THE CREEP BENDING OF SHORT RADIUS PIPE BENDS

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SUMMARY

There is some existing theoretical work on the creep behaviour of smooth curved pipes which refers to long radius bends where the radius of curvature \( R \) is assumed to be much greater than the cross-sectional radius \( r \). The present paper extends the previous work and presents a theoretical analysis of the stationary creep of a short radius pipe bend under in-plane bending loading.

Pipe bends are known to be rather important components in the design of high duty pipework systems. At temperatures sufficiently elevated to cause creep the influence of the bends becomes even more important partly because of the increased flexibility and partly because of their tendency to attract strain to themselves. In many systems the influence of the bends may well dominate. In existing and proposed liquid metal fast breeder reactor design the pipework has considerable importance. Parts of the LMFBR include thin walled short radius bends which are expected to operate in the creep regime. In linear elasticity it is known that the assumption of long radius bends is not too severe as far as the flexibility characteristics are concerned although some modifications are necessary for accurate determination of the stresses. No data exists for nonlinear creep. Current work is aimed at elucidating the effect of the various assumptions common to linear elastic theory in so far as they affect the creep characteristics of bends on systems. Herein an energy based analysis using a simple \( n \) power constitutive law for stationary creep is employed to derive basic design data for flexibilities and stresses which will be necessary before complete systems can be assessed for creep.

The analysis shows on comparison with the long radius work that the assumption of \( R > r \) is not much more restrictive in creep than for linear elasticity. Flexibilities for short radius bends appear to be well approximated by the long radius values. Thus the attractive reference stress information already derived may be used directly to find deformations without a complete knowledge of the constitutive relationship. However, stresses are somewhat different. Fortunately the maximum deviation occurs at relatively low levels of stress, the peak stresses being in fair agreement. When \( n = 1 \) the present results reduce essentially to those obtained from existing linear elastic theory.
1. Introduction

The geometry of a pipe bend depends somewhat on the method of manufacture. For example, cold bending from straight pipe can only be used successfully for bends having radii of curvature relatively large compared with the radii of cross-section. On the other hand, hot forming over mandrels produces consistently good bends where the ratio \( \frac{R}{r} \) is not very large. Values of \( \frac{R}{r} = 2.0 \) are routinely forged by this procedure and smaller ratios are occasionally made. Being hot forged and formed on a mandrel, bends of this type are usually close to the ideally circular, constant thickness shape within fine tolerances.

In linear elastic analyses, the most commonly discussed refinement to the classical energy approach for pipe bends was to consider including the effect of the radius ratio in the theory. The bends were then referred to as 'short radius' bends. A number of investigators considered short radius bends but two are appropriate to the present work since they used energy analyses and considered the matter in some detail. They are Symonds and Purdue [1] (see the appendix of reference [2]) and Jones [3].

Bearing in mind the above remarks on manufacturing technique, it is considered that short radius bends may be reasonably taken as constant thickness and circular in cross-section for the purposes of investigating the effect of the radius ratio. It is not obvious that the small influence of the radius ratio in the linear work will not be magnified under creep conditions.

