ELASTO/VISCO-PLASTIC DYNAMIC RESPONSE
OF SHELLS OF REVOLUTION

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SUMMARY

The dynamic response problem of shell structures is closely related to the reactor vessel
technology in many aspects, e.g. fast breeder reactor accident analysis or containment
shells under the influence of gas-cloud explosion. On the dynamic response of shell struc-
tures to time-varying loads many investigations have been carried out for not only the
simple geometries such as cylindrical shells and spherical shells but also the general
axisymmetrical shells. The dynamic response investigations of the shell structures, how-
ever, are almost limited to ones based on elastic (visco-elastic) theory or the conventional
plastic theory in which time does not enter directly into consideration, in spite of the fact
that the analytical results from the conventional plastic theory are at variance with ex-
perimental ones.

In this paper the authors study the large deflection elasto/visco-plastic dynamic re-
ponse of shells of revolution to strong blast loads, where the viscosity of the material
is considered in the plastic range. The equations of motion and the relations between
the strain and the displacement are derived from the Sanders nonlinear theory for thin shells.
The constitutive relation for shell response is linear elastic, visco-plastic. In the linear elas-
tic range Hooke’s law is used. In the plastic range the elasto/visco-plastic equations by
Fyfe based on the model developed by Perzyna are employed. The criterion for yielding
used in this analysis is the von Mises yield theory.

The numerical method selected for integration of the equations of motion is a method
using finite difference in both space and time. The differential equations are written in
finite difference form on the basis of the parabola method. For the time integration of
the equations of motion the second-order finite difference in time is used. The equations
of motion are thus expressed in finite difference form if we divide the shell into segments
along meridional length and around the circumference. Resultant forces and resultant mo-
ments are given from numerical integration by Simpson’s 1/3 rule. The loadings which
are considered in this paper may be either impulsive or of finite time duration.

As the numerical example a cylindrical shell subjected to external impulsive loads of
importance in practical use is analyzed. The impulsive load with uniform distribution
along the length and cosine distribution around the half circumference is considered. The
ends of the cylinder are assumed to be fixed. The material of the cylinder is 6061-T6 al-
uminum. Some of the essential features of the elasto/visco-plastic solution under external
impulsive loads are shown in this paper.
1. Introduction

On a dynamic response of shell structures to time-dependent loads, many investigations have been carried out, not only for simple geometries such as cylindrical shells [1]–[5] and spherical shells [6]–[8] but also for the general axisymmetrical shells [9]–[11]. These investigations, however, are almost limited to ones based on the elastic (visco-elastic) theory or the conventional plastic theory which time does not enter directly into consideration. It may be easily observed in experiments on pressure vessels, pipelines, etc. that influence of viscosity appears in metals in a plastic range at room temperature as well as high polymers [12]. The influence of viscosity in a plastic range becomes remarkable in a dynamic response of shells subjected to impulsive loads owing to high strain rates.

Then the authors study the elasto-visco-plastic dynamic response of shells of revolution to strong blast loads, where a viscosity of a material in a plastic range is considered. The equations of motion and the relations between a strain and a displacement are derived from the Sanders nonlinear theory for thin shells [13]. For the constitutive relations the elasto-visco-plastic equations by Fyfe [14] based on the model developed by Perzyna [15] are employed. The criterion for yielding used in the analysis is the von Mises yield theory.

The numerical method selected for this problem is a method using finite difference in both space and time.

As the numerical example a cylindrical shell subjected to external impulsive loads is analyzed.

2. Analytical Formulations

If the middle surface of axisymmetrical shells is given by \( r = r(s) \), where \( r \) is the distance from the axis and \( s \) is the meridional distance measured from a boundary along the middle surface, as shown in Fig. 1, the relations among the non-dimensional curvatures \( \omega_{x} = (a/R) \), \( \omega_{y} = (a/R) \) and the non-dimensional radius \( \varphi = (r/R) \) become:

\[
\begin{align*}
\omega_{x} &= \frac{\dot{r} + \ddot{r}}{\omega_{y}}, \\
\omega_{y} &= \frac{\dot{r} - \ddot{r}}{\omega_{x}}, \\
\varphi &= \frac{\omega_{x}}{\omega_{y}}, \\
\dot{\varphi} &= \frac{\ddot{r}}{\omega_{x}}, \\
\ddot{\varphi} &= \ddot{r}/\omega_{x} \\
\end{align*}
\]

where \( a \) is the reference length.

