NUMERICAL STUDIES OF RANDOM PRESSURE INDUCED VIBRATIONS OF PIN ENDED RODS AND TUBES

Z.J. HOLY, J.J. THOMPSON

School of Nuclear Engineering,
The University of New South Wales, Kensington, N.S.W., Australia

SUMMARY

This work is concerned with the elastic vibration of pin ended rods and tubes due to stationary random surface pressure fluctuation. The particular problem of interest is the significance and validity of Bernoulli-Euler, i.e. simple beam theory, for the prediction of displacements and strains, or the interpretation of measured differential axial surface strain in the general field of flow induced vibration of rods and tubes. It is possible to measure the displacement by optical means out of core, but in reactors and when dealing with rod clusters this becomes very difficult. The most common approach is to use strain gauges. The interpretation is usually based upon Bernoulli-Euler theory, but here immediately the following questions arise: (1) How valid is simple beam theory for deflections where the surface pressures fluctuate very rapidly in time and space? (2) What is the relationship between the power spectral density of surface displacements and the spectral density of the differential axial strain as measured by diametrically opposed strain gauges? Simple beam theory does not allow for localised loads, such as effects of locally uncorrelated applied pressure pulses or lateral Poisson's ratio effects. Further it does not allow for the details of surface pressure fluctuation around the circumference of the surface. It utilises only the resultant. Although all these must be taken into account when investigating the problem of a solid cylindrical pin, they become of the utmost importance when dealing with thin walled tubes such as used in canned fuel elements.

Bearing in mind the limitations of the Bernoulli-Euler theory, an idealised problem was set up, which is amenable to the exact three dimensional elasticity theory solution and which takes into account all these effects. The present work is an extension of a theoretical treatment of this problem already presented, and is intended to give numerical results showing the application of the theory to both rods and tubes. These numerical examples and graphs should illustrate and clarify the problems encountered when the interpretation of experimental results is based on simple beam theory and not on exact elasticity theory.

A number of graphs are presented for a short rod and tube showing typical response spectra calculated from the exact and simple beam theory models for various input parameters. Further, a number of results are given in tabulated form, for short and long rods and tubes, to investigate the effect of axial location, for two types of power spectral density of the pressure fluctuation, maintaining the mean square value constant. Band limited white noise and truncated Gaussian distributions are considered. The results show the expected agreements and disagreements between the exact and approximate models, attributable to the neglect in the approximate model of the higher modes of elastic distortion. They also indicate that beam theory behaves poorly as far as the prediction of differential axial strain is concerned.

1. Introduction

The prediction of displacements and strains or the interpretation of measured results in the general field of flow induced vibration of rods and tubes is based as a rule on simple beam theory. This paper is concerned with the significance and validity of this practice. Since there is as yet insufficient knowledge to define completely the correct random space/time characteristics of the pressure field in turbulent axial flow, this study considers a hypothetical stationary homogeneous random surface pressure field characterised by three functions: the local pressure power spectral density, and the frequency independent axial and circumferential correlation functions. The rods and tubes are assumed to vibrate in an infinite stationary fluid in order to calculate the vertical mass effects from potential flow theory, and viscous damping is described using local radial, circumferential and axial damping coefficients.

The theoretical development leads to expressions for the power spectral density of the radial displacement, axial displacement and differential axial strain as functions of position, in terms of the local pressure density and the Fourier coefficients of the correlation functions. To utilise these expressions it was necessary to develop a substantial computer programme. An essential part of this was the calculation of all possible free vibration modes and frequencies in the range of interest, for both the exact elastic and approximate beam theory models.

2. Formulation

An "exact" elasticity theory solution for a pin ended rod in a fluid medium has been developed [1,2] for stationary homogeneous random surface pressure fluctuations with a power spectral density of the form

$$\Phi_p(\theta_1,z_1,\theta_2,z_2,\omega) = \Phi_p(\omega)\Psi_\theta(|\theta_1-\theta_2|,\omega)\chi_z(|z_1-z_2|,\omega)$$

With $\chi_n(\omega)$ and $\Psi_n(\omega)$ denoting the axial Fourier cosine series components of $\chi(\omega)$ and $\Psi(\omega)$, the general form of the power spectral density of a surface response of type k, (e.g., displacement in direction k) is:

$$\phi_k(z,\omega) = \Phi_p(\omega)\sum_n\sum_s \tilde{\Omega}_k^{n,s}(z)\Psi_s(\omega)\chi_n(\omega)$$

where

$$\tilde{\Omega}_k^{n,s} = |\tilde{\Delta}_k^{n,s}|^2 \sin^2\left(\frac{n\pi z}{L}\right) + \left|\sum_m I_{m,n}\sin\left(\frac{m\pi z}{L}\right)\tilde{\Delta}_k^{m,s}\right|^2$$

and

$$\tilde{\Delta}_k^{n,s} = \sum_j V_k^{j,n,s}(r_o)\, V_r^{j,n,s}(r_o) \,/\, \tilde{Z}_{n,s}^j$$

Here $I_{m,n} = 4m/\pi(m^2-n^2)$ for m and n of different parity, $V_k^{j,m,s}(r_o)$ denotes the surface displacement, at r_o, in direction k, associated with the normalised elastic free vibration displacement eigenfunction for the j^{th} radial mode, with axial and circumferential wave numbers m and s. The function $\tilde{Z}_{m,s}^j(\omega)$ depends on solid and fluid densities, the speed of sound in the fluid, the rod dimensions, the eigenvalues $\omega_{m,n}^j$, the surface eigenfunction values and parameters defining fluid damping. For the latter, a viscous damping coefficient K^j, $(j = r,\theta,z)$ relates viscous stress to surface velocity v_j. With $\phi_k(z,\omega)$ available the mean square response at location z is the integral

$\int_0^\infty \phi_k(z,\omega)d\omega$.