2. Notation

- \( B \) constant or function of time in creep law
- \( \dot{c} \) series coefficient in deformation rate
- \( h \) half the pipe wall thickness
- \( I \) second moment of area of cross section \((2hnr^3)\)
- \( K \) flexibility factor in creep \( \dot{\gamma}/\dot{\gamma}_0 \)
- \( K_{\theta, \phi} \) curvature rate, circumferential or meridional
- \( \mathbf{M} \) applied in plane moment
- \( n \) stress index in the creep law
- \( P \) number of terms in series
- \( r \) radius of cross section mid surface
- \( R \) radius of pipe bend centre line
- \( \dot{\Upsilon}, \dot{\Phi}, \dot{\Psi} \) circumferential, meridional and radial displacement rates
- \( a \) initial bend angle
- \( \beta \) \((\dot{\phi}a)/(\dot{\gamma}a)\)
- \( \dot{\gamma} \) rate of change of angle of bend
- \( \dot{\delta} \) shift in neutral axis
- \( \dot{\varepsilon}_{\theta, \phi} \) strain rate: circumferential or meridional
- \( \dot{\varepsilon} \) \(\frac{1}{2}(\dot{\varepsilon}_1 - \dot{\varepsilon}_2)^2 + (\dot{\varepsilon}_2 - \dot{\varepsilon}_3)^2 + (\dot{\varepsilon}_3 - \dot{\varepsilon}_1)^2\)
\[ \dot{\varepsilon}_o \] arbitrary or reference strain rate
\[ \xi \] reference stress parameter \( M/(hr^2\delta_o) \)
\[ \lambda \] \( 2hr/r^2 \)
\[ \sigma \] stress
\[ \bar{\sigma} \] \( \frac{1}{2} \left( (\sigma_1 - c_2)^2 + (\sigma_2 - c_3)^2 + (\sigma_3 - c_1)^2 \right)^{\frac{1}{2}} \)
\[ \sigma_o \] arbitrary or reference stress
\[ (\cdot) \] \( d(\cdot)/dt \) where \( t \) is time

Other symbols are defined in Fig. 1 or in the text.

3. Theoretical Analysis

The following analysis is generally similar to that given in [4] for long radius bends. Use is made of Odqvists [5] type I energy theorem for steady creep: a compatible velocity field is assumed and inserted in a suitable formulation of the energy. Subsequent optimisation produces the required results. The problem is defined geometrically in Fig. 1. It is desired to find a solution for the deformation of the bend as characterised by the end rotation rate \( \dot{\theta} \). The load is in-plane bending constant with time. A stationary creep constitutive relationship is assumed of the form.

\[ \dot{\varepsilon} = B\sigma^n \]  
(1)

where \( n \) and \( B \) are material characteristics. In generalised non-dimensional form it can be written as

\[ \left( \frac{\dot{\varepsilon}}{\varepsilon_o} \right) = \left( \frac{\sigma}{\sigma_o} \right)^n \]  
(2)

3.1 An Assumed Velocity Field

Specific assumptions are as follows:

1. The bend is initially stress free.
2. The initially circular cross-section deforms into an oval shape as shown in Fig. 1. Provision is made for a shift in the position of the neutral axis from the centre of the circle. The deformation is taken to be the same at every cross-section of the bend. Thus no end effects are considered.
3. The bend is 'thin'. The radius of the cross-section is much greater than the thickness.
4. The mid-surface meridional strain is zero.
5. The circumferential strain does not vary much through the thickness. Consequently the circumferential curvature can be neglected in energy considerations. Physically the assumption seems reasonable. It is not necessary to assume it identically zero except for convenience when formulating stresses.

In so far as these assumptions actually influence the results they are more or less identical to those of [2] and [3]

The appropriate strain rate/displacement rate equations for a rotationally symmetric shell of revolution modified slightly for the pipe bend case can be written as
\[ \varepsilon_\phi = \frac{1}{r} \left( \frac{2y}{3} + \omega \right) = 0 \]
\[ \varepsilon_\theta = \left[ \frac{2u}{3} + v \cos \phi + \omega \sin \phi \right] / (r + s \sin \phi) \]
\[ K_\phi = -\frac{1}{r^2} \left( \frac{2v}{3} + \frac{2s}{3} \right), \quad K_\theta = 0 \]

(3)

where the radial, tangential and circumferential displacements of the cross section midsurface are \( v \), \( w \) and \( u \) respectively. The radial displacement (rate) may be described by a finite series. Symonds and Purdue 40 chose the tangential one as

\[ v = -A \left[ \sum a_p \sin \phi + \sum b_q \cos \phi \right], \quad p = 2, 4, 6, ..., \quad q = 3, 5, 7, ... \]

where \( A \) is a constant and \( a_p, b_q \) are coefficients. This can be written via eqn 3a) as

\[ \omega = A \left[ \sum a_p \cos \phi \sin \phi + \sum b_q \sin \phi \cos \phi \right] \]

(4)

Jones [3] used

\[ \omega = \sum a_p \cos \phi \]

(5)

but his angle of reference \( \gamma \) was measured from \(-\pi/2\) in Fig. 1. These equations in fact give identical deformation modes. Here the same representation will be used, written as

\[ \omega = \sum a_p \cos \phi \left( \pi/2 + \pi/2 \right) \]

(6)

which has the necessary symmetry about the axis in the plane of the bend.

