A PRACTICAL METHOD FOR COMPUTING THE DEFLECTION OF BEAMS AND TUBES CAUSED BY CREEP

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SUMMARY

In many cases, the effect of creep contributes substantially to the deflection of structural elements that comprise nuclear reactors. The purpose of this paper is to present a method of analysis for computing these deflections due to bending. Probably the most useful equation for evaluating the strains due to creep bending is $\varepsilon = K\omega^n$. Here it is assumed that elastic strains and transient creep are negligible in comparison to steady-state creep. In a more exact analysis, the deflection function is determined by two successive integrations using this creep equation and the moment-curvature relation. Even when the moment can be expressed as a very simple expression of position, the exact analysis can still be involved and many indeterminate cases cannot be obtained, since for metals the exponent $n$ is much greater than one, and often not an integer.

The method of analysis considered in the paper is based upon the idea that the effect of creep bending is localized, like plastic yielding, resulting in the formation of "creep hinges" which are generated at points of maximum moment. This simplification allows two simplifications of the analysis; namely, only one integration of the moment-curvature relation is needed rather than two, and if necessary, approximate expressions may be used in place of exact expressions in regions of maximum stress.

A series of statically determinate and indeterminate examples are worked to illustrate the method. In particular, it is shown that the analysis is most advantageous for statically indeterminate problems.

The accuracy of the technique is shown to be dependent upon the creep exponent $n$, —the higher the exponent, the greater the accuracy; and the shape of the bending-moment diagram, —sharp peaks enhance the effect of localized bending. Due to the simplicity of application, the method offers a usefulness for determining deflections with a degree of accuracy suitable for design and analysis.
When beams and long slender tubes are subjected to a long-time high temperature, primary deformation occurs by creep. Most often, the creep strains are evaluated on the basis of Norton's law $\dot{e}_c = K\varepsilon^n$ where $\varepsilon_c$ is the creep rate, $\dot{e}$ the stress and $K$ and $n$ are constants.\[1\]

The usual method of analysis for determining the creep strains is to integrate the strain equation over the beam cross-section to obtain a moment-curvature relationship.\[2\] This equation has the form

$$\frac{d^2w}{dx^2} = kt \frac{M^n}{e}$$ \hspace{1cm} (1)

where the constant $k$ depends upon the geometry of the cross-section. In particular, for a rectangular section of width $b$ and height $h$,

$$k = \frac{K}{b^2} \left(1 + \frac{L}{b^2}\right)^n \left(\frac{2}{b^2}\right)^{2n+1}$$

The deflection is determined by a double integration of eq. (1), satisfying the appropriate boundary conditions. For most cases, this exact analysis will be difficult since $M$ is not a simple function of $x$ and the creep exponent $n$ is usually much greater than one. As an example, for most metals used for reactor applications $n=6$ or 7.\[3\]

In order to solve creep-bending problems in a more approximate sense, the method presented in this paper is based on the idea that since $n$ is usually high, most of the bending is concentrated in a short segment of the beam, located at points of maximum moment. This localized creeping gives rise to the concept of a creep-hinge,\[4\] which assumes that most deformation occurs as localized rotations at the hinge points, while the rest of the beam length remains straight. The concept is analogous to the kinematics of a plastic-hinge used in plastic analysis. The basic assumption provides a greater simplification of the analysis, since only one integration is necessary to obtain the slope. Deflections are then obtained in the basis of the slope of beam segments. The method will be explained on the basis of three examples of beams subjected to a concentrated load $P$.

A most elementary case: consider a cantilevered beam of length $L$ and supporting a load $P$ at its end. If $x$ is measured from the point of maximum moment (at the wall), the internal moment is $M=P(L-x)$, thus eq. (1) becomes

$$\frac{d^2w}{dx^2} = kt (P(L-x))^n$$

so that by integration, the slope is

$$\frac{dw}{dx} = \frac{ktP^n}{(n+1)} \left[L^{n+1} - (L-x)^{n+1} \right]$$ \hspace{1cm} (2)
The maximum slope occurs at the free end so that

$$\frac{dw}{dx} = \frac{ktn}{(n+1)} \left(\frac{L}{n+1}\right)^{n+1}$$

Since the creep hinge occurs very near to the wall, (point of maximum moment) the change in slope occurs here, so that the deflection at the free end is

$$w_L = L \frac{dw}{dx} = \frac{ktL^2 (PL)^n}{(n+1)}$$ (3)

In this particular example, the approximate method has little value since a second integration of eq. (2) is easily performed. This yields the exact solution

$$w_L = \frac{kt (PL)^n L^2}{(n+2)}$$ (4)

By comparison, with n=6, eq. (3) overestimates the deflection by 8%. For larger values of n greater accuracy is achieved. It should also be pointed out that, clearly the slope is about constant beyond the creep hinge, a small distance x from the fixed-end. For example, at x=L/2, n=6, the slope is in error by less than one percent of its value at the free end.