For comparison, the corresponding response spectra using simple beam theory may readily be derived from the governing equation

$$(m_r + m_f) \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial z^4} + C \frac{\partial y}{\partial t} = q(z,t)$$

where q(z,t) is the lateral load corresponding to the random surface pressure distribution of the above elasticity theory, m_f is the virtual mass of the fluid, and C the damping coefficient.

The surface responses of particular interest are the radial and circumferential displacements, and the differential axial strain

$$\Delta\varepsilon_{zz} = \varepsilon_{zz}(r_o,\theta,z) - \varepsilon_{zz}(r_o,\theta+\pi,z)$$

since these three are directly amenable to experimental determination.

3. Numerical Studies

The results quoted here were obtained on an IBM 360/50 computer using Fortran IV language. A marching process, combined with Regula Falsi with interval halving, was used to find the roots of the characteristic frequency equation, and the results were checked where possible against the tables of Armenakas et al.[3]. Simpson's Rule was applied for calculation of the mean square values. The study reported here was confined to the special case where the axial and circumferential correlation functions χ_z and ψ_θ are independent of frequency. The results refer to a solid steel rod and a steel tube, both of 20 mm outer diameter, vibrating in water, the tube having 1 mm wall thickness. The lengths for both the rod and tube studied in detail, were 200 mm and 1000 mm. Damping coefficients $K^r = K^\theta$ were obtained from the corresponding beam damping coefficient C, estimated from experimental results of Chen and Wambsganss [4] and $K^z = 0$. The values were estimated for 200 mm length, with the same values utilised for 1000 mm length. In the evaluations of response spectra, as well as mean square values for displacements and axial strain, only the first five values of the Fourier expansion coefficients of the correlation functions χ_z and ψ_θ were used, together with five roots for each set, involving a total of 125 free vibration frequencies, as it was verified that additional terms were unnecessary.

Only the effect of the variation of the axial correlation length for a particular fixed circumferential function was investigated, because of the amount of computation involved. The correlation functions are: $\chi_z = \exp(-\alpha z/L)$; and $\psi_\theta = \cosh(\pi-\theta)/\cosh \pi$. A detailed study was carried out for parameter $\alpha = 1$ and $\alpha = 10$. $\alpha = 1$ gives a "flat" type distribution, very well correlated, while $\alpha = 10$ results in a "peaked" type distribution which is less well correlated. The same Fourier expansion coefficients, calculated for the 200 mm length, were utilised for the longer rod and tube as well.

To illustrate the response, white noise and truncated Gaussian distribution were chosen for the power spectral density of the pressure fluctuation. Typical response spectra calculated from the exact and simple beam theory models are shown in figures 1 to 8 for the rod and tube respectively. These figures, which deal only with the 200 mm length, at axial location z/L = 0.25, refer to the broad band limited white noise and also Gaussian input normalised to unity at the origin, both covering a range of 32 kHz. Results for "flat" and "peaked" axial correlations are illustrated. The range of 32 kHz was chosen as it covers the first five most important frequencies.

The figures show in detail the expected agreements and disagreements between the exact and approximate models, attributable to the neglect in the approximate model of the higher modes of elastic distortion. For the rod the radial and tangential displacements are not significantly different, but significant differences do occur for a thin walled tube. All the results indicate that the beam theory behaves poorly as far as the prediction of differential axial strain is concerned, and this is of considerable consequence, as experimental interpretation, which is as a rule based on the simple beam theory approach, would give a completely false picture even when measuring the response at the first dominant frequency. As seen, the displacements in all cases for both the exact and beam theory are practically identical, while the axial strains are significantly different. The effect of the degree of correlation of the axial function on the magnitude of the displacements and axial strains follows the expected pattern. The "flat" well correlated distribution gives lower magnitudes than the "peaked" less correlated distribution, the effect being more pronounced for the differential axial strain.

