A GENERAL SOLUTION OF THE PLANE PROBLEM IN THERMOELASTICITY IN POLAR COORDINATES

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SUMMARY

A general solution, in polar coordinates, of the plane problem in thermoelasticity is obtained in terms of a stress and displacement function. The solution is valid for arbitrary temperature distribution $T(r, \theta)$. The characteristic feature of the paper is the forthright determination of the displacement components brought about by the introduction of a displacement function.

Solution of the problem consists of solving the compatibility equation

$$\nabla^2\nabla^2\phi = -\Gamma\nabla^2\alpha T \tag{1}$$

with

$$\Gamma = E \text{ for plane stress,} \qquad \Gamma = \frac{E}{(1-\mu)} \text{ for plane strain}.$$

The displacement components u and v are obtained by the use of a displacement function ψ. We write

$$\nabla^2\phi = \frac{\partial^2 r\psi}{\partial r\partial\theta} \tag{2}$$

an equation which defines ψ except for functions of integrations $g(r)$ and $f(\theta)$.

It is shown that the function ψ found from Eq. (2) must be so adjusted, by means of the functions of integration of $g(r)$ and $f(\theta)$ that it satisfies the equation

$$\nabla^2\psi = \frac{\Gamma}{r^3}\left\{-r\int \alpha T\,\mathrm{d}\theta + \iint \alpha T\,\mathrm{d}r\mathrm{d}\theta + r^2\int \frac{\partial\alpha T}{\partial r}\,\mathrm{d}\theta + \int\frac{\partial\alpha T}{\partial\theta}\,\mathrm{d}r\right\} \tag{3}$$

Equation (3) is a general equation for the determination of the displacement components in plane thermoëlasticity. It is arrived at by substituting Eq. (2) in the expressions for the direct strains. Upon integration, the displacement components u and v are obtained in terms of the stress and displacement functions. These expressions must be such as to satisfy the expression for the shear strain. With the temperature function set to zero, Eq. (3) becomes the equation for the plane problem in elasticity.

The usefulness of the newly introduced displacement function is the simplification it brings to the process for calculating the displacement components. It is a powerful tool for obtaining the general expression for the displacement components.

1. Introduction

A general solution, in polar coordinates, of the plane problem in thermoelasticity is obtained in terms of a stress and displacement function. The solution is valid for arbitrary temperature distribution T (r,θ).

A characteristic of the paper is the forthright determination of the displacement components brought about by the introduction of a displacement function.

Application of the general solution is illustrated by the treatment of a circular plate subject to two unequal but constant temperatures T_1 and T_2 affecting two complementary segments of the plate shown in Fig. 1.

2. Determination of the General Stress Function

Consider the general case of an arbitrary temperature distribution

$$T = T\,(r,\ \theta) \tag{1}$$

Solution of this problem requires a general solution of the compatibility equations

$$\nabla^4\,\phi = -\,\Gamma\,\nabla^2\,(\alpha T) \tag{2}$$

where

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{2}{\partial r} + \frac{2}{r^2}\frac{\partial^2}{\partial \theta 2}\right)$$

and

$$\Gamma = \begin{cases} E & \text{for plane stress} \\ \dfrac{E}{1-\mu} & \text{for plane strain} \end{cases}$$

The temperature T (r, θ) may be represented by a Fourier series

$$\alpha T = \sum_{n=0}\left[T_n\cos n\theta + \overline{T}_n\sin n\theta\right] \tag{3}$$

where

$$T_n(r) = \sum_{j=-\infty} a_{nj}\,r^j$$

$$\overline{T}_n(r) = \sum_{j=-\infty} \overline{a}_{nj}\,r^j$$

Then the complete solution of eq. (2) is

$$\phi = A_o \log r + B_o r^2 \log r + C_o r^2 + D_o - \Gamma F_o(r)$$

$$+ \left[A_1 r + B_1 r^3 + C_1 r^{-1} + D_1 r \log r - \Gamma F_1(r)\right] \cos \theta$$

$$+ \left[\overline{A}_1 r + \overline{B}_1 r^3 + \overline{C}_1 r^{-1} + \overline{D}_1 r \log r - \Gamma \overline{F}_1(r)\right] \sin \theta$$

