EIGENVALUE SOLUTIONS IN FINITE ELEMENT THERMAL TRANSIENT PROBLEMS

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SUMMARY

The eigenvalue economiser concept can be useful in solving large finite element transient heat flow problems in which the boundary heat transfer coefficients are constant. The usual economiser theory is equivalent to applying a unit thermal "force" to each of a small sub-set of nodes on the finite element mesh, and then calculating sets of resulting steady state temperatures. Subsequently it is assumed that the required transient temperature distributions can be approximated by a linear combination of this comparatively small set of master temperatures. The use of reduced eigenvectors as generalised co-ordinates diminishes the transient calculation to almost trivial proportions; the bulk of the numerical work being expended previously in the reduction and eigenvector computations.

The accuracy of a reduced eigenvalue calculation depends upon a good choice of master nodes, which presupposes at least a little knowledge about what sort of shape is expected in the unknown temperature distributions. There are some instances, however, where a reasonably good idea exists of the required shapes, permitting a modification to the economiser process which leads to greater economy in the number of master temperatures. An example of the kind of situation envisaged is a component consisting basically of, say, an axially symmetric thick shell but with various non axisymmetric features such as nozzles or axial flanges. An initial transient thermal analysis might be carried out using a comparatively cheap axisymmetric finite element model, ignoring the three dimensional detail. This would give a rough idea of what time in the transient was most critical and also what sort of shapes the temperature distribution would have at that time, therefore providing guidance in applying an economiser procedure to a detailed three dimensional finite element model.

The suggested new approach is to use manually prescribed temperature distributions as the master distributions, rather than using temperatures resulting from unit thermal forces. Thus, in the case of our example, if it were found that at the critical time the through-thickness temperature was very nearly linear, then the master distributions would be chosen such that only linear variations in temperature were possible in that direction. Similarly, we might know from the preliminary analysis that in the axial direction, temperatures vary smoothly. In such a case there would be no point in using master distributions sufficiently detailed to be able to represent irregular temperature variations.

Thus, with a little pre-knowledge one may write down a set of master distributions which, as a linear combination, can represent the required solution over the range of interest to a reasonable engineering accuracy, and using the minimum number of variables. The proposed modified eigenvalue economiser technique then uses the master distributions in an automatic way to arrive at the required solution.

The technique is illustrated by some simple finite element examples.

1. Introduction

Heat conduction analysis for complicated structures can be expensive, and so there is an incentive for examining ways of reducing the cost of such calculations. One procedure, commonly used in structural dynamic calculations, (see e.g. ref. 1) is to employ a reduced eigenvalue (or eigenvalue economiser), method to drastically reduce the number of variables in the problem. This is equally applicable to heat conduction analysis, although it has been found that the standard reduction process can be improved if the temperatures are represented as linear combinations of basic distributions generated intuitively by the analyst at the start of the calculation.

2. Thermal Eigenvalues

The problem of calculating transient temperature distributions in a body leads, in matrix form, to the following equation

$$H\phi + P\dot{\phi} + F = 0 \qquad (2.1)$$

ϕ is a column matrix of temperatures

H is a heat conduction matrix

P is a heat capacity matrix

F is a column matrix of thermal "forces", a function of time

$\dot{\phi} = \left[\frac{\partial\phi}{\partial t}\right]$, a column matrix of time derivatives of temperature

Consider the case of a body with no applied thermal forces, that is, having a null F matrix. We assume that the body is cooling uniformly so that at all times the temperatures are given by

$$\phi = S\phi_i \qquad (2.2)$$

ϕ_i defines some temperature distribution, and the scalar S, (a function of time), sets the scale of the temperatures.

From 2.2 we can write

$$\dot{\phi} = \dot{S}\phi_i \qquad (2.3)$$

Substituting 2.2 and 2.3 into 2.1 we obtain the standard matrix eigenvalue equation

$$\left(\left(\frac{-S}{\dot{S}}\right)I - H^{-1}P\right)\phi_i = 0 \qquad (2.4)$$

The eigenvectors ϕ_i which satisfy 2.4 are those temperature distributions which cool uniformly. The associated eigenvalues, λ_i, define the natural cooling rates.