The circumferential deformation may be taken approximately as

\[ u = \left( \delta + r \sin \phi \right) \frac{\gamma}{\delta} \]

(7)

Substitution of eqns 6 and 7 into 3 gives

\[ \varepsilon_\phi = \left[ \left( \delta + r \sin \phi \right) \frac{\gamma}{\delta} + \sum \xi_p \cos \phi \left( \pi/2 + \pi/2 \right) \sin \phi - \sum \xi_p \sin \phi \cos \phi \right] / (r + s \sin \phi) \]

\[ K_\phi = \frac{1}{r^2} \sum \xi_p \left( \pi/2 - 1 \right) \cos \phi \left( \pi/2 + \pi/2 \right) \]

(8)

3.2 The Energy Formulation

The total potential energy rate is given approximately in this case by

\[ V = \frac{n}{n+1} \left( \frac{\pi}{2} \right)^2 2 h \sigma_0 \frac{\xi}{\delta} \sum_{\phi_1} \int_{\phi_{11}} [ \sum \xi \varepsilon_\phi \delta + N h \kappa_\phi^2 / (\pi^2) ] \, d\omega \, d\phi - M \dot{\gamma} \]

(9)

where \( N = \frac{n}{(2n+1)} \left( 2n / (\pi r^2) \right) \)

Inserting eqn 8 and integrating leads to

\[ V = \frac{n}{n+1} \left( \frac{\pi}{2} \right)^2 2 h \sigma_0 \frac{\xi}{\delta} \sum_{\phi_1} \int_{\phi_{11}} [ \sum \xi \varepsilon_\phi \delta + \frac{N h}{\pi^2} \kappa_\phi^2 / (\pi^2) ] \, d\omega \, d\phi - M \dot{\gamma} \]

(10)
where
\[
D = \int_{0}^{2\pi} \left\{ \left( \frac{\beta_{p}}{R} + SW \phi \right) + \frac{N \beta_{p}}{2} (\beta_{p} - i) \cos \varphi (\phi + \pi\nu) - \frac{1}{2} \cos \phi SW \phi (\phi + \pi\nu) \right\}^2 \cos \phi \cos \varphi \sin \phi d\phi + \frac{N \lambda^2}{2} \left[ \frac{\beta_{p}}{R} (\beta_{p} - i) \sin \varphi (\phi + \pi\nu) \right] \left( \frac{\lambda^{(n+1)/2}}{\lambda} \right)^2 (1 + \frac{1}{2} \frac{R}{SW} \sin \phi) d\phi,
\]
\[
\lambda = \frac{2 \lambda R}{\pi^2}, \quad \beta_{p} = \frac{C_{p} \lambda}{\sqrt{V}}.
\]
The energy rate has now been expressed in terms of several parameters namely \( \beta_{p}, \dot{\nu}, \lambda R, \).

It is routine to follow the procedure outlined in [4] to minimise the energy and arrive at an equation relating the end rotation rate to the bending moment in terms of the minimum value of the function \( D \) namely \( D_{\text{min}} \). Thus
\[
\dot{\nu} = \left( \frac{n}{4} \right)^{1/2} \frac{R \lambda}{
\frac{\lambda}{\pi} \left( \frac{M \lambda^{\nu}}{2 \lambda^{2} \lambda^{2} \lambda^{2}} \right)^{n} \frac{1}{D_{\text{min}}}.
\]

In fact the results were evaluated in a slightly different manner by evaluating flexibility factors directly from the energy function as described in the following section.