$$+ \sum_{n=2} \left[A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_r r^{-n+2} - \Gamma F_n(r)\right] \cos n\theta$$

$$+ \sum_{n=2} \left[\overline{A}_n r^n + \overline{B}_n r^{-n} + \overline{C}_n r^{n+2} + \overline{D}_n r^{-n+2} - \Gamma \overline{F}_n(r)\right] \sin n\theta \qquad (4)$$

where

$$F_n(r) = \sum_{j=-\infty} a_{nj} f_{nj} \qquad n = 0, 1, 2 \ldots \qquad (4\text{-a})$$

$$\overline{F}_n(r) = \sum_{j=-\infty} a_{nj} f_{nj} \qquad n = 1, 2, 3 \qquad (4\text{-b})$$

with

$$f_{nj} = \frac{r^{j+2}}{(j+2)^2 - n^2} \qquad \text{for } n \geq 0 \quad ; j \neq \pm\, n-2 \qquad (4\text{-c})$$

$$f_{nj} = \frac{r^{j+2} \log r}{2\,(j+2)} \qquad \text{for } n > 0 \quad ; j = \pm\, n-2 \qquad (4\text{-d})$$

$$f_{oj} = \frac{(\log r)^2}{2} \qquad \text{for } n = 0 \quad j = -2 \qquad (4\text{-e})$$

3. General Expressions for the Stresses

The stresses corresponding to the general stress function may be represented by a complete Fourier series. We write

$$\sigma_r = R_o + R_1 \cos\theta + \bar{R}_1 \sin\theta + \sum_{n=2} \left[R_n \cos n\theta + \bar{R}_n \sin n\theta \right]$$

$$\sigma_\theta = S_o + S_1 \cos\theta + \bar{S}_1 \sin\theta + \sum_{n=2} \left[S_n \cos n\theta + \bar{S}_n \sin n\theta \right] \qquad (5)$$

$$\tau_{r\theta} = P_1 \sin\theta + \bar{P}_1 \cos\theta + \sum_{n=2} \left[P_n \sin n\theta + \bar{P}_n \cos n\theta \right]$$

the coefficients in eq. (5) being obtained by using eq. (6) in conjunction with eq. (4)

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \qquad (6)$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}$$

Thus

$$R_o = \frac{A_o}{r^2} + B_o\,(1 + 2\log r) + 2\,C_o - \frac{\Gamma}{r} F_o'(r)$$

$$R_1 = 2\,B_1\,r - 2\,\frac{C_1}{r^3} + \frac{D_1}{r} - \frac{\Gamma}{r^2}\left(r\,F_1'(r) - F_1(r)\right)$$

$$\bar{R}_1 = 2\,\bar{B}_1\,r - 2\,\frac{\bar{C}_1}{r^3} + \frac{\bar{D}_1}{r} - \frac{\Gamma}{r^2}\left(r\,\bar{F}_1'(r) - \bar{F}_1(r)\right)$$

$$R_n = -\,n\,(n-1)\,A_n\,r^{n-2} - n\,(n+1)\,B_n\,r^{-n-2} + (-n^2 + n + 2)\,C_n\,r^n \qquad (7)$$

$$+ (-n^2 - n + 2)\,Dn\,r^{-n} + \frac{\Gamma}{r}\left(n^2\,F_n(r) - F_n'(r)\right)$$

$$\bar{R}_n = -\,n\,(n-1)\,\bar{A}_n\,r^{n-2} - n\,(n+1)\,\bar{B}_u\,r^{-n-2} + (-n^2 + n + 2)\,\bar{C}_n\,r^n$$

$$+ (-n^2 - n + 2)\,\bar{D}_n\,r^{-n} + \frac{\Gamma}{r^2}\left(n^2\,\bar{F}_n(r) - r\bar{F}_n'(r)\right)$$