Since

$$\frac{-S}{\dot{S}} = \lambda_i \qquad (2.5)$$

then

$$S = \exp\left(\frac{-t}{\lambda_i}\right) \qquad (2.6)$$

If we take S to be unity when time t = 0; then for $t = \lambda_i$ we see that $S = e^{-1} = .3679$. Thus λ_i may be regarded as the time required for the ith natural thermal mode to decay exponentially to 37% of its original value.

3. Transient solutions using Eigenvalues

Consider again the full transient heat conduction problem

$$H\phi + P\dot{\phi} + F = 0 \tag{3.1}$$

and assume that we have previously calculated the complet set of eigenvalues λ and eigenvectors θ of the matrix $H^{-1}P$.

By definition $\theta\lambda = H^{-1}P\theta$ (3.2)

where θ is a square matrix containing all the eigenvectors, and λ is a diagonal matrix of the corresponding eigenvalues.

First rewrite 3.1 in the form

$$\phi + H^{-1}P\dot{\phi} + H^{-1}F = 0 \tag{3.3}$$

Now suppose that the unknown temperatures, ϕ , can be represented by a linear combination of the eigenvectors, as follows:-

$$\phi = \theta L \tag{3.4}$$

L is a column matrix of multipliers

From 3.4, we can write

$$\dot{\phi} = \theta \dot{L} \tag{3.5}$$

Similarly, we can express the thermal forcing function as a linear combination of the eigenvectors, thus

$$H^{-1}F = \theta Q \tag{3.6}$$

where Q is another column of multipliers.

Substituting 3.2, 3.4, 3.5 and 3.6 into 3.3 gives

$$\theta(L + \lambda\dot{L} + Q) = 0 \tag{3.7}$$

But, since the columns of θ are linearly independent, equation 3.7 implies that

$$L + \lambda\dot{L} + Q = 0 \tag{3.8}$$

We now have a set of uncoupled first order differential equations to solve

If $L = [1_i]$, $Q = [q_i]$ and $\dot{L} = [{}^{dl}/dt]$ then we solve n equations of the form

$$\frac{dl_i}{dt} + \frac{l_i}{\lambda_i} + \frac{q_i}{\lambda_i} = 0 \tag{3.9}$$

for which we can write down a simple and direct exponential solution.

So, by investing in the numerical work of eigenvalue calculation, the final stage of the problem is made almost trivial. There is no need for step-by-step calculations through a large number of small time intervals, the solution at any time t is given directly by substituting the value of t into a straightforward sum of exponential terms.

However it must be recognised that the problem dealt with is somewhat restricted, since a constant heat conduction matrix H is implied. This in turn means that boundary heat transfer coefficients must be constant with time, although piecewise linear timewise variations of boundary temperatures are easily allowed for.

Nevertheless, a large number of practical engineering problems do fall within this restricted category, and it may often be advantageous to use an eigenvalue solution. But perhaps the biggest advantage to be gained from eigenvalue methods is the possibility of employing the eigenvalue economiser idea to reduce the size of large problems in order to obtain, economically, approximate solutions.

4. Approximate Eigenvalues from Reduced Matrices

In reference 2, Wright and Miles show, without appealing to any physical arguments that the eigenvalue problem

$$(\lambda H - P)\,\phi = 0 \qquad (4.1)$$

can be replaced by a reduced problem for the purpose of obtaining an approximate solution.

The matrices are partitioned to correspond with a division of the variables into master and slave categories, identified by suffices 1 and 2 respectively.

$$\text{Thus } H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \quad \text{and } \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \qquad (4.2)$$

(H is a symmetric matrix)

Then by truncating a series expansion, Wright and Miles produced the following reduced equation

$$(\lambda H_r - P_r)\,\phi_1 = 0 \qquad (4.3)$$

$$\text{where } H_r = H_{11} - H_{12} H_{22}^{-1} H_{21} \qquad (4.4A)$$

$$\text{and } P_r = [\, I \;\vdots\; - H_{12} H_{22}^{-1} \,]\, P \begin{bmatrix} I \\ \cdots\cdots \\ -H_{22}^{-1} H_{21} \end{bmatrix} \qquad (4.4B)$$

(I = the unit matrix)

The solution of the reduced equation 4.3 gives values for λ and ϕ_1 which approximately satisfy 4.1 providing that a good choice has been made for the master variables.

