for unsymmetrical loading cases the lateral displacement u was assumed to have five terms.

In order to solve for the corresponding constants in the displacement functions eleven equations are required for symmetrical loads and sixteen for unsymmetrical loads. The equations were found from rigid body motion requirements, loading conditions at the nodes and element boundary conditions for both displacement and equilibrium. The number of degrees of freedom having been chosen to be exactly equal to the total number of equations to be satisfied a determinate system for a shell divided into n elements results. The total number of equations involved is 11n or 16n, for symmetrical and unsymmetrical loads respectively. These were to be satisfied simultaneously and if n was small 'in core' solutions were practical and no special solution methods were required due to computer storage limitations. The storage requirement for the matrix corresponding to these simultaneous equations increases as n^2, neglecting banding, or as n, if the band width only is stored in a banded formulation.

The purpose of the present paper is to show that a number of the equations correspond to conditions which are invariant with respect to the inter-nodal boundary conditions. Hence prior solution for these equations is possible and the number of equations to be solved simultaneously may be greatly reduced.

2. The Equations to be Satisfied

Only axi-symmetric deformation will be considered to demonstrate the principles involved. Clearly with the addition of the appropriate extra variables the same solution procedure can be extended to symmetrical deformation cases as in [1] and [2].

Using a common notation and sign convention as e.g. Flügge [3], the equations for the equilibrium of an element, the Force displacement relationships and the Forces at the nodes are:

$$Q_\phi = f_1(SR),\quad p_r = f_2(SR),\quad p_\phi = f_3(SR); \qquad 1(a)$$

$$N_\phi = f_4(DIS),\quad N_\theta = f_5(DIS),\quad M_\phi = f_6(DIS),\quad M_\theta = f_7(DIS) \qquad 1(b)$$

$$V = f_8(SR),\quad H = f_9(SR) \qquad 1(c)$$

(Forces per unit length along and perpendicular to the axis)

where f(SR) and f(DIS) denote functions of the stress resultants and displacements respectively, as given by any consistent formulation of thin shell equations e.g. the equations given explicitly in [1] based on Flügge [3].

Now, as eqs. (1b) can be expressed in terms of the assumed series functions in the co-ordinate S, hence eqs. (1a) and (1c) can also in turn be so expressed.

The required eleven equations per element in terms of the constants in the displacement functions are:

(i) One axial constraint or rigid body motion condition.

(ii) Three loading conditions corresponding to V, p_r and p_ϕ at each node i.e. six per element.

(iii) Four edge boundary conditions and/or inter-element force and displacement compatibility conditions.

3. Re-formulation of the Displacement Relations

As in [1] the assumed displacement expansions are:

$$v = k_1 + k_2S + k_3S^2 + k_4S^3 + k_5S^4 \tag{2}$$

$$w = k_6 + k_7S + k_8S^2 + k_9S^3 + k_{10}S^4 + k_{11}S^5$$

and using eqs. (1) the conditions determined by (i) and (ii) Section 2 can be written in the form:

$$\begin{aligned}
V_o &= A_1k_1 + A_2k_2 \;\cdots\cdots\; A_{11}k_{11} \\
V_\ell &= A_{12}k_1 + \;\cdots\cdots\; A_{22}k_{11} \\
p_{ro} &= A_{23}k_1 + \;\cdots\cdots\; A_{33}k_{11} \\
p_{r\ell} &= A_{34}k_1 + \;\cdots\cdots\; A_{44}k_{11} \\
p_{\phi o} &= A_{45}k_1 + \;\cdots\cdots\; A_{55}k_{11} \\
p_{\phi\ell} &= A_{56}k_1 + \;\cdots\cdots\; A_{66}k_{11} \\
\delta_{ao} &= A_{67}k_1 + \;\cdots\cdots\; A_{77}k_{11}
\end{aligned} \tag{3}$$

where o and ℓ refer to values at the ends of an element, where S = 0 and S = ℓ respectively, and δ_{ao} is an axial displacement condition, e.g. axial displacement at S = 0 zero. A_1 to A_{77} are the coefficients of the constants in the displacement functions and are calculable for a given shell geometry by using eqs. (2) and (1).

For a defined loading and axial constraint situation the left hand sides of eqs. (3) must be known. Hence eqs. (3) are seven equations in eleven unknowns. These must be satisfied whatever the edge boundary conditions (iii) in Section 2 may be.

If, therefore, four of the basic constants in eqs. (2) are taken as the unknowns to be determined by the four homogeneous edge boundary conditions the remaining seven can be expressed in terms of these four. Hence selecting k_2, k_6, k_7 and k_8 as these four: (this choice appears to be arbitrary)

$$\begin{aligned}
k_1 &= N_1 + N_2k_2 + N_3k_6 + N_4k_7 + N_5k_8 \\
k_3 &= N_6 + N_7k_2 + \;\cdots\cdots\; N_{10}k_8 \\
k_4 & \\
&\cdot \\
k_{10} & \\
k_{11} &= N_{31} + N_{32}k_2 + \;\cdots\; N_{35}k_8
\end{aligned} \tag{4}$$

where N_1 to N_{35} are numbers calculable in principle from eqs. (3), and putting (4) in (2):

$$\begin{aligned}
v = {} & (N_1 + N_6S^2 + N_{11}S^3 + N_{16}S^4) + k_2(N_2 + S + N_7S^2 + N_{12}S^3 + N_{17}S^4) \\
& + k_6(N_3 + N_8S^2 + N_{13}S^3 + N_{18}S^4) + k_7(N_4 + N_9S^2 + N_{14}S^3 + N_{19}S^4) \\
& + k_8(N_5 + N_{10}S^2 + N_{15}S^3 + N_{20}S^4)
\end{aligned}$$

$$w = (N_{21}S^3 + N_{26}S^4 + N_{31}S^5) + k_2\,(N_{22}S^3 + N_{27}S^4 + N_{32}S^5) \qquad (5)$$
$$+ k_6(1 + N_{23}S^3 + N_{28}S^4 + N_{33}S^5) + k_7(S + N_{24}S^3 + N_{29}S^4 + N_{34}S^5)$$
$$+ k_8(S^2 + N_{25}S^3 + N_{30}S^4 + N_{35}S^5)$$