Figures 9 and 10 and Tables 1 to 4 give the results of a number of studies carried out to investigate the effect of axial location, length and input spectrum on the mean square responses of radial displacements and differential axial strain. Two types of power spectral density of the pressure fluctuations were considered, maintaining the mean square value constant while varying ω_m :

(a) Band limited white noise $\quad \Phi_p(\omega) = \text{const} \quad ; \quad 0 \leq \omega \leq \omega_m$

$\quad = 0 \quad ; \quad \omega > \omega_m$

(b) Truncated Gaussian distribution $\quad \Phi_p(\omega) \propto \exp(\beta\omega^2) \quad ; \quad 0 \leq \omega \leq \omega_m$

$\Phi_p(\omega_m) / \Phi_p(0) = 0.01$

The relevant bandwidths for both lengths are given in Tables 1 to 4, which are self-explanatory. Figures 9 and 10 graphically illustrate information contained in Tables 1 to 4 which show in detail the effect of axial location, length, and axial correlation for a rod and tube on the mean square differential axial strain, using both approximate and exact theory. The diagrams deal with white noise responses with bandwidths of 2 kHz for 200 mm length and 100 Hz for 1000 mm length, both covering the first dominant frequency. For the particular set of parameters and functions used in these applications, the validity of simple beam theory is confirmed as regards the prediction of displacement for any axial location. The maximum difference does not exceed 10% for the short length, while for the greater length the discrepancy is negligible. On the other hand, the prediction of differential axial strain using simple beam theory is entirely unsatisfactory for any set of parameters investigated. Strains calculated exactly are significantly smaller, presumably as a result of correctly accounting for the local pressure. Further, there seems to be a greater discrepancy between the results for tubes than for rods, and also greater differences for a short length than a greater length. There does not seem to be any clear cut indication of the effect of the type of axial correlation on mean square responses.

It is obvious that any attempt to interpret measured differential axial strains by means of simple beam theory, e.g., to estimate mean square displacements, is fruitless. While the current investigation has been concerned with a pin ended rod, which is amenable to an exact solution, these conclusions would apply also to vibration problems of a bundle of fuel elements.

4. Conclusion

Numerical results obtained to date have tended to confirm by comparison with simple beam theory, the correctness both of the more exact elasticity theory, and of the associated computer programme. One significant aspect of the work is the availability of a programme to calculate exactly the natural frequencies and the normalised displacement mode for pin ended rods and tubes of arbitrary dimensions, which it is hoped will be utilised in a number of theoretical and experimental studies on shock and vibration problems relevant to reactor structural mechanics.

5. Acknowledgement

Acknowledgement is due to the Australian Institute of Nuclear Science and Engineering for financial assistance with the computing.

6. References

[1] Thompson, J.J. and Holy, Z.J. "Random pressure induced vibration of pin ended cylindrical rods". Paper D3/6, 2nd International Conference on Structural Mechanics in Reactor Technology, Berlin. September 1973.

[2] Thompson, J.J. and Holy, Z.J. "Random pressure induced vibration of pin ended cylindrical rods and tubes". Paper submitted for publication in Nucl. Eng. Design.

[3] Armenakas, A.E., Gazis, D.C. and Herrmann, G. "Free vibrations of circular cylindrical shells". Pergamon (1968).

[4] Chen, S. and Wambsganss, M.W. "Parallel-flow-induced vibration of fuel rods". Nucl. Eng. Design 18, 253-278 (1972).

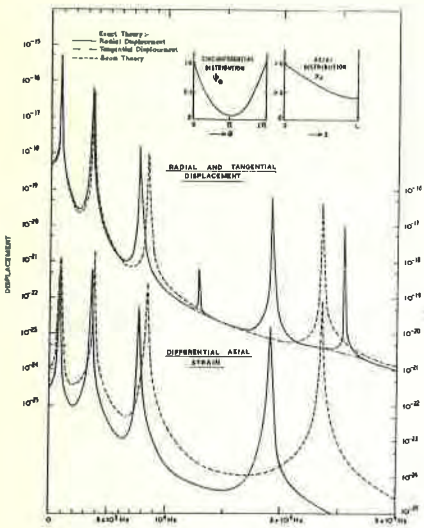

Figure 1 Solid rod response at z/L = 0.25 to white noise input. (L = 200 mm, α = 1)

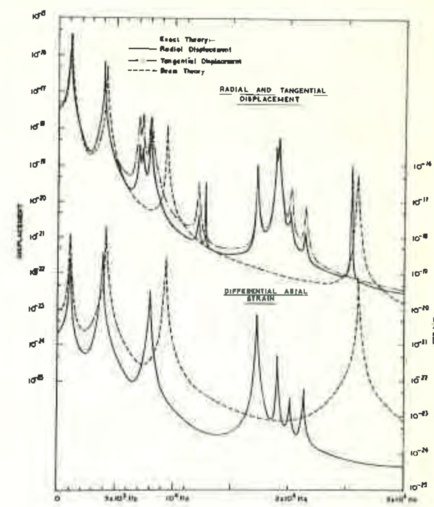

Figure 2 Tube response at z/L = 0.25 to white noise input. (L = 200 mm, α = 1)

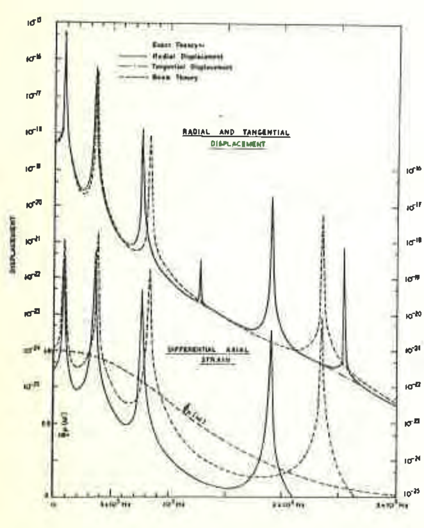

Figure 3 Solid rod response at z/L = 0.25 to Gaussian noise input. (L = 200 mm, α = 1)