However, equations 4.4 may not be the best means of computing the reduced matrices. A practical scheme as put forward in ref. 2 is to solve the steady state thermal conduction problem

$$HJ = F \qquad (4.5)$$

using a set of unit thermal forces corresponding to the master variables. That is we put

$$F = \begin{bmatrix} I \\ \cdots \\ 0 \end{bmatrix} \qquad (4.6)$$

and solve for the temperatures J which correspond to unit master forces. If we partition J into master and slave variables so that

$$J = \begin{bmatrix} J_{11} \\ \cdots\cdots \\ J_{21} \end{bmatrix} \qquad (4.7)$$

then we find, using 4.5, 4.6 and 4.7

$$(H_{11} - H_{12} H_{22}^{-1} H_{21}) = J_{11}^{-1} \qquad (4.8A)$$

$$\text{and } -H_{22}^{-1} H_{21} = J_{21} J_{11}^{-1} \qquad (4.8B)$$

From which, by comparing with 4A and B, it is seen that the reduced matrices may be alternatively expressed as

$$H_r = J_1^{-1} \qquad (4.9A)$$

$$\text{and } P_r = [\, I \;\vdots\; J_{11}^{-1} J_{21}^{t} \,]\, P \begin{bmatrix} I \\ \cdots\cdots\cdots \\ J_{21} J_{11}^{-1} \end{bmatrix} \qquad (4.9B)$$

($J_{21}^{t} = J_{21}$ transposed)

J_{11} and J_{21} are, of course, submatrices of the inverse of matrix H; J_{11} being symmetric.

A simple physical argument which can be used to derive the reduction process is as follows:-

Assume that the solution is to be approximated by a linear combination of the steady state temperatures corresponding to thermal forces at the master positions only. From the steady state equation, this implies a relationship between the complete set of variables and the master set, defined by

$$\begin{bmatrix} \phi_1 \\ \cdots \\ \phi_2 \end{bmatrix} = \begin{bmatrix} I \\ \cdots\cdots \\ J_{21}\; J_{11}^{-1} \end{bmatrix} \phi_1 \qquad (4.10)$$

If we substitute this into the eigenvalue equation 2.4, and further make the decision that we will now satisfy the equation exactly at the master points, but only approximately elsewhere, then

$$\left(\lambda \begin{bmatrix} I \\ \cdots\cdots \\ J_{21}\; J_{11}^{-1} \end{bmatrix} - H^{-1} P \begin{bmatrix} I \\ \cdots\cdots \\ J_{21}\; J_{11}^{-1} \end{bmatrix} \right) \phi_1 = \begin{bmatrix} 0 \\ \cdots \\ \varepsilon \end{bmatrix} \qquad (4.11)$$

where ε is a matrix of errors.

But since $H^{-1} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$ and $J_{12} = J_{21}^{t}$

then we can split 4.11 into two equations, the second of which is discarded for practical computations, thus

$$\left(\lambda I - [J_{11}\; J_{21}^{1}]\, P \begin{bmatrix} I \\ \cdots\cdots \\ J_{21}\; J_{11}^{-1} \end{bmatrix} \right) \phi_1 = 0 \qquad (4.12)$$

$$\left(\lambda\, [J_{21}\; J_{11}^{-1}] - [J_{21}\; J_{22}]\, P \begin{bmatrix} I \\ \cdots\cdots \\ J_{21}\; J_{11}^{-1} \end{bmatrix} \right) \phi_1 = \varepsilon \qquad (4.13)$$

Equation 4.12, slightly rearranged gives

$$\left(\lambda I - J_{11}\, [I \,\vdots\, J_{11}^{-1}\; J_{21}^{1}]\, P \begin{bmatrix} I \\ \cdots\cdots \\ J_{21}\; J_{11}^{-1} \end{bmatrix} \right) \phi_1 = 0 \qquad (4.14)$$

or $\quad (\lambda I - H_r^{-1} P_r)\, \phi_1 = 0 \qquad (4.15)$

which is the same result for the reduced matrices equations 4.9.