4. Deformation Results
4.1 Flexibility Factors

The primary deformation, indicative of the behaviour of the pipe bend, is the end rotation rate. It is useful to define a flexibility factor for creep conditions as
\[
K = \frac{\dot{\nu}}{\dot{\nu}_{o}} = \text{the end rotation rate of a pipe bend in creep} \overline{\text{the end rotation rate of a corresponding straight}}
\]

where \( \dot{\nu}_{o} \) refers to the straight pipe of length \( R_{o} \) of the same cross-sectional properties as the bend and loaded by the same moment \( M \). The straight pipe result has already been developed in [4] and is
\[
\dot{\nu}_{o} = \frac{R_{o}}{4} \left( \frac{M \lambda^{\nu}}{2 \lambda^{2} \lambda^{2} \lambda^{2}} \right)^{n} \frac{1}{D_{o}}
\]
where \( D_{o} = 4 \int_{0}^{\pi/2} \left( \frac{M \lambda^{\nu}}{2 \lambda^{2} \lambda^{2} \lambda^{2}} \right)^{n} d\phi \).

Dividing eqn 10 by \( \dot{\nu}_{o} \) and using the definition of \( K \) leads to
\[
\frac{\dot{\nu}}{\dot{\nu}_{o}} = \frac{n}{M \lambda^{\nu}} \left( \frac{4}{3} \right)^{(n+1)/2n} \frac{K^{(n+1)/n}}{D_{o}} \frac{D}{D_{o}} - K
\]

Eqn 13 was minimised directly on a computer in terms of the variables \( \beta_{p}, K \) and \( \nu R_{o} \).

In this way flexibility factors emerge automatically. It is not so easy to control the accuracy of \( K \) directly using this technique since it is obviously better to control the function value. However, by using various numbers of terms in the series, it is relatively simple to check convergence. Obviously there is no particular advantage in evaluating equation (13) instead of an equation like (11); however, the method was used to check the computer procedures which had to be employed in another problem.

Using the direct minimisation procedure flexibility factors were evaluated for specific \( \lambda, n \) and \( \nu R_{o} \) values of the radius ratio of 0, 0.2 and 0.5 being covered, for the usual range of the other variables. Once the extra parameter \( F/R \) is included it seems hardly worthwhile retaining the parameter \( \lambda \) since the two separate geometric quantities \( F/R \)
and \( 2\pi \) may be used. In view of the secondary influence of \( T/R \) throughout the results, as will become apparent, it was decided to retain the usual pipe bend parameter and use the pair \( T/R \) and \( \lambda \). Certain combinations of \( T/R \) and \( \lambda \) are not necessarily valid.

Fig. 2 shows the flexibility factors. The graph is virtually identical to that in [4] since the values for different \( T/R \) were so close as to be almost indistinguishable on the graph. Only at \( n = 5 \), \( T/R = 0.5 \) has it been possible to show any difference at all and this is irrelevant because of the thin shell limitation as indicated. In the special case for \( T/R = 0 \), the results were indistinguishable from [4]. Most of the results were evaluated with 7 terms in the series (corresponding to 4 in the long radius case) and some for \( \lambda > 0.5 \) with 5 terms in the series. Some were checked for convergence at low by investigating term by term.

4.2 Reference Stresses for Deformation

The reference stress technique may also be used in the manner described previously. Consider the definition of the flexibility factor written as

\[ \hat{V} = K \hat{V}_0 \]  

Then by putting \( n = 1 \)

\[ \hat{V}_1 = K_1 \hat{V}_{o1} \]  

Division gives

\[ \frac{\hat{V}}{\hat{V}_1} = \frac{K \hat{V}_0}{K_1 \hat{V}_{o1}} \]  

Using eqn 12 generally and also with \( n = 1 \) for \( \hat{V}_0 \) and \( \hat{V}_{o1} \) results in

\[ \frac{\hat{V}}{\hat{V}_1} = \frac{K_1 \pi}{K_1 \phi_0} \left( \frac{M}{2 \pi \phi \phi_0} \right)^{n-1} \]  

(16)