Consider now the case of a simply supported beam of length L and carrying a load P at its center. With x having an origin at the center, again the point of maximum moment, the internal moment is

$$M = \frac{P}{2} (L/2 - x)$$

so that eq. (1) becomes

$$\frac{d^2w}{dx^2} = kt \left(\frac{PL}{4}\right)^n \left[1 - \frac{2x}{L}\right]^n$$

The function on the right may be difficult to integrate for large or for non-integral values of n. A simplification may be made, however, based on the creep-hinge concept. Since the slope change is localized at the hinge, (center of the beam) the curvature can be approximated by a more suitable function for small values of x. By comparison of power series, we can use

$$\frac{d^2w}{dx^2} = kt \left(\frac{PL}{4}\right)^n \frac{-2xn}{L}$$

The upper limit of integration is immaterial since the curvature is negligible at points distant from the creep hinge. Thus, taking the upper limit as infinity and integrating from zero, yields

$$\frac{dw}{dx} = kt \left(\frac{PL}{4}\right)^n \int_0^\infty e^{-\frac{2xn}{L}} dx = \frac{ktL}{2n} \left(\frac{P}{2}\right)^n$$
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so that the maximum deflection (at the center) is

\[ w_C = \frac{L}{2} \left( \frac{\partial w}{\partial x} \right)_x = \frac{ktL^2}{4n} \left( \frac{P}{E} \right)^n \]

A third case used to show the method of application will be a fixed-end statically-indeterminate beam of length \( L \), loaded at the center. Since peak moments occur at the beams center, and at the wall, two expressions of internal moment must be determined. For the center \( C \), with origin of \( x_1 \) at this point, \( M = \frac{P}{2} x_1 - M_C \) where \( M_C \) is the induced moment at the center. Measuring \( x_2 \) from the wall, where the moment is \( M_A \) we have \( M = \frac{P}{2} x_2 - M_A \). From statics we also require

\[ M_A = -M_C + \frac{PL}{8} \]  \( (5) \)

With the above expressions, the turning of \( C \) can be determined.

\[ \frac{d^2 w}{dx_1^2} = kt \left( \frac{P}{2} x_1 - M_C \right)^n \]

\[ \frac{\partial w}{\partial x_1} = \frac{M_C^n}{kt} \int_0^{L/4} \left( 1 - \frac{P}{8M_C} x_1 \right)^n dx_1 \]

Here the limits on the integral extend only to \( L/4 \) since in theory interference with the hinge at \( A \) is to be avoided. Again, the integral may be approximated by an exponential, so that the solution becomes

\[ \frac{d^2 w}{dx_1^2} = \frac{M_C^n}{kt} \int_0^{\infty} e^{-\frac{Pn}{8M_C}} dx_1 = \frac{2kt M_C^{n+1}}{Pn} \]  \( (6) \)

In a similar fashion, the turning at \( A \) produces

\[ \frac{d^2 w}{dx_2^2} = \frac{2kt M_A^{n+1}}{Pn} \]

By kinematics, eq. (6) and (7) may be equated, and using eq. (5) we get

\[ M_A = M_C = \frac{PL}{8} \]

This is the same value as determined by plastic analysis. The deflection at the center becomes

\[ w_C = \frac{L}{2} \left[ \frac{2kt M_A^{n+1}}{Pn} \right] = \frac{ktL^2}{4n} \left( \frac{PL}{8} \right)^n \]

The examples given here show that the approximate method is easier to apply than that of the more exact approach. Actually, for many other types of loadings it is not possible to obtain exact solutions. This is particularly true for statically indeterminate problems. As noted, accuracy of the method depends upon a large value for the creep exponent and a bending-moment diagram having sharp peaks. In practice, however, most materials and loadings are suitable for a application of this method. The results can be expected to fall within allowable limits for predicting deflections of beams and tubes by creep.
References