The values for λ and ϕ_1 obtained from 4.15 are not exactly the same as would be derived from the full equations 2.4, because the basic relationship (equation 4.10) leading to the reduction, imposes constraints which prevent the temperatures from reaching precisely their correct values at the salve points. However, by making a good choice for the master variables, values of sufficient accuracy for practical purposes may be obtained for the lower order modes, with a comparatively small number of variables.

5. Transient Solutions using Reduced Eigenvalues

Let us suppose that we are dealing with problems in which only the lower order eigenvectors and eigenvalues make a significant contribution to the solution. In that case we reduce equation 3.1 by using the fundamental relationship of equation 4.10.

$$H \begin{bmatrix} I \\ \cdots\cdots \\ J_{21}\; J_{11}^{-1} \end{bmatrix} \phi_1 + P \begin{bmatrix} I \\ \cdots\cdots \\ J_{21}\; J_{11}^{-1} \end{bmatrix} \dot{\phi}_1 + F = 0 \qquad (5.1)$$

where ϕ_1 and $\dot{\phi}_1$ now contain entries for the master variables only; hence the use of suffix 1.

Multiplying by H^{-1} gives

$$\begin{bmatrix} I \\ \cdots\cdots \\ J_{21}\ J_{11}^{-1} \end{bmatrix} \phi_1 + \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} P \begin{bmatrix} I \\ \cdots\cdots \\ J_{21}\ J_{11}^{-1} \end{bmatrix} \dot{\phi}_1 + \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} F = 0 \qquad (5.2)$$

As before, if we decide to satisfy the equations at the master points only, we have to solve

$$\phi_1 + J_{11} \left[I \vdots J_{11}^{-1} J_{12} \right] P \begin{bmatrix} I \\ \cdots\cdots \\ J_{21}\ J_{11}^{-1} \end{bmatrix} \dot{\phi}_1 + [J_{11}\ J_{12}] F = 0 \qquad (5.3)$$

or $$\phi_1 + H_r^{-1} P_r \dot{\phi}_1 + [J_{11}\ J_{12}] F = 0 \qquad (5.4)$$

But, if we have calculated the eigenvalues λ_r and eigenvectors θ of the reduced matrices, we can make the substitution

$$\left.\begin{aligned} \theta &\simeq \phi_1 \\ \theta \lambda_r &\simeq H_r^{-1} P_r \phi_1 \end{aligned}\right) \qquad (5.5)$$

and also use equations 3.4 and 3.5 giving

$$\theta (L + \lambda_r \dot{L}) + [J_{11}\ J_{12}] F = 0 \qquad (5.6)$$

finally we decompose the forcing term into modal components, i.e.

$$[J_{11}\ J_{12}] F = \theta Q$$

hence $$Q = \theta^{-1} [J_{11}\ J_{12}] F \qquad (5.7)$$

and so finally end up with

$$\theta (L + \lambda_r L + Q) = 0 \qquad (5.8)$$

Since θ is a set of eigenvectors, the matrix is non singular, and so we conclude it to be necessary that

$$L + \lambda_r L + Q = 0 \qquad (5.9)$$

This is exactly the same set of equations as was obtained for the full eigenvalue solution, except that they now refer to a reduced number of variables. The reduced system will only give acceptable accuracy, however, if a sufficient number of master variables has been selected to allow all the important thermal modes to appear. For example, in order to produce good results for early times in a transient it would be necessary to have a large number of master variables so that some higher order, rapidly decaying modes were captured in the analysis. For times later in a transient, a much smaller number of variables could be used because only a few of the lower order, slowly decaying modes would be significant.

6. Example of Reduced Eigenvalue Solutions

The first example (Fig. 1) is of heat conduction along a bar insulated everywhere except at the end $x = 0$. The temperature at this end varies linearly from zero at $t = 0$ to $500^{\circ}F$ at $t = 0.1$ hours, after which it remains constant. For calculation purposes the bar was divided into 11 unequal finite elements. Temperature distributions were calculated using firstly all 12 nodes, then using the reduced eigenvalue approach with 6 and then 3 degrees of freedom. Figs. 2 to 6 show some typical results. After half an hour the 3 degrees of freedom solution is unsatisfactory; a negative temperature appearing at the insulated end. At later times the results improve in accuracy. Comparison was also made with an analytic solution which was for all practical purposes identical to the 12 degree of freedom finite element results, plotted as full lines in the figures.