Adding the inertia terms to the equilibrium equations in the Sanders nonlinear theory for thin shells [13] and eliminating the transverse shear forces \( Q_{y} \) and \( Q_{z} \) from these, where the rotary inertia terms are omitted, the following equations of motion are obtained:

\[
\begin{align*}
\frac{\partial}{\partial s} \left( \frac{2}{a} \left( \frac{\partial N_{x}}{\partial \varphi} + \frac{\partial M_{x}}{\partial \varphi} \right) - \frac{2}{a} \frac{\partial M_{x}}{\partial \varphi} \right) + \frac{1}{2} \left( \omega_{x} - \omega_{y} \right) \frac{\partial^{2}}{\partial \varphi^{2}} \left( M_{y} \right) - \omega_{x} \frac{\partial^{2}}{\partial \varphi^{2}} \left( M_{x} \right) + \frac{\partial}{\partial s} \left( R_{x} - R_{y} \frac{\partial \varphi}{\partial t} \right) = 0 \\
\frac{\partial}{\partial s} \left( \frac{2}{a} \left( \frac{\partial N_{y}}{\partial \varphi} + \frac{\partial M_{y}}{\partial \varphi} \right) + \frac{\partial M_{y}}{\partial \varphi} \right) + \omega_{x} \frac{\partial^{2}}{\partial \varphi^{2}} \left( M_{y} \right) + \frac{\partial}{\partial s} \left( R_{y} - R_{x} \frac{\partial \varphi}{\partial t} \right) = 0 \\
\frac{\partial}{\partial s} \left( \frac{2}{a} \left( \frac{\partial N_{z}}{\partial \varphi} + \frac{\partial M_{z}}{\partial \varphi} \right) + \frac{\partial M_{z}}{\partial \varphi} \right) - \omega_{x} \frac{\partial^{2}}{\partial \varphi^{2}} \left( M_{x} \right) + \frac{1}{2} \frac{\partial}{\partial s} \left( \frac{\partial^{2}}{\partial \varphi^{2}} \left( M_{x} \right) + \frac{\partial}{\partial \varphi} \frac{\partial^{2}}{\partial \varphi^{2}} \left( M_{y} \right) \right) = 0 \\
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial s} \left( \frac{2}{a} \left( \frac{\partial N_{z}}{\partial \varphi} + \frac{\partial M_{z}}{\partial \varphi} \right) + \frac{\partial M_{z}}{\partial \varphi} \right) - \omega_{x} \frac{\partial^{2}}{\partial \varphi^{2}} \left( M_{x} \right) + \frac{1}{2} \frac{\partial}{\partial s} \left( \frac{\partial^{2}}{\partial \varphi^{2}} \left( M_{x} \right) + \frac{\partial}{\partial \varphi} \frac{\partial^{2}}{\partial \varphi^{2}} \left( M_{y} \right) \right) = 0 \\
\end{align*}
\]
where the notations $\rho$, $t$ and $\rho_s$ in the inertia terms are thickness of shell, time and mass density, respectively, and the others are shown in Fig. 2.

In the fairly large deflection problem, the membrane strains of the middle surface are given by

\[
\begin{align*}
\varepsilon_{ym} &= \frac{1}{a} \left( \frac{\partial U_t}{\partial z} + \omega_s W \right) + \frac{1}{2} \frac{\partial}{\partial z} \varepsilon_z^2 \\
\varepsilon_{em} &= \frac{1}{a} \left( \frac{1}{a} \frac{\partial U_e}{\partial z} + \gamma U_r + \omega_s W \right) + \frac{1}{2} \frac{\partial}{\partial z} \varepsilon_r^2 \\
\varepsilon_{om} &= \frac{1}{2a} \left( \frac{\partial U_o}{\partial z} + \frac{1}{a} \frac{\partial U_s}{\partial a} - \gamma U_t \right) + \frac{1}{2} \frac{\partial}{\partial a} \varepsilon_a \varepsilon_a 
\end{align*}
\]

where $\varepsilon_{ym}$ is half the usual engineering shear strain, and the rotations $\varepsilon_z$, $\varepsilon_r$ are:

\[
\varepsilon_z = \frac{1}{a} \left( -\frac{\partial W}{\partial z} + \omega_s U_t \right), \quad \varepsilon_r = \frac{1}{a} \left( -\frac{\partial W}{\partial z} + \omega_s U_r \right)
\]