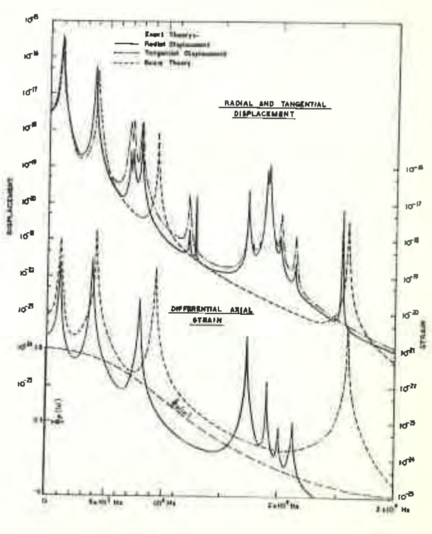

Figure 4 Tube response at z/L = 0.25 to Gaussian noise input. (L = 200 mm, α = 1)

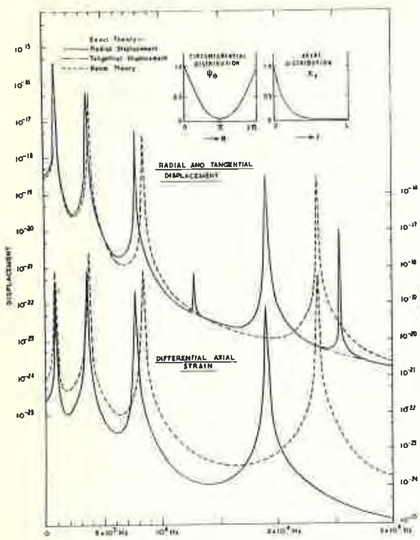

Figure 5 Solid rod response at z/L = 0.25 to white noise input. (L = 200 mm, α = 10)

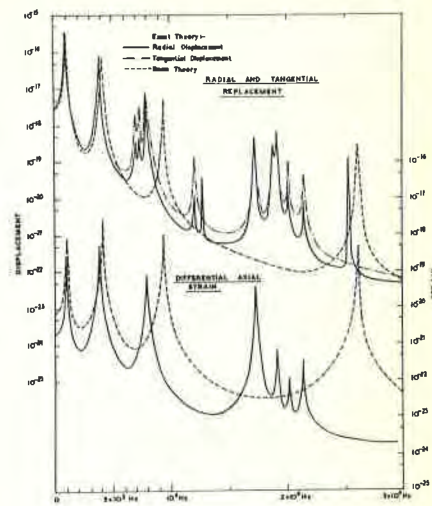

Figure 6 Tube response at z/L = 0.25 to white noise input. (L = 200 mm, α = 10)

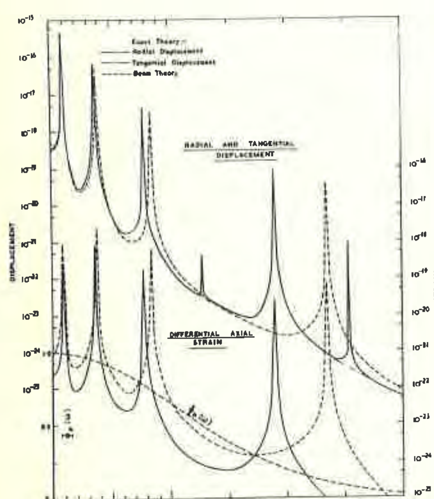

Figure 7 Solid rod response at z/L = 0.25 to Gaussian noise input. (L = 200 mm, α = 10)

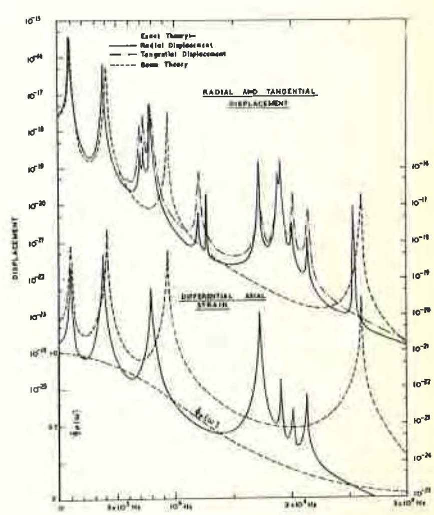

Figure 8 Tube response at z/L = 0.25 to Gaussian noise input. (L = 200 mm, α = 10)

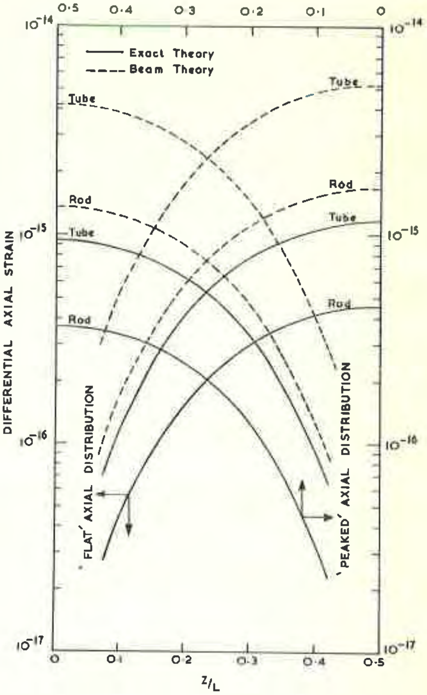

Figure 9 Strain response to white noise input which spans fundamental frequency only. (L = 200 mm)