The second illustration is really the same bar problem as the previous one, but this time, the bar is idealised as a single solid 20 node brick type of finite element, (see Fig. 7). In this case, we know that the temperature distribution is constant across any transverse cross-section. In order to represent this feature by linear combinations of unit thermal forces, it was found to be necessary to retain all the nodes in the analysis, otherwise the calculation produced considerable deviations from the desired constant temperatures. Thus, it seemed that for this particular problem the reduced eigenvalue approach based on unit thermal forces could not be used. However the problem did suggest a modification to the reduction process which made use of our pre-knowledge about the shapes of temperature distribution which the calculation must describe. This modification consists of manually writing down an arbitrary transformation between the complete set of temperatures and the master set, as an alternative to deriving the transformation from unit thermal forces.

7. The Eigenvalue Method with an Arbitrary Reduction Transformation

The starting point is to write down a matrix relationship between the complete set, ϕ, and the master set of temperatures, ϕ_1, as follows

$$\phi = \begin{bmatrix} B_1 \\ \cdots \\ B_2 \end{bmatrix} \phi_1 \qquad (7.1)$$

The matrix B is partitioned into temperatures at master nodes, and temperatures at slave nodes. The master nodes are those at which we choose to satisfy the heat conduction equations exactly. The columns of B are simply any temperature distributions which, as linear combinations, are considered capable of representing the correct shapes.

Following the same procedures as described previously, it is found that the reduced eigenvalue problem becomes

$$(\lambda_r I - H_r^{-1} P_r)\theta = 0 \qquad (7.2)$$

where $H_r = B_1$ (7.3)

and $$P_r = [J_{11} \; J_{12}] \, P \begin{bmatrix} B_1 \\ \cdots \\ B_2 \end{bmatrix} \qquad (7.4)$$

After solving the eigenvalue problem, the transient equations become, as before

$$L + \lambda_r \dot{L} + Q = 0 \qquad (7.5)$$

but now we calculate Q using

$$Q = \theta^{-1} B_1^{-1} [J_{11} \; J_{12}] F \qquad (7.6)$$

The standard unit thermal force reduction transformation can be seen to be a special case of matrix B, corresponding to

$$B_1 = I \text{ and } B_2 = J_{21} J_{11}^{-1} \qquad (7.7)$$

The method has been tried out on the previously mentioned 20 node brick problem. Three basic distributions were used, all having constant temperatures across any transverse section, as illustrated in Fig. 8. The first two varied from zero at one end to unity at the other. The third distribution consisted of a sine curve, zero at both ends. One master node was selected at each end, and one from the centre cross-section.

The problem was therefore reduced from 20 degrees of freedom to 3. Some results from the transient calculation are compared with the analytic solution in Figs. 9 to 13. These show that 3 degrees of freedom gives surprisingly good results for times greater than one hour. As a further test the calculation was repeated using only two degrees of freedom so that the lengthwise temperature distribution was restricted to linear variations. These results are also shown on Figs. 9 to 13. Even with only two degrees of freedom the temperature at the insulated end usually has an error of less than 5% of the temperature peak.

8. <u>Conclusion</u>

The examples, although almost trivial problems, demonstrate that the eigenvalue reduction process can give useful approximate solutions with comparatively small numbers of variables. Hence the cost of very large, complex problems can be reduced if some degree of approximation is permissible. This is particularly true when pre-knowledge of the shapes of the final solution, allows a reduction transformation matrix to be constructed manually, rather than by means of a unit force steady state calculation.