$n = 2, 3, 4, \ldots,$

$$S_o = \frac{A}{r^2} + B_o\ (3 + 2 \log r) + 2\ C_o - \Gamma\ F_o''(r)$$

$$S_1 = 6\ B_1 r + 2 \frac{C_1}{r^3} + \frac{D_1}{r} - \Gamma\ F_1''(r)$$

$$\overline{S}_1 = 6\ \overline{B}_1 r + 2 \frac{\overline{C}_1}{r^3} + \frac{\overline{D}_1}{r} - \Gamma\ \overline{F}_1''(r) \tag{8}$$

$$S_n = n\ (n-1)\ A_n\ r^{n-2} + n\ (n+1)\ B_n\ r^{-n-2} + (n+2)\ (n+1)\ C_n\ r^n$$
$$+ (n-2)\ (n-1)\ D_n\ r^{-n} - \Gamma\ F_n''(r)$$

$$\overline{S}_n = n\ (n-1)\ \overline{A}_n\ r^{n-2} + (n+1)\ \overline{B}_n\ r^{-n-2} + (n+2)\ (n+1)\ \overline{C}_n\ r^n$$
$$+ (n-2)\ (n-1)\ \overline{D}_n\ r^{-n} - \Gamma\ \overline{F}_n''(r)$$

$n = 2, 3, 4, \ldots,$

$$P_1 = 2\ B_1\ r - 2 \frac{C_1}{r^3} + \frac{D_1}{r} - \frac{\Gamma}{r^2} \left(rF_1'\ (r) - F_1(r) \right)$$

$$\overline{P}_1 = 2\ \overline{B}_1\ r + 2 \frac{\overline{C}_1}{r^3} - \frac{\overline{D}_1}{r} - \frac{\Gamma}{r^2} \left(\overline{F}_1(r) - r\overline{F}_1'(r) \right)$$

$$P_n = n\ (n-1)\ A_n\ r^{n-2} - n\ (n+1)\ B_n\ r^{-n-2} + n\ (n+1)\ C_n\ r^n$$
$$+ n\ (1-n)\ D_n\ r^{-n} - \frac{P_n}{r^2} \left(rF_n'\ (r) - F_n(r) \right) \tag{9}$$

$$\overline{P}_n = n\ (1-n)\ \overline{A}_n\ r^{n-2} + n\ (n+1)\ \overline{B}_n\ r^{-n-2} - n\ (n+1)\ \overline{C}_n\ r^n$$
$$+ n\ (n-1)\ \overline{D}_n\ r^{-n} - \frac{P_n}{r^2} \left(\overline{F}_n(r) - r\overline{F}_n'(r) \right)$$

$n = 2, 3, 4, \ldots,$

4. Determination of the General Displacement Function

From the known results of the theory of linear thermoelasticity we have, for the case of plane stress, the equations

$$2G\ \frac{\partial u}{\partial r} = - \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{(1+\nu)} \nabla^2 \phi + \frac{E\alpha T}{(1+\nu)} \tag{10}$$

$$2G \left(\frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) = - \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{(1+\nu)} \nabla^2 \phi + \frac{E\alpha T}{(1+\nu)} \tag{11}$$

Let us write

$$\nabla^2\phi = \frac{\partial^2(r\psi)}{\partial r\ \partial\theta} \tag{12}$$

An equation which defines ψ except for an arbitrary function for r and one for θ. We view ψ as a displacement function. Substituting eq. (12) in eq. (10) and integrating with respect to r we obtain

$$2Gu = -\frac{\partial\phi}{\partial r} + \frac{1}{(1+\nu)} r \frac{\partial\psi}{\partial\theta} + \frac{E}{1+\nu}\int\alpha T dr \tag{13}$$

the arbitrary function of integration being included in ψ.

Using eqs. (13) and (12) in eq. (11) and integrating with respect to θ we obtain

$$2Gv = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \frac{1}{1+\nu}\ r^2\frac{\partial\psi}{1+\nu} + \frac{E}{1+\nu}\left[r\int\alpha T d\theta - \iint\alpha T dr d\theta\right] \tag{14}$$

the arbitrary function of integration being included in ψ.