Thus the displacement functions have been re-formulated in terms of two separate sets of constants; $k_{2,6,7,8}$ and N_{1-35}. Of these the constants k depend solely on the inter-element boundary conditions. The constants N depend solely on the loading and axial constraint conditions and therefore may be found for each element separately.

4. Prior Condensation of the System of Equations for a Complete Shell Structure

Whatever the constants $k_{2,6,7,8}$ are, eqs. (3) must be independently satisfied at the nodes. The values of N_{1-35} must therefore be calculable from the conditions corresponding to eqs. (3). Therefore if all the constants $k_{2,6,7,8}$ are zero or successively all but one are zero eqs. (3) must still be satisfied. The corresponding five applications of eqs. (3) each give rise to seven equations in seven of the constants N_{1-35}.

Solutions of these five cases of seven equations can then be substituted back into eqs. (5).

The resulting equations for v and w are now in terms of $k_{2,6,7,8}$ as the only unknowns.

5. Final Matrix Assembly for a Complete Shell

For each element the calculations of section 4 can be performed independently of inter-element boundary conditions and these calculations involve the simultaneous solution of a maximum of seven algebraic equations.

For a shell divided into n elements there remain 4n unknowns to be determined. The conditions for the corresponding 4n equations are the inter-element compatibility of stress resultants and displacements and the defined edge conditions at the outer edges of the outer elements.

At inter-element boundaries δ, β, M_ϕ and H must be equated, where δ and β are the displacement perpendicular to the axis of symmetry and the rotation of a tangent to the meridian respectively.

At the outer edges of the complete shell a total of four conditions for δ, β, M_ϕ and H must be known for a determinate problem. Hence a total of 4n equations can be obtained and solved for the constants $k_{2,6,7,8}$ for all the n elements. Back substitution in eqs. (2) and (1) then gives the values of all displacements and stress resultants at all nodes.

6. Solution Procedure

For the five cases of eqs. (3) to be solved the displacements may be written:

$$N_p + N_q S^2 + N_r S^3 + N_s S^4 = v - x$$
$$N_t S^3 + N_u S^4 + N_v S^5 = w - y \qquad (6)$$

where x and y take the values:

Case	k_2	k_6	k_7	k_8	x	y
1	0	0	0	0	0	0
2	1	0	0	0	S	0
3	0	1	0	0	0	1
4	0	0	1	0	0	S
5	0	0	0	1	0	S^2

For case 1 the left hand sides of eqs. (3) are given by the loading and axial support conditions at $S = 0$ and $S = \ell$. For cases 2-5 these values must be independent of $k_{2,6,7,8}$ and so the left hand sides of eqs. (3) must be taken as zero.

The form of eqs. (6) being common to all five solution cases, seven algebraic equations in seven unknowns with multiple right hand sides result when eqs. (6) are used to derive eqs. (3). Therefore only one inversion of the corresponding matrix is required.

Having obtained the values of all the constants N for all elements and all five solution cases, equations for δ, β, H and M_ϕ can be written in terms of $k_{2,6,7,8}$ with coefficients derived from the terms in brackets in eqs. (5). 4n edge conditions then give the required 4n constants to complete the calculation for v and w and so for all variables of displacement or stress resultants derived from v and w.

7. Application of the Method

The example given in [1] Section 2.2 and shown in Fig. 3 in [1] gave the stress analysis of a large, very thin, hemispherical tank with varying fluid pressure loading. Using the same division into only seven elements, varying from 35 cm to 1172 cm length, for a constant thickness of 2.3 cm, the present method reproduces exactly the same distributions of stress and deflection. This is because using the same shell equations the calculations performed should be the same. However the order of performing the calculations is altered, in particular, the maximum number of equations to be solved simultaneously is reduced from 77 to 28.

8. Further Developments

Having tested the detailed application of the method on a relatively small problem, the analysis of very large shell structures becomes possible without requiring the simultaneous solution of large numbers of equations. This is the case even though the order of the displacement functions remains comparatively high i.e. equivalent to eqs. (2). Furthermore the conditions of eqs. (3) could be applied at a number of 'internal nodes' without increasing the maximum number of equations to be solved simultaneously. This would entail the introduction of even higher order displacement functions and so the possibility of even fewer elements being required.

For each 'internal node three equations for p_r, p_ϕ and V would have to be satisfied and so three more terms in the displacement functions would be required.

9. Conclusions

A development of a Collocation method for the stress analysis of thin shells using finite elements has demonstrated the use of high order displacement functions without the need for very large computer storage requirements.

By retaining the loading as distributed, rather than reducing it to equivalent concentrated forces at the nodes, the conditions which may be applied to each element before final matrix assembly are made explicit.

References

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