An example of the sort of situation where the approach might be useful is in the design of a basically axisymmetric component, which has to be checked for thermal distortion. The component may have a substantial amount of non-axisymmetric detail, but nevertheless a purely axisymmetric finite element model would be used in the early stages to provide quick and cheap temperature information for design purposes. This initial work would be sufficient to describe the peak local stresses occurring during thermal transients, but because of non-axisymmetric parts such as, say nozzles or axial flanges, a full three dimensional analysis could well be necessary as a distortion check on the final design. The temperature distribution calculation in such a case could benefit greatly by the knowledge gained from the early calculations. It should be possible, in the light of the axisymmetric calculations, to construct a matrix B with the minimum number of columns consistent with allowing sufficient freedom for the temperatures to attain their correct shape.

To illustrate this point further, consider the solid finite element example again. Suppose that this was not a single element, but a part of a larger structure, of such a shape that several elements were necessary to represent its geometry properly. Suppose also that it was known that at the most interesting time in the thermal transient the temperature in this part could be regarded as constant in two directions and perhaps varying slightly in the third direction. Then as has been shown, only 3 columns in the reduction matrix, B, would be required to describe the temperature distribution adequately, even though the original finite element model of the part may require a large number of nodes. It is unlikely that a reduction matrix based on three unit thermal forces could produce results anywhere near as good as three columns written down by the engineer.

One final comment should be made. Whenever the eigenvalue economiser method is to be used, inevitably an engineering judgement must be made in choosing the number and positions of master nodes. That such decisions can be made at all implies some knowledge about what the final results should look like. The construction of an intuitive B matrix seems a much more direct way of putting that knowledge into an analysis then relying on temperature distributions due to unit thermal forces.

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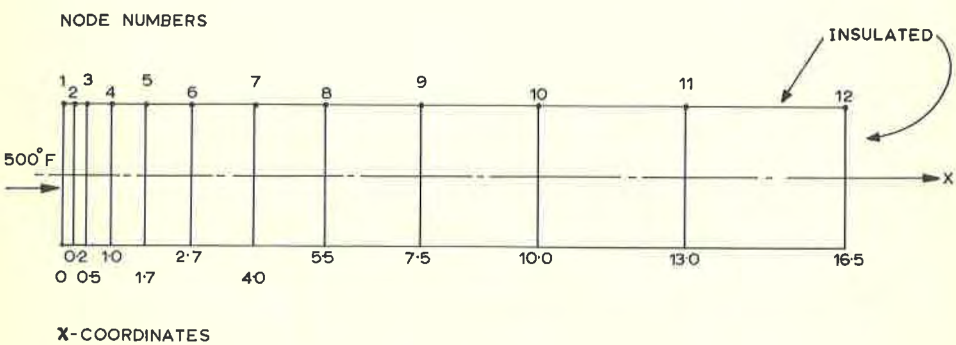
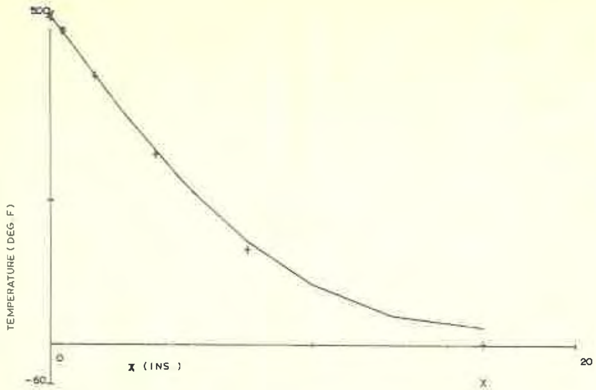
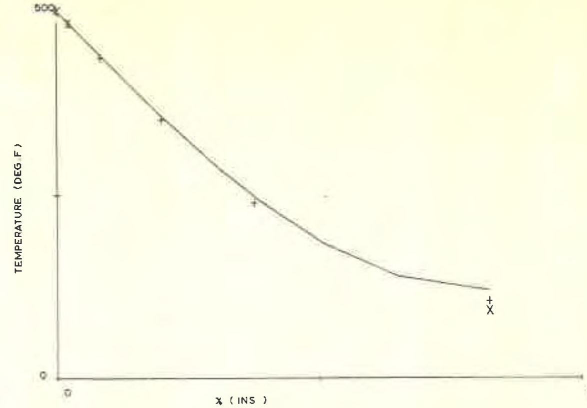
(2) WRIGHT G.C., MILES G.A., An Economical Method for Determining the Smallest Eigenvalues of Large Linear Systems, Int. Journal for Numerical Methods in Engineering, Vol. 3, pp.25-33 (1971).