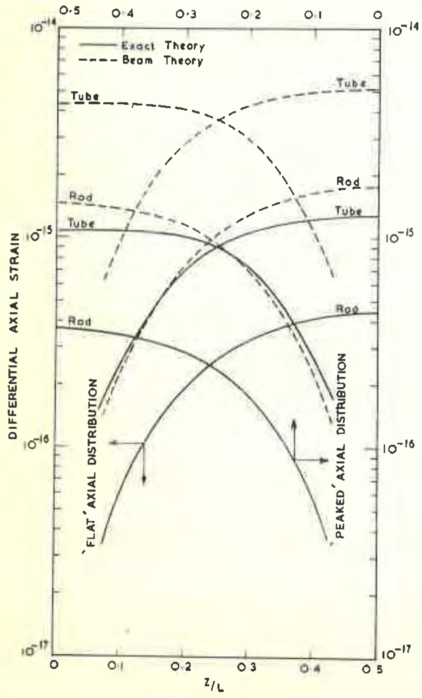

Figure 10 Strain response to white noise input which spans fundamental frequency only. (L = 1000 mm)

TABLE I

MEAN SQUARE RESPONSES, WHITE NOISE SPECTRUM, 20 cm LENGTH

Z/L	Band Width	Radial Displacement								Differential Axial Strain							
		Flat Axial Correlation				Peaked Axial Correlation				Flat Axial Correlation				Peaked Axial Correlation			
		Solid		Tube		Solid		Tube		Solid		Tube		Solid		Tube	
		Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam
0.1	(a)	0.55	0.51	1.59	1.55	0.53	0.50	1.56	1.50	0.18	0.93	0.38	2.55	0.46	2.41	0.84	6.67
	(b)	1.10	1.02	3.17	3.08	1.05	1.00	3.09	2.91	0.35	1.71	0.73	3.40	0.78	3.95	1.48	6.72
	(c)	2.18	2.00	6.22	6.07	2.05	1.82	5.72	5.47	0.52	1.81	1.02	5.11	1.02	2.28	1.42	6.38
	(d)	7.30	6.55	20.64	20.31	5.91	5.30	16.65	16.46	0.43	1.61	1.14	5.09	0.35	1.31	0.94	4.20
0.2	(a)	1.87	1.73	5.39	5.28	1.68	1.57	4.88	4.74	0.36	1.60	0.81	4.46	0.68	2.75	1.23	7.53
	(b)	3.75	3.46	10.77	10.55	3.36	3.15	9.75	9.47	0.69	3.19	1.61	8.44	1.16	5.48	2.46	13.59
	(c)	7.49	6.87	21.41	20.99	6.69	6.10	19.01	18.46	1.32	5.15	2.89	14.57	2.09	6.27	3.63	17.56
	(d)	26.41	23.70	74.28	73.45	21.40	19.20	60.20	59.51	1.67	5.81	4.10	18.20	1.28	4.73	3.36	14.92
0.25	(a)	2.63	2.41	7.53	7.39	2.28	2.12	6.59	6.45	0.37	1.68	0.91	4.83	0.54	2.46	1.21	7.16
	(b)	5.26	4.82	15.04	14.78	4.56	4.24	13.18	12.88	0.75	3.32	1.81	9.45	1.01	4.64	2.32	12.94
	(c)	10.62	9.63	30.02	29.50	9.13	8.39	26.09	25.50	1.49	6.18	3.49	17.62	2.01	7.32	4.10	20.58
	(d)	38.22	34.29	107.47	106.27	30.96	27.77	87.08	86.10	2.28	8.39	5.90	26.20	1.85	6.82	4.82	21.41
0.3	(a)	3.34	3.05	9.53	9.37	2.83	2.61	8.14	7.98	0.40	1.79	0.98	5.16	0.54	2.64	1.21	7.55
	(b)	6.68	6.09	19.04	18.74	5.66	5.21	16.24	15.92	0.78	3.45	1.92	9.41	0.95	4.41	2.23	10.89
	(c)	13.36	12.18	38.04	37.46	11.31	10.40	32.37	31.79	1.51	6.83	3.71	18.55	1.71	7.30	4.05	20.70
	(d)	50.03	44.88	140.63	139.08	40.52	36.35	113.94	112.67	2.98	10.96	7.69	34.13	2.42	8.90	6.27	27.83
0.4	(a)	4.41	3.98	12.49	12.32	3.64	3.30	10.34	10.17	0.38	1.63	0.93	4.76	0.51	2.35	1.10	6.58
	(b)	8.82	7.95	24.96	24.62	7.28	6.59	20.66	20.29	0.77	3.25	1.86	8.26	1.02	4.70	2.21	9.31
	(c)	17.63	15.88	49.83	49.19	14.52	13.06	41.00	40.34	1.38	5.19	3.36	15.59	1.51	5.01	3.22	14.84
	(d)	69.13	62.02	194.25	192.13	55.98	50.23	157.35	155.62	4.11	15.11	10.56	46.88	3.33	12.25	8.58	38.10
0.5	(a)	4.79	4.30	13.53	13.35	3.92	3.52	11.08	10.93	0.33	1.28	0.84	4.10	0.52	2.27	1.06	6.67
	(b)	9.58	8.60	27.04	26.71	7.85	7.04	22.15	21.85	0.67	2.56	1.63	7.77	0.89	3.67	1.92	10.61
	(c)	19.17	17.16	53.98	53.30	15.68	13.90	43.83	43.17	1.33	4.18	3.01	13.01	1.74	3.41	2.75	10.64
	(d)	76.42	68.57	214.71	212.39	61.89	55.53	173.91	172.02	4.54	16.70	11.65	51.74	3.68	13.53	9.46	41.99