Since \( \phi_0 \) has not been fixed, \( \phi_0 \) can be selected for a fixed \( n, \lambda \) and \( T/R \) to make the right hand side of eqn (16) unity. For that particular value of \( \phi_0 \) it is then true that

\[ \hat{V} = \hat{V}_1 \]  

(17)

From eqn (16) the reference stress can be extracted as the non dimensional group

\[ \frac{M}{\phi_0 \phi} \left( \frac{K_1}{K} \right)^{n-1} \left( \frac{\tau_0}{\phi} \right)^{n-1} = \xi \]

(18)

Values of the reference stress parameter \( \xi \) can now be evaluated directly from the flexibility factors; a few typical values are given in the table below.

The reference stress parameter is practically independent of \( n \) or \( T/R \). Some of the values in the table can be compared with those in [4]. There may be marginal differences for \( T/R = 0 \) due to difference in the method of evaluation. A single value of \( \xi \), say \( \xi_o \), can be chosen as representative of a particular \( \lambda \) over the whole range of \( n \) and \( T/R \) with little error (about \( \pm 4\% \)). The differences between the selected \( \xi_o \) values and those in [4] for long radius bends are of the order of 12. Consequently the values of the reference stress parameters already given in [4] may be taken as suitable for short
radius bends of any radius ratio

<table>
<thead>
<tr>
<th>$\lambda/\lambda^*$</th>
<th>$n = 1.5$</th>
<th>$n = 3$</th>
<th>$n = 5$</th>
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<tr>
<td>$\lambda R$</td>
<td>0</td>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>1.0</td>
<td>7.26</td>
<td>7.24</td>
<td>7.20</td>
</tr>
<tr>
<td>0.5</td>
<td>5.02</td>
<td>5.00</td>
<td>4.94</td>
</tr>
<tr>
<td>0.2</td>
<td>2.81</td>
<td>2.80</td>
<td>2.78</td>
</tr>
<tr>
<td>0.1</td>
<td>1.77</td>
<td>1.76</td>
<td>1.75</td>
</tr>
<tr>
<td>$\lambda R$</td>
<td>0</td>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>7.33</td>
<td>7.30</td>
<td>7.16</td>
<td></td>
</tr>
<tr>
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<td>5.05</td>
<td>4.81</td>
<td></td>
</tr>
<tr>
<td>2.85</td>
<td>2.84</td>
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</tr>
<tr>
<td>1.80</td>
<td>1.80</td>
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</tr>
</tbody>
</table>

The end rotation rate can be written as

$$\dot{\gamma} = \frac{1}{2\pi} \frac{Eh}{\sqrt{\pi}} K_1 \varepsilon_n \dot{\epsilon}_n$$

where $K_1$ is the flexibility factor with $n = 1$, $\varepsilon_n$ is the reference parameter as defined and $\dot{\epsilon}_n$ is the reference strain rate corresponding to a reference stress of $\sigma_n = M/(\pi h R^2)$

Eqn 19 is identical to eqn 23 of [4]

5. Stationary Stresses

It is worth examining the stresses resulting from the short radius bend analysis since it is known from the linear elastic work that the parameter $R/R$ has a somewhat greater effect on the stress levels than on deformation. Only stationary stresses can be given.

5.1 Formulation

The development of the stress equations follows in a similar manner and will not be repeated in detail. The main stresses expressed as stress concentration factors based on the maximum linear elastic beam stress can be shown to be

$$\sigma_{\theta D} \frac{R}{h} = \left( \frac{4}{3} \right)^{(n+1)/2n} K \frac{\pi}{h} \varepsilon_n \left[ A_1 + \frac{n^2}{2} A_2 \right]$$

$$\sigma_{\phi D} \frac{R}{h} = \left( \frac{4}{3} \right)^{(n+1)/2n} \frac{n}{2(n+1)} K \frac{\pi}{h} \varepsilon_n \left[ A_1 + \frac{n^2}{2} A_2 \right] \frac{A_2}{2}$$

where

$$A_1 = \left\{ (\varepsilon_n + \sin \phi) + \frac{E\beta}{h^2} \left[ \sin \phi \cos \phi (\phi + \frac{\pi}{2}) - \frac{1}{2} \cos \phi \sin \phi (\phi + \frac{\pi}{2}) \right] \right\}$$

$$A_2 = \frac{E\beta}{h^2} \left( \phi^2 - \frac{1}{2} \right) \cos \phi (\phi + \frac{\pi}{2})$$