The values of u and v given by eqs (13) and (14) must be such as to satisfy the thermoelastic stress-strain equation in shear. Expressed in terms of a stress function ϕ and the displacement components, the equation assumes the form

$$\left(\frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) = \frac{1}{G}\left(\frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right) \tag{15}$$

Applying eqs (13) and (14) to eq (15) we obtain

$$\nabla^2\psi = \frac{E}{r^3}\left\{-\iint\alpha T dr d\theta - \int\frac{\partial(\alpha T)}{\partial\theta} dr - r^2\int\frac{\partial(\alpha T)}{\partial r} d\theta + r\int\alpha T d\theta\right\} \tag{16}$$

In treating the case for plane strain the analytic process adopted for plane stress is followed. The equations for the displacements and the displacement function ψ are found to be of the same functional form as those for plane stress. We thus have for the plane problem in linear thermoelasticity the equations

$$2Gu = -\frac{\partial\phi}{\partial r} - \zeta_1 r\frac{\partial\psi}{\partial\theta} + \Gamma\zeta_1\int\alpha T\ dr \tag{17}$$

$$2Gv = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \zeta_1 r^2\frac{\partial\psi}{\partial r} + \Gamma\zeta_1\left[r\int\alpha T d\theta - \iint\alpha T dr d\theta\right] \tag{18}$$

$$\nabla^2\psi = \frac{\Gamma}{r^3}\left\{r\int\alpha T d\theta - \iint\alpha T dr d\theta - r^2\int\frac{\partial\alpha T}{\partial r} d\theta - \int\frac{\partial\alpha T}{\partial\theta} dr\right\} \tag{19}$$

Where

$\zeta_1 = \frac{1}{1+\nu}$; $\Gamma = E$ for plane stress

$\zeta_1 = (1-\nu)$; $\Gamma = \frac{E}{1-\nu}$ for plane strain.

The function ψ found from eq. (12) must be so adjusted by means of the functions of integration that it satisfies eq. (19). With ψ having been found in this manner the displacement components u and v are determined by using eqs. (17) and (18).

Equations (2) and (19) are the general equations for the solution of the plane problem in linear thermoelasticity by the use of a stress and displacement function.

5. General Expressions for the Displacements.

With the stresses known, formation of the expressions for the displacements can be proceeded with. We have

$$\nabla^2\phi = \sigma_r + \sigma_\theta \tag{20}$$

Applying eq. (20) to eq. (12) and integrating we obtain

$$\psi = \eta\ (r,\theta) + g(r) + \frac{1}{r} f(\theta) \tag{21}$$

where $\eta\ (r,\theta)$ in a known function, $g(r)$ and $f(\theta)$ are functions of integration to be determined. The function ψ given by eq. (21) must be so adjusted by the functions of integration that it satisfies eq. (19). Thus

$$\left.\begin{aligned} & f''(\theta) + f(\theta) - 4D_1 \sin\theta + 4\,\overline{D}_1 \cos\theta = o \\ & g''(r) + \frac{1}{r}\, g'(r) = o \end{aligned}\right\} \tag{24}$$

solutions of which yield

$$f(\theta) = H_1 \cos\theta + H_2 \sin\theta + 2\, D_1\theta \sin\theta - 2\,\overline{D}_1\theta \cos\theta \tag{25}$$

$$g(r) = K_1 \log r + K_2 \tag{26}$$

where H_1, H_2, K_1 and K_2 are constants of integration.

The first two terms in eq. (25) depict rigid body movement. They do not contribute to distortion and we may therefore write

$$H_1 = H_2 = o \tag{27}$$

The last two terms in eq. (25) are not single valued and must therefore be removed, thus

$$D_1 = \overline{D}_1 = o \tag{28}$$

The function g(r) given by eq. (26) is irrelevant to the general deformation, eq. (18). This is plainly the case since being a function of r only it depicts axisymmetric deformation where the circumferential displacement V is zero. Consequently

$$K_1 = K_2 = o \tag{29}$$

Thus, by virtue of eqs. (27), (28) and (29) the expression for the displacement function becomes

$$\psi = \nu\ (r,\theta) \tag{30}$$

Where

$$\eta(r,\theta) = Q_o\theta + Q_1\sin\theta - \overline{Q}_1\cos\theta + \sum_{n=2} \frac{1}{n}\left(Q_n \sin n\theta - \overline{Q}_n \cos n\theta\right) \tag{31}$$

in which

$$Q_o = 4 B_o \log r + 4 C_o - \frac{\Gamma}{r}\int T_o (r)\, dr$$

$$Q_1 = 4 B_1 r + 2 D_1 r^{-1} \log r - \frac{\Gamma}{r}\int T_1 (r)\, dr$$

$$\overline{Q}_1 = 4 \overline{B}_1 r + 2 \overline{D}_1 r^{-1} \log r - \frac{\Gamma}{r}\int \overline{T}_1 (r)\, dr \qquad (32)$$

$$Q_n = 4 C_n r^n + 4 D_n r^{-n} - \frac{\Gamma}{r}\int T_n (r)\, dr$$

$$\overline{Q}_n = 4 \overline{C}_n r^n + 4 \overline{D}_n r^{-n} - \frac{\Gamma}{r}\int \overline{T}_n (r)\, dr$$