Bandwidth: (a) = 32 kHz ; (b) = 16 kHz ; (c) = 8 kHz ; (d) = 2 kHz

TABLE II

MEAN SQUARE RESPONSES, GAUSSIAN SPECTRUM, 20 cm LENGTH

Z/L	Band Width	Radial Displacement								Differential Axial Strain							
		Flat Axial Correlation				Peaked Axial Correlation				Flat Axial Correlation				Peaked Axial Correlation			
		Solid		Tube		Solid		Tube		Solid		Tube		Solid		Tube	
		Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam
0.1	(a)	1.22	1.13	3.52	3.42	1.14	1.08	3.35	3.18	0.33	1.49	0.71	3.63	0.69	3.11	1.33	6.90
	(b)	2.34	2.16	6.70	6.52	2.11	1.97	6.10	5.84	0.45	1.97	1.04	5.16	0.73	3.13	1.56	7.65
	(c)	4.22	3.83	11.81	11.49	3.60	3.29	10.08	9.73	0.51	2.14	1.20	5.59	0.59	2.47	1.33	6.20
	(d)	6.89	6.10	17.32	16.58	5.68	4.94	14.02	13.42	0.41	1.50	0.86	4.16	0.33	1.22	0.78	3.40
0.2	(a)	4.19	3.86	12.02	11.77	3.72	3.48	10.77	10.46	0.72	3.23	1.70	8.94	1.13	5.15	2.46	13.44
	(b)	8.15	7.47	23.20	22.70	7.10	6.58	20.36	19.74	1.21	5.26	2.86	14.28	1.64	7.20	3.69	18.63
	(c)	14.95	13.52	41.80	40.84	12.57	11.44	35.21	34.18	1.58	6.47	3.73	17.18	1.71	7.14	3.94	18.27
	(d)	24.94	22.08	62.65	59.99	20.20	17.88	50.73	48.58	1.49	5.41	3.45	14.87	1.21	4.39	2.81	12.13
0.25	(a)	5.89	5.40	16.82	16.52	5.09	4.72	14.85	14.31	0.81	3.56	1.96	10.05	1.08	4.84	2.45	13.19
	(b)	11.52	10.51	32.68	32.07	9.85	9.07	28.10	27.41	1.43	6.18	3.46	17.12	1.74	7.65	4.05	20.46
	(c)	21.35	19.28	59.66	58.43	17.80	16.15	49.82	48.54	2.02	8.20	4.85	22.19	2.11	8.75	4.93	22.77
	(d)	36.10	31.95	90.64	86.80	29.23	25.88	73.41	70.30	2.15	7.81	4.87	21.41	1.74	6.34	4.05	17.46
0.3	(a)	7.50	6.83	21.34	21.00	6.34	5.82	18.14	17.79	0.85	3.65	2.09	10.39	1.02	4.48	2.37	11.91
	(b)	14.72	13.36	41.65	40.96	12.37	11.31	35.14	34.44	1.52	6.42	3.75	18.13	1.56	7.21	4.00	19.80
	(c)	27.58	24.85	77.01	75.61	22.80	20.41	63.73	62.34	2.32	9.22	5.63	25.59	2.29	9.37	5.46	25.07
	(d)	47.25	41.83	118.62	113.60	38.27	33.88	96.10	92.02	2.81	10.21	6.48	27.89	2.28	8.29	5.28	22.75
0.4	(a)	9.92	8.94	28.04	27.67	8.17	7.38	23.14	22.76	0.81	3.25	1.99	9.46	0.96	4.03	2.18	10.27
	(b)	18.57	17.62	55.14	54.40	16.05	14.47	45.27	44.58	1.45	5.63	3.62	16.58	1.45	5.74	3.50	16.15
	(c)	37.29	33.47	104.02	102.53	30.38	27.30	84.77	83.46	2.48	9.35	6.20	27.70	2.17	8.31	5.38	24.14
	(d)	65.29	57.80	163.86	156.84	52.88	46.81	132.78	127.16	3.88	14.09	8.91	38.32	3.15	11.43	7.26	31.23
0.5	(a)	10.78	9.68	30.40	30.03	8.81	7.90	24.85	24.50	0.73	2.75	1.82	8.36	0.94	3.69	1.99	10.22
	(b)	21.33	19.13	59.96	59.24	17.34	15.55	48.77	48.10	1.35	4.99	3.39	15.14	1.39	5.19	3.24	14.70
	(c)	40.88	38.64	113.99	112.52	33.11	29.67	92.33	91.13	2.43	8.92	6.19	27.42	2.00	7.23	5.04	22.28
	(d)	72.18	63.90	181.14	173.49	58.44	51.76	146.79	140.57	4.29	15.56	9.84	42.29	3.48	12.63	8.00	34.44