The bending distortions $K_x$, $K_y$, $K_{xy}$ are:

\[
\begin{align*}
K_x &= \frac{1}{a} \frac{\partial ^2 z}{\partial a^2}, \quad K_y = \frac{1}{a} \left( \frac{1}{a} \frac{\partial ^2 z}{\partial a^2} + \gamma \phi \right) \\
K_{xy} &= \frac{1}{2a} \left( \frac{1}{a} \frac{\partial ^2 z}{\partial a^2} + \frac{\partial ^2 z}{\partial a^2} - \frac{\partial ^2 z}{\partial a^2} - \gamma \phi \right)
\end{align*}
\]

Under the Kirchhoff-Love hypothesis and the neglect of terms of order $\phi / R_0$ and $\phi / R_s$ relative to unity, the strains at the distance $\phi$ from the middle surface, $\varepsilon_{x}$, $\varepsilon_{y}$, $\varepsilon_{xy}$, are:

\[
\begin{align*}
\varepsilon_{x} &= \varepsilon_{ym} + \frac{1}{a} K_x \\
\varepsilon_{y} &= \varepsilon_{em} + \frac{1}{a} K_y \\
\varepsilon_{xy} &= \varepsilon_{om} + \frac{1}{a} K_{xy}
\end{align*}
\]

respectively.

Now, we shall use the visco-plastic equations by Pyfe [14] for constitutive relations. The function $\Phi(F)$ in the Perzyna visco-plastic equations:

\[
\dot{\varepsilon}_i = \frac{1-\nu}{E} \dot{\varepsilon}_i - \frac{1-2\nu}{E} \dot{\varepsilon}_i \delta_{ij} + \eta_0 \langle \Phi(F) \rangle S_{ij} J_z^{1/2}
\]

is determined from many dynamic plastic experimental results by Pyfe as follows:

\[
\Phi(F) = C \exp\left( - \frac{F}{30} \right)
\]

where the dot denotes the partial derivative with respect to time, $S_{ij}$, $S$ and $\dot{S}_{ij}$ are the deviatoric stress, the mean stress, and the strain, respectively, while $E$, $\nu$, $\gamma$, $\alpha$ are the Young's modulus, Poisson's ratio, and material constants and $J_z$ is the second invariant of the deviatoric stress tensor. The symbol $\Phi(F)$ is defined as follows:

\[
\langle \Phi(F) \rangle = 0 \quad \text{when} \quad F \leq 0 \quad \text{and} \quad \langle \Phi(F) \rangle = \Phi(F) \quad \text{when} \quad F > 0
\]

where function $\Phi(F)$ is:

\[
F = \left( J_z^{1/2} - \frac{\gamma}{\eta_0} \right) / \gamma_0
\]

and $F=0$ denotes the von Mises yield surface, while $\gamma$ and $\gamma_0$ are the static strain hardening and initial yield stress in simple shear, respectively.

In the plane stress state ordinarily assumed in the shell theories, the constitutive relation proposed by Pyfe may be expressed from eqs.(7),(8) and (10) as follows:

\[
\begin{align*}
\dot{\varepsilon}_z &= \frac{E}{1-\nu^2} \left( \dot{\varepsilon}_z + \nu \dot{\varepsilon}_r - \dot{\varepsilon}_r + \nu \dot{\varepsilon}_p \right) \\
\dot{\varepsilon}_r &= \frac{E}{1-\nu^2} \left( \nu \dot{\varepsilon}_z + \dot{\varepsilon}_r - \nu \dot{\varepsilon}_r + \dot{\varepsilon}_p \right) \\
\dot{\varepsilon}_{xy} &= \frac{E}{1+\nu} \left( \dot{\varepsilon}_{xy} - \dot{\varepsilon}_{xy}^p \right)
\end{align*}
\]