With the stress and displacement functions known, the general expressions for the displacement can be obtained using eqs. (4) and (30), the general expressions for the displacements, eq. (17) and (18) assume the form

$$2Gu = U_o + U_1 \cos\theta + \overline{U}_1 \sin\theta + \sum_{n=2} \left[U_n \cos n\theta + \overline{U}_n \sin n\theta \right] \qquad (33)$$

$$2Gv = V_1 \sin\theta + \overline{V}_1 \cos\theta + \sum_{n=2} \left[V_n \sin n\theta + \overline{V}_n \cos n\theta \right] \qquad (34)$$

Where

$$U_o = 2 (2\zeta_1 - 1) C_o r + \frac{A_o}{r} + B_o \left[2 (2\zeta_1 - 1) r \log r - r \right] + \Gamma F_1'(r)$$

$$U_1 = A_1 + (4\zeta_1 - 3) r^2 B_1 + C_1 r^{-2} + D_1 \left[(2\zeta_1 - 1) \log r - 1 \right] + \Gamma F_1'(r)$$

$$\overline{U}_1 = - \overline{A}_1 + (4\zeta_1 - 3) r^2 \overline{B}_1 + \overline{C}_1 r^{-2} + \overline{D}_1 \left[(2\zeta_1^{-1}) \log r - 1 \right] + \Gamma \overline{F}_1'(r)$$

$$U_n = -\mu A_n r^{n-1} + n B_n r^{-n-1} + (4\zeta_1 - n - 2) C_n r^{n+1} + (4\zeta_1 + n - 2) D_n r^{-n+1} + \Gamma F_n(r)$$

$$V_1 = A_1 + (4\zeta_1 + 1) B_1 r^2 + C_1 r^{-2} - \Gamma r^{-1} F_1(r)$$

$$\overline{V}_1 = -\overline{A}_1 - (4\zeta_1 + 1) \overline{B}_1 r^2 - \overline{C}_1 r^{-2} + \Gamma r^{-1} \overline{F}_1(r)$$

$$V_n = n A_n r^{n-1} - n B_n r^{-n-1} + (4\zeta_1 + n) C r^{n+1} + (n - 4\zeta_1) D_n r^{-n+1} - n \Gamma r^{-1} F_n(r)$$

$$\overline{V}_n = - n \overline{A}_n r^{n-1} + n \overline{B}_n r^{-n-1} - (4\zeta_1 + n) \overline{C}_n r^{n+1} - (n - 4\zeta_1) \overline{D}_n r^{-n+1} + n \Gamma r^{-1} \overline{F}_n(r) \qquad n = 2,3 \ldots,$$

This completes the general solution in polar coordinates of the plane problem of thermoelasticity.

6. The Problem of a Circular Plate with a Hot Sector.

By way of application of the general solution consider the case of a thin solid circular plate of radius 'a' subject to constant temperatures T_1 and T_2 over the regions shown in fig. 1.

We have

$$T = T_1 \qquad \text{for} \qquad -\frac{\beta}{2} \leq \theta \leq \frac{\beta}{2}$$

$$T = T_2 \qquad \text{for} \qquad \frac{\beta}{2} < \theta < 2\pi - \frac{\beta}{2}$$

Expanding (αT) in Fourier series with due regard for symmetry with respect to $\theta = o$, eq. (3) becomes

$$\alpha T = \alpha \left[(T_1 - T_2) \frac{\beta}{\pi} + T_2 \right] + \sum_{n=1} \alpha \frac{(T_1 - T_2)}{n\pi} \sin n\beta \cos n\theta \tag{35}$$

The stress function is obtained from eq. (4) retaining only even functions of θ since the temperature is symmetric with respect to $\theta = o$. Next we determine the 'F' functions.