Dispersion: (a) = 14.91 kHz ; (b) = 7.46 kHz ; (c) = 3.73 kHz ; (d) = 0.93 kHz .

TABLE III

MEAN SQUARE RESPONSES, WHITE NOISE SPECTRUM, 100 cm LENGTH

Z/L	Band Width	Radial Displacement								Differential Axial Strain							
		Flat Axial Correlation				Peaked Axial Correlation				Flat Axial Correlation				Peaked Axial Correlation			
		Solid		Tube		Solid		Tube		Solid		Tube		Solid		Tube	
		Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam
0.1	(a)	0.33	0.33	0.97	0.97	0.32	0.32	0.93	0.93	0.21	0.88	0.62	2.57	0.56	2.29	1.57	6.65
	(b)	0.66	0.66	1.94	1.93	0.64	0.64	1.86	1.86	0.40	1.60	1.12	4.63	0.90	3.65	2.52	10.50
	(c)	1.32	1.32	3.83	3.83	1.25	1.25	3.58	3.58	0.58	2.33	1.62	6.62	1.13	4.54	2.98	12.10
	(d)	4.43	4.42	13.04	13.03	3.63	3.62	10.87	10.86	0.59	2.38	2.44	9.87	0.57	2.29	2.68	10.86
0.2	(a)	1.13	1.13	3.30	3.30	1.02	1.02	2.98	2.98	0.38	1.53	1.09	4.47	0.64	2.59	1.83	7.52
	(b)	2.27	2.27	6.60	6.60	2.04	2.04	5.93	5.92	0.78	3.06	2.18	8.09	1.28	5.17	3.82	14.91
	(c)	4.54	4.53	13.16	13.15	4.07	4.06	11.74	11.73	1.44	5.77	4.02	16.33	2.29	9.20	6.05	24.67
	(d)	15.93	15.91	46.48	46.44	13.02	13.00	38.47	38.43	1.91	7.64	7.13	28.72	1.76	7.04	7.33	29.56
0.25	(a)	1.58	1.58	4.62	4.62	1.39	1.38	4.02	4.02	0.41	1.66	1.18	4.83	0.61	2.46	1.73	7.14
	(b)	3.18	3.17	9.24	9.23	2.77	2.77	8.05	8.04	0.81	3.25	2.32	9.45	1.11	4.46	3.18	12.89
	(c)	6.36	6.35	18.45	18.44	5.54	5.53	16.03	16.02	1.61	6.47	4.54	18.43	2.19	8.81	5.98	24.35
	(d)	22.98	22.95	66.67	66.62	18.74	18.71	54.93	54.89	2.60	10.40	9.10	36.64	2.33	9.34	9.05	36.46
0.3	(a)	2.02	2.01	5.86	5.85	1.72	1.71	4.98	4.98	0.44	1.77	1.28	5.15	0.64	2.60	1.81	7.54
	(b)	4.03	4.03	11.72	11.71	3.43	3.42	9.96	9.96	0.85	3.40	2.43	9.86	1.06	4.27	3.04	12.44
	(c)	8.06	8.05	23.40	23.39	6.85	6.84	19.87	19.88	1.61	6.45	4.59	18.59	1.86	7.48	5.32	21.58
	(d)	29.98	29.93	86.52	86.47	24.40	24.36	70.97	70.92	3.22	12.88	10.58	42.56	2.82	11.30	10.15	40.88
0.4	(a)	2.65	2.64	7.70	7.70	2.19	2.18	6.36	6.38	0.41	1.63	1.16	4.74	0.56	2.26	1.60	6.60
	(b)	5.30	5.29	15.40	15.39	4.38	4.37	12.73	12.72	0.81	3.27	2.31	9.42	1.12	4.52	3.14	12.93
	(c)	10.59	10.58	30.75	30.74	8.73	8.72	25.33	25.32	1.41	5.68	4.02	16.25	1.58	6.36	4.44	17.99
	(d)	41.21	41.16	117.99	117.94	33.44	33.39	98.06	98.01	4.12	16.48	12.24	49.22	3.46	13.84	10.86	43.69
0.5	(a)	2.87	2.87	8.35	8.35	2.35	2.35	6.84	6.84	0.35	1.41	1.01	4.11	0.57	2.30	1.61	6.70
	(b)	5.75	5.74	16.70	16.69	4.70	4.69	13.67	13.64	0.67	2.68	1.93	7.80	0.91	3.68	2.64	10.82
	(c)	11.49	11.48	33.36	33.36	9.40	9.38	27.24	27.22	1.32	5.30	3.69	14.87	1.78	7.14	4.57	18.57
	(d)	45.46	45.41	129.81	129.74	36.88	36.80	105.38	105.33	4.44	17.76	12.72	51.13	3.66	14.64	10.83	43.57

Bandwidth: (a) = 1600 Hz ; (b) = 800 Hz ; (c) = 400 Hz ; (d) = 100 Hz .