In addition

$$\sigma_{\theta D} = \frac{1}{2} \sigma_{\phi D}, \quad \sigma_{\phi D} = \frac{1}{2} \sigma_{\phi D}$$
As in previous type 1 analyses it is advisable to modify the direct meridional stress from considerations of equilibrium. It has rather more relevance in the present context of short radius bends where $\tau/R$ is not neglected. The modification is

$$
\sigma_{dD} = -\frac{1}{R} \cos \phi \int_0^{\pi/2} \sigma_{dD} d\phi
$$

and emerges naturally from a type 2 analysis [6].

Since all the necessary coefficients or parameters $n, \lambda, \tau/R, \kappa, \beta_p, \sqrt{\tau}$ have been used or evaluated, it is a relatively straightforward matter to compute stresses for any particular angle of cross-section.

### 5.2 Selected Stress Results

No attempt will be made to present all the information contained in the stress equations. Only typical results are given. The maximum meridional bending stress values are practically independent of the radius ratio and therefore virtually identical to those given in [4] or [6] for all $\tau/R$. They are not repeated here. The maximum values of circumferential direct stress are slightly dependent on the radius ratio and are shown for $n = 1$ in Fig. 3. They are almost identical for other $n$. The graphs have been omitted below $(\frac{\tau}{2R}) = 5$. The position of the maximum $\sigma_{dD}$ varies weakly with $\tau/R$ and it is necessary to have its distribution to obtain the location of the maximum. Although the maxima were not affected, the distributions of the meridional bending stress are influenced slightly by the radius ratio as seen in Figs. 4 and 5 for one $n$ and two $\lambda$ values. The distributions are no longer symmetrical in each quadrant of the cross-section and values are given from $-\kappa/2$ to $+\kappa/2$.

Despite the fact that the maximum values of the circumferential direct stress vary with $\tau/R$ the distributions are such that the modification 22, is almost independent of $\tau/R$. Modified $\Delta_{dD}$ can therefore be taken as the same as in [6].

### 6. Discussion

The present analysis is a slight generalisation of the first type 1 creep analysis for bends given in [4]. The present short radius bend includes the long radius bend as a special case. Generally the complication of including the radius ratio in the analysis is not justified since deformations and maximum stresses are largely unaffected by $\tau/R$. It is nevertheless useful to know that the long radius results may reasonably be adapted for short radius bends even under creep conditions.

All the results discussed or presented can be taken to have effectively converged. The points raised in [4] regarding time independent problems, the errors in the deformation at large values, the limitations due to thin shell theory and large displacements are all equally applicable here. The deformation results may be described as lower bounds. The bounding is not immediately obvious because of the method of evaluation but it is clear from the method used in [4] or [6]. The series representation of eqn 6 is worth some
comment. As might be expected in the light of the results, the dominant coefficients are
the ones associated with even \( p \) which correspond directly to the long radius analysis.
The odd \( p \) coefficients and the neutral axis shift are always small by comparison.

Despite the volume of literature available it is difficult to find information on short
radius bends for comparative purposes. Some is given in \([7]\) and \([3]\). Where
comparisons have been made for both flexibilities and stresses the agreement has been exact
provided details such as the \((1 - \nu^2)\) term - see ref. \([4]\) - are taken into account.
Distributions of stresses have not been compared in detail but appear to be similar.

References

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**Fig. 1.** Deformation of short radius bend.

**Fig. 2.** Short radius bend flexibility factors.

**Fig. 3.** Maximum circumferential direct stress concentration factor.
Fig. 4. Stress distribution: Meridional Bending.

Fig. 5. Stress distribution: Circumferential Direct.