where
\[
\dot{\varepsilon}_p = \gamma_0 \left[ e \times p \left\{ \frac{1}{2} \left( \frac{\alpha}{\sigma} - \frac{1}{2} \sigma \right) \right\} \right] \frac{2}{3 \sqrt{F_s}} \left( \sigma_y - \frac{1}{2} \sigma_0 \right) \\
\dot{\varepsilon}_\theta = \gamma_0 \left[ e \times p \left\{ \frac{1}{2} \left( \frac{\alpha}{\sigma} - \frac{1}{2} \sigma \right) \right\} \right] \frac{2}{3 \sqrt{F_s}} \left( \sigma_\theta - \frac{1}{2} \sigma_0 \right) \\
\dot{\varepsilon}_\phi = \gamma_0 \left[ e \times p \left\{ \frac{1}{2} \left( \frac{\alpha}{\sigma} - \frac{1}{2} \sigma \right) \right\} \right] \frac{1}{3 \sqrt{F_s}} \sigma_\phi \\
\]

The membrane forces and the resultant moments per unit length are:

\[ 
N_3 = \int_0^{\frac{\theta}{2}} \sigma_3 \theta d\sigma \ , \ N_\theta = \int_0^{\frac{\theta}{2}} \sigma_\theta \theta d\sigma \ , \ N_\phi = \int_0^{\frac{\theta}{2}} \sigma_\phi \theta d\sigma \\
M_3 = \int_0^{\frac{\theta}{2}} \sigma_3 \phi d\sigma \ , \ M_\theta = \int_0^{\frac{\theta}{2}} \sigma_\theta \phi d\sigma \ , \ M_\phi = \int_0^{\frac{\theta}{2}} \sigma_\phi \phi d\sigma 
\]

A complete set of field equations for 26 independent variables: \( N_3, N_\theta, N_\phi, M_3, M_\theta, M_\phi, U_3, U_\theta, U_\phi, W, \dot{E}_3, \dot{E}_\theta, \dot{E}_\phi, \dot{E}_m, E_3, E_\theta, E_\phi, M_3, M_\theta, M_\phi, \sigma_3, \sigma_\theta, \sigma_\phi, \sigma_m, \dot{U}_3, \dot{U}_\theta, \dot{U}_\phi, \dot{W}, \dot{E}_3, \dot{E}_\theta, \dot{E}_\phi \) is now given by 26 equations, (2)-(6) and (11)-(13).

3. Numerical method

If we solve the above 26 equations with suitable boundary conditions, we have solutions for this problem. Since it's too difficult to solve these equations analytically, we shall use the finite difference method.

Let the shell be divided into \((M-1)\) equal segments along its meridian and \((N-1)\) equal segments around the circumference as shown in Fig.3. Then the increments \(\Delta \tilde{z}\) and \(\Delta \tilde{\theta}\) in the nondimensional variables \(\tilde{z}\) and \(\tilde{\theta}\) are as follows:

\[ 
\Delta \tilde{z} = \frac{S_0}{\alpha (M-1)} , \quad \Delta \tilde{\theta} = \frac{\pi}{(N-1)} 
\]

where \(S_0\) is the length of the meridian. In eqs. (14) we treat the case of symmetry with respect to \(\theta = 0^\circ\) and other cases are similarly.

In order to use Simpson's 1/3 rule for integration of eqs. (13), we divide the thickness \(\tilde{A}\) of the shell into \((L-1)\) equal layers, running from 1 at the inner surface to \(L\) at the outer surface, then the positions of arbitrary points of the shell may be written as \((\tilde{z}, \tilde{\theta}, \tilde{A})\).

Denoting some quantities at the mesh point \((\tilde{z}, \tilde{\theta}, \tilde{A})\) by \(\tilde{f}\) and employing three point difference formulas for the boundary and central difference formulas for other points, nine finite difference equations related to four boundary lines, four boundary corner points and the region except the boundary can be obtained for each derivative: \(\partial^3 f/\partial z^3\), \(\partial^3 f/\partial z^2\partial \theta\), \(\partial^3 f/\partial z\partial \theta^2\), \(\partial^3 f/\partial \theta^3\). For the second derivatives with respect to time in the inertia terms in eqs. (2) the next relations are employed:

\[ 
\frac{\partial^2 f}{\partial t^2} = \frac{1}{(\Delta t)^2} \left( \int_{t-\Delta t}^{t+\Delta t} f(t) \right) - 2 \int_{t-\Delta t}^{t} f(t) + \int_{t}^{t+\Delta t} f(t) 
\]