TABLE IV

MEAN SQUARE RESPONSES, GAUSSIAN SPECTRUM, 100 cm LENGTH

Z/L	Band Width	Radial Displacement								Differential Axial Strain							
		Flat Axial Correlation				Peaked Axial Correlation				Flat Axial Correlation				Peaked Axial Correlation			
		Solid		Tube		Solid		Tube		Solid		Tube		Solid		Tube	
		Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam	Exact	Beam
0.1	(a)	0.74	0.74	2.16	2.16	0.70	0.70	2.03	2.03	0.39	1.56	1.09	4.48	0.86	3.46	2.33	9.66
	(b)	1.44	1.44	4.20	4.19	1.32	1.32	3.83	3.83	0.56	2.24	1.61	6.57	0.97	3.90	2.74	11.20
	(c)	2.68	2.68	7.82	7.82	2.34	2.34	6.83	6.82	0.72	2.89	2.22	9.02	0.92	3.71	2.90	11.80
	(d)	6.66	6.65	22.37	22.35	5.40	5.39	18.28	18.26	0.77	3.08	3.18	12.82	0.68	2.73	3.10	12.53
0.2	(a)	2.55	2.54	7.40	7.40	2.27	2.27	6.60	6.60	0.81	3.24	2.30	9.38	1.29	5.20	3.62	14.87
	(b)	5.00	4.99	14.50	14.49	4.40	4.40	12.74	12.73	1.40	5.64	4.02	16.33	2.01	8.08	5.65	23.06
	(c)	9.41	9.40	27.40	27.38	8.05	8.04	23.43	23.41	2.05	8.21	6.07	24.56	2.43	9.73	7.24	29.34
	(d)	24.04	23.99	80.38	80.30	19.51	19.47	65.58	65.52	2.61	10.44	10.02	40.31	2.24	8.98	9.28	37.38
0.25	(a)	3.57	3.57	10.37	10.37	3.10	3.10	9.00	9.00	0.89	3.97	2.54	10.32	1.21	4.88	3.43	14.01
	(b)	7.03	7.02	20.40	20.39	6.06	6.05	17.56	17.55	1.62	6.51	4.62	18.75	2.05	8.23	5.80	23.60
	(c)	13.38	13.34	38.86	38.84	11.28	11.27	32.83	32.81	2.51	10.07	7.32	29.59	2.82	11.31	8.28	33.53
	(d)	34.74	34.66	115.85	115.75	28.20	28.14	94.44	94.35	3.65	14.61	13.46	54.13	3.11	12.45	12.22	49.18
0.3	(a)	4.53	4.53	13.17	13.16	3.85	3.84	11.18	11.17	0.93	3.72	2.64	10.76	1.17	4.69	3.28	13.37
	(b)	8.95	8.94	25.98	25.96	7.56	7.55	21.93	21.92	1.68	6.75	4.79	19.38	1.91	7.66	5.46	22.13
	(c)	17.18	17.13	49.87	49.84	14.31	14.29	41.62	41.60	2.74	10.97	7.89	31.86	2.89	11.60	8.44	34.16
	(d)	45.40	45.31	151.05	150.92	36.85	36.77	123.03	122.81	4.65	18.58	16.50	66.31	3.93	15.71	14.70	59.15
0.4	(a)	5.97	5.96	17.34	17.33	4.92	4.91	14.30	14.29	0.85	3.42	2.42	9.83	1.07	4.30	3.00	12.27
	(b)	11.84	11.82	34.35	34.33	9.72	9.71	28.23	28.22	1.52	6.10	4.34	17.53	1.63	6.52	4.64	18.80
	(c)	22.97	22.93	66.75	66.72	18.77	18.74	54.62	54.59	2.55	10.61	7.65	30.81	2.44	9.87	7.30	29.44
	(d)	62.60	62.47	207.49	207.32	50.79	50.68	168.70	168.55	6.18	24.72	20.84	83.74	5.15	20.58	17.91	71.99
0.5	(a)	6.48	6.47	18.82	18.81	5.29	5.28	15.38	15.37	0.74	2.87	2.13	8.61	1.01	4.05	2.83	11.56
	(b)	12.86	12.84	37.34	37.33	10.47	10.45	30.42	30.40	1.37	5.50	3.95	15.92	1.53	6.14	4.37	17.73
	(c)	25.08	25.05	72.91	72.88	20.34	20.31	59.20	59.18	2.48	8.93	7.24	29.12	2.16	8.65	6.51	26.22
	(d)	69.15	69.00	220.89	220.71	56.10	56.00	185.95	185.80	6.78	27.00	22.38	89.93	5.58	22.33	18.90	75.98

Dispersion: (a) = 746 Hz ; (b) = 373 Hz ; (c) = 186.5 Hz ; (d) = 46.6 Hz .