Observing that since the temperature is constant, T_n is not a function of r and consequently $j = o$ and eqs. (4-c) and (4-e) become

$$f_{no} = \frac{r^2}{2^2 - n^2} \qquad \text{for} \quad n = 1, 3, 4, 5 \ldots$$

and (36)

$$f_{no} = \frac{r^2 \log r}{4} \qquad \text{for} \quad n = 2$$

Comparing eqs. (35) and (31) and noting that $j = o$ we have

$$a_{oo} = \alpha \left[(T_1 - T_2) \frac{\beta}{\pi} + T_2 \right] \tag{37}$$

$$a_{no} = \frac{\alpha (T_1 - T_2)}{n\pi} \sin n\beta \qquad n = 1, 2, 3 \ldots, \tag{38}$$

substitution of eqs. (36), (37) and (38) in eq. (4-a) leads to

$$f_o(r) = \frac{\alpha}{4} \left[(T_1 - T_2) \frac{\beta}{\pi} + T_2 \right] r^2$$

$$f_1(r) = \frac{\alpha}{3} \left[\frac{(T_1 - T_2)}{\pi} \sin \beta \right] r^2$$

$$f_2(r) = \frac{\alpha}{4} \left[\frac{(T_1 - T_2)}{2\pi} \sin 2\beta \right] r^2 \log r \tag{39}$$

$$f_n(r) = \frac{\alpha}{2^2 - n^2} \left[\frac{(T_1 - T_2)}{n\pi} \sin n\beta \right] r^2 \qquad n = 3, 4 \ldots,$$

If the edge of the plate is free, the non-vanishing constants of integration appearing in the expression for the stress function, eq. (4) are as follows

$$\begin{aligned}
C_o &= \Gamma \frac{\alpha}{4}\left[(T_1 - T_2)\frac{\beta}{\pi} + T_2\right] \\
B_1 &= \Gamma \frac{\alpha}{6a}\left[(T_1 - T_2)\frac{\beta}{\pi} \sin\beta\right] \\
A_2 &= \Gamma \frac{\alpha}{8}\left[\frac{(T_1 - T_2)}{2\pi} (2\log a - 1)\sin 2\beta\right] \\
C_2 &= \Gamma \frac{\alpha}{4a^2}\left[\frac{(T_1 - T_2)}{2\pi} \sin 2\beta\right] \\
A_n &= \Gamma\alpha \frac{n}{2(2^2 - n^2)a^{n-2}}\left[\frac{T_1 - T_2}{n\pi} \sin n\beta\right] \\
C_n &= \Gamma\alpha \frac{1}{2(2+\nu)a^n}\left[\frac{(T_1 - T_2)}{n\pi} \sin n\beta\right] \qquad n = 3, 4, 5 \ldots ,
\end{aligned} \tag{40}$$

Using eqs. (5) and (40) the expressions for the stresses are

$$\begin{aligned}
\sigma_r = \Gamma\alpha \Bigg\{ & \frac{1}{3}\left[\frac{(T_1 - T_2)}{\pi} \sin\beta \left(\frac{r}{a} - 1\right)\right] \cos\theta + \frac{1}{2}\left[\frac{(T_1 - T_2)}{2\pi} \sin 2\beta \log\frac{r}{a}\right] \cos 2\theta \\
& + \sum_{n=3} \frac{(T_1 - T_2)}{n\pi} \frac{1}{2(n^2 - 2^2)} \left[n^2 (n - 1)\left(\frac{r}{a}\right)^{n-2} - (n - 2)^2 (n + 1)\left(\frac{r}{a}\right)^n \right. \\
& \left. - 2n^2 + 4 \right] \sin n\beta \cos n\theta \Bigg\}
\end{aligned}$$