By the use of above finite difference equations the equations of motion (eqs. (2)) are transformed into the following equations at the point \((\tilde{z}, \tilde{\theta})\):

\[ 
\begin{aligned}
\mathbf{U}_j (t + \Delta t) = & - \mathbf{U}_j (t - \Delta t) + 2 \mathbf{U}_j (t) + \frac{\Delta t}{\alpha J^2} \left( A_1 N_j (t) + A_2 \frac{\partial N_j (t)}{\partial \tilde{z}} + A_3 \frac{\partial N_j (t)}{\partial \tilde{\theta}} + A_4 M_j (t) + A_5 \frac{\partial M_j (t)}{\partial \tilde{z}} + A_6 \frac{\partial M_j (t)}{\partial \tilde{\theta}} + A_7 \frac{\partial M_j (t)}{\partial \tilde{A}} + A_8 \frac{\partial M_j (t)}{\partial \tilde{\theta}} + A_9 \frac{\partial M_j (t)}{\partial \tilde{A}} \right) \\
& + A_0 \mathbf{U}_j (t) + A_9 \frac{\partial \mathbf{U}_j (t)}{\partial \tilde{\theta}} + A_{10} \mathbf{P}_j (t) 
\end{aligned}
\]
and so on,
where the coefficients $\mathcal{A}_i(j, i)$ consist of the constants determined from the shell form
and the rotations at time $T = t$. From eqs. (15), if each displacement at time $T = t - \Delta t$ and
$T = t$, and each internal force at time $T = t$ are known at every spatial mesh point $(i, j)$ at
each mesh point the displacements at time $T = t + \Delta t$ can be calculated.

When each displacement at $T = t + \Delta t$ is obtained, the increments of displacements are
evaluated from the next equations,
\[
\delta u_j(i, j, t) = u_j(i, j, t + \Delta t) - u_j(i, j, t)
\]
and so on.

The incremental strains $\delta \varepsilon$ at any point $(i, j)$ are as follows from eqs. (3)-(6):
\[
\delta \varepsilon_j(i, j, t) = \frac{1}{\alpha} \left[ \varepsilon \frac{\partial \delta u_j(i, j, t)}{\partial \xi} + \omega_y \delta w(i, j, t) + \alpha \left( \delta \varepsilon_y(i, j, t) \right) \right] + \left( \frac{\alpha - 1}{\alpha - 1} - \frac{1}{2} \right) \frac{\partial \delta \varepsilon_y(i, j, t)}{\partial \eta}
\]
and so on.

From the incremental strains $\delta \varepsilon$ and the stresses at time $T = t$, incremental stresses
at every point can be calculated by the use of eqs. (11) and adding these to the stresses
at time $T = t$, the stresses at time $T = t + \Delta t$ can be obtained. Each internal force at
time $T = t + \Delta t$ is calculated from numerical integration of eqs. (13) by Simpson's 1/3 rule.
Substituting these internal forces into eqs. (16), the displacements at the next time can
be obtained.

Now the initial displacements (incremental displacements) at time $T = \Delta t$ must be given
for this calculation. In order to reduce the error in the initial approximation, we divide
the first time increment into several equal parts: $\Delta t_o = \Delta t / \eta$ (e.g. $\eta = 10$). The loading
which are considered in this analysis may be either impulsive or of finite duration. A
general case of impulsive loading is denoted by impulse $I(x, \theta)$ per unit area. Since all
displacements are assumed zero to begin, initial incremental displacements may then be given by
\[
\begin{align*}
\delta u_j &= \frac{I_j(x, \theta)}{p_0 \alpha} \Delta t_o = \dot{u}_j \Delta t_o \\
\delta u_\theta &= \frac{I_\theta(x, \theta)}{p_0 \alpha} \Delta t_o = \dot{u}_\theta \Delta t_o \\
\delta w &= \frac{I_w(x, \theta)}{p_0 \alpha} \Delta t_o = \dot{w} \Delta t_o
\end{align*}
\]
If the shell is excited by applied surface loads, initial conditions become as follows:
\[
\begin{align*}
\delta u_j &= \frac{R_j \Delta t_o^2}{2 p_0 \alpha} \\
\delta u_\theta &= \frac{R_\theta \Delta t_o^2}{2 p_0 \alpha} \\
\delta w &= \frac{R_w \Delta t_o^2}{2 p_0 \alpha}
\end{align*}
\]
4. Numerical example

As a numerical example a cylindrical shell subjected to external impulsive loads with uniform distribution along the length and cosine distribution around the half circumference as shown in Fig. 4 is treated. The ends of the cylinder are assumed to be fixed. The material of the cylinder is 6061-T6 aluminum.

The Geometrical parameters of a quarter of this shell are as follows:

\[
\begin{align*}
\rho &= R, \quad \Delta z = L/2R(M-1), \quad \Delta \theta = \pi/(N-1), \quad \beta = 1 \\
\varphi &= 0, \quad \gamma = 0, \quad \omega_\theta = 1, \quad \omega_z = \omega_z' = 0
\end{align*}
\]

The boundary conditions are, respectively,

for the fixed edge AC: \[ U_\beta = U_\theta = W = \frac{\partial W}{\partial \beta} = 0 \]

for the symmetrical surface MN: \[ \frac{\partial U_\beta}{\partial \beta} = \frac{\partial W}{\partial \beta} = \frac{\partial^2 W}{\partial \beta^2 \partial \theta} = 0 \]

for the meridian AM, CN: \[ \frac{\partial U_\beta}{\partial \theta} = \frac{\partial W}{\partial \theta} = \frac{\partial^2 W}{\partial \beta \partial \theta} = 0 \]

The initial conditions may be given by

\[
\begin{align*}
t &= 0 : \quad U_\beta = U_\theta = W = 0 \\
\dot{U}_\beta = \dot{U}_\theta = 0, \quad \dot{W} = \frac{I_\beta(\xi, \theta)}{J_0 R} = -\frac{I_\beta \cos \theta}{J_0 R} = -W_0 \cos \theta
\end{align*}
\]

The material constants given by Fyfe [14]:

\[
E = 7270 \text{ kg/mm}^2, \quad \nu = 0.33, \quad J_0 = 2.76 \times 10^{10} \text{ kg·} \text{mm}^4, \quad \tau_0 = 17.3 \text{ kg/mm}^2
\]

(yield stress in tension \( \sigma_0 = 30.3 \text{ kg/mm}^2 \)), \( \alpha = 2 \), \( \gamma = 5000/\text{s} \), \( \tau^* = \tau_0 \)

are employed in calculations.

Calculations for three kinds of impulsive loads are carried out. The mesh point number and the division number along the direction of the thickness are \( M = 21 \), \( N = 31 \) and \( L = 11 \), respectively. The following time intervals are employed for three loads:

\[
\begin{align*}
\Delta t &= 5.0 \times 10^{-3} \text{ ms } (W_0 = 10 \text{ m/s}) \\
\Delta t &= 1.0 \times 10^{-3} \text{ ms } (W_0 = 50 \text{ m/s}) \\
\Delta t &= 5.0 \times 10^{-4} \text{ ms } (W_0 = 100 \text{ m/s})
\end{align*}
\]

These values are determined in consideration of convergence of the solutions, capacity of the computer and computing time.

Some of the essential features of the solutions for the case of \( W_0 = 50 \text{ m/s} \) are shown in Figs. 5-11.

5. Conclusions

In this paper we have described the elasto-visco-elastic dynamic response of shells of revolution to blast loads. The equations of motion and the strain-displacement relations have been derived from the Sanders nonlinear theory for thin shells, and the constitutive equations by Fyfe have been employed. The criterion for yielding used in this analysis is von Mises yield theory. The numerical method selected for this problem is a method using finite difference in both space and time.

As the numerical example a cylindrical shell subjected to the unsymmetrical external impulsive loads has been treated and the aspects of displacements and internal forces with time have been discussed.
References


Fig. 3 Mesh points

Fig. 4 Cylindrical shell under impulsive load

\[ l = 800 \text{ mm} \]
\[ R = 400 \text{ mm} \]
\[ h = 3 \text{ mm} \]
Fig. 5 Deformations in cross sections with time \( \dot{W}=50 \text{ m/s} \)

Fig. 6 Meridional distributions of \( W \) and \( U_\theta \) with time
Fig. 7 Meridional distributions of internal forces with time
Fig. 8 Circumferential distributions of internal forces with time
Fig. 9 Variations of $W$ and $U_9$ with time

Fig. 10 Variations of internal forces with time
Fig. 11 Progression of yield with time