$$\begin{aligned}
\sigma_\theta = \Gamma\alpha \Bigg\{ & \frac{(T_1 - T_2)}{\pi} \left(\frac{r}{a} - \frac{2}{3}\right)\sin\beta\cos\theta + \frac{(T_1 - T_2)}{2\pi}\left[3\left(\frac{r}{a}\right)^2 - \frac{1}{2}\log\frac{r}{a} - \frac{3}{4}\right] \\
& \times \sin 2\beta \cos 2\theta + \sum_{n=3} \frac{(T_1 - T_2)}{n\pi} \frac{1}{2(2^2 - n^2)} \\
& \left[n^2 (n - 1)\left(\frac{r}{a}\right)^{n-1} + (n + 1)(2^2 - n^2)\left(\frac{r}{a}\right)^n - 4\right] \sin n\beta \cos n\theta \Bigg\}.
\end{aligned} \tag{41}$$

$$\tau_{r\theta} = \Gamma\alpha \left\{ \frac{1}{3} \frac{(T_1 - T_2)}{\pi} \frac{r}{a} - 1 \quad \sin\beta \cos\theta + \frac{1}{2} \left[\frac{(T_1 - T_2)}{2\pi} \sin 2\beta \right] \right.$$

$$\left[3 \frac{r}{a}^2 - \log\frac{r}{a} - 3 \right] \cos 2\theta + \sum_{n=3} \frac{(T_1 - T_2)}{n\pi} \frac{1}{2(2^2 - n^2)}$$

$$\left. \left[n(n-1) \frac{r}{a}^{n-2} + (n+1)(2-n) \frac{r}{a}^{n-2} \right] \sin n\beta \cos n\theta \right\}$$

Substituting eqs. (39) and in the expressions for the displacements are obtained

$$2Gu = \Gamma\alpha \left\{ \left[\frac{(T_1 - T_2)}{\pi} + T_2 \right] \zeta_1 r + \left[\frac{(T_1 - T_2)}{\pi} \right] \left[\frac{2}{3}\zeta_1 - \frac{1}{2} \frac{r}{c}^2 + \frac{2}{3} r \right] \sin\beta\cos\theta , \right.$$

$$+ \left[\frac{(T_1 - T_2)}{2\pi} \right] \left[-(\zeta_1 - 1)\frac{r^3}{a^2} + \frac{r}{4}\log\frac{r}{a} + \frac{r}{2} \right] \sin 2\beta \cos 2\theta$$

$$+ \sum_{n=3} \left[\frac{(T_1 - T_2)}{n\pi} \right] \left[\frac{2r}{2^2 - n^2} - \frac{n^2 r^{n-1}}{2(2^2 - n^2)a^{n-2}} + \frac{(4\zeta_1 - n - 2)}{2(2+n)} \cdot \frac{r^{n+1}}{a^n} \right]$$

$$\left. \sin n\beta \cos n\theta \right\} \qquad (42)$$

$$2Gv = \Gamma\alpha \left\{ \frac{(T_1 - T_2)}{\pi} \left[\frac{2\zeta_1}{3} + \frac{1}{6} \frac{r^3}{a} - \frac{r}{3} \right] \sin\beta \sin\theta + \frac{(T_1 - T_2)}{2\pi} \right.$$

$$\left[-\frac{r}{4}\log\frac{r}{n} - \frac{r}{8} + \zeta_1 + \frac{1}{2} \frac{r^3}{a^2} \right] \sin 2\beta \sin 2\theta$$

$$\left. + \sum_{n=3} \frac{(T_1 - T_2)}{n\pi} \left[\frac{n^2}{2(2^2 - n^2)} \cdot \frac{r^{n-1}}{a^{n-2}} + \frac{(4\zeta_1 + n)}{2(2+n)} \cdot \frac{r^{n+1}}{a^n} - \frac{r}{(2^2 - n^2)} \right] \sin n\theta \right\}$$

This completes the solution of the problem.

7. Conclusion

The solution obtained is a general solution in polar coordinates of the plane problem in thermoelasticity and is directly applicable to obtaining solutions of a wide range of problems of practical importance.

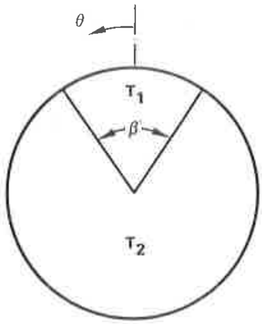

Figure 1. PLATE WITH FREE EDGE SUBJECT TO CONSTANT TEMPERATURE T_1 AND T_2