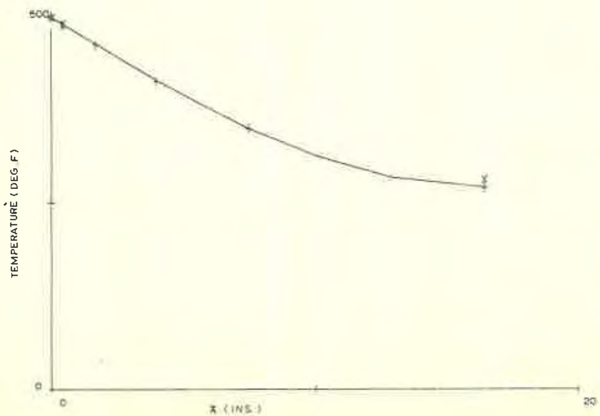
(1) One dimensional heat conduction example.



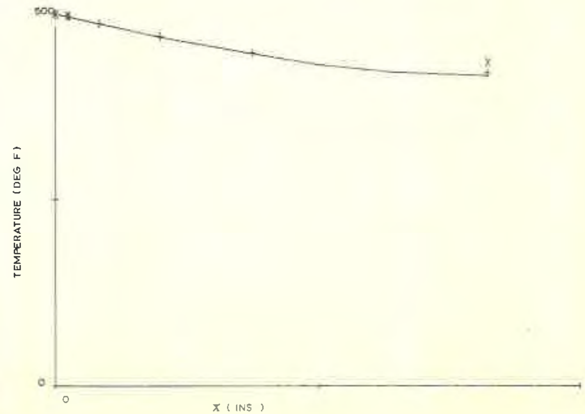
(2) Transient temperature in bar after 0.5 hours.
x = 3 degrees of freedom ; + = 6 degrees of freedom



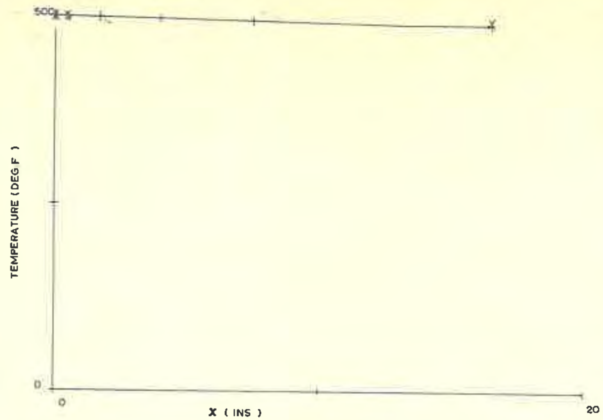
(3) Transient temperature in bar after 1.01 hours.
x = 3 degrees of freedom ; + = 6 degrees of freedom



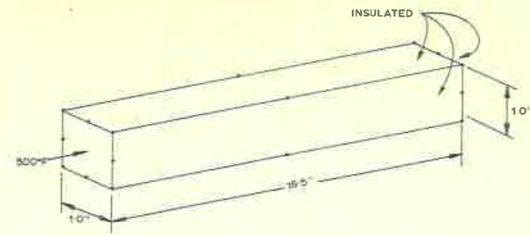
(4) Transient temperature in bar after 2.01 hours.
x = 3 degrees of freedom ; + = 6 degrees of freedom



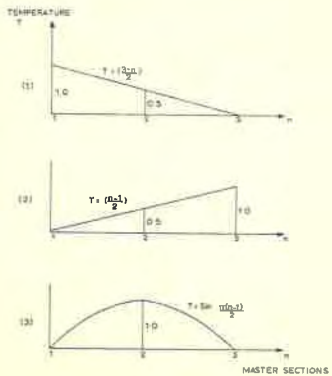
(5) Transient temperature in bar after 4.01 hours.
x = 3 degrees of freedom ; + = 6 degrees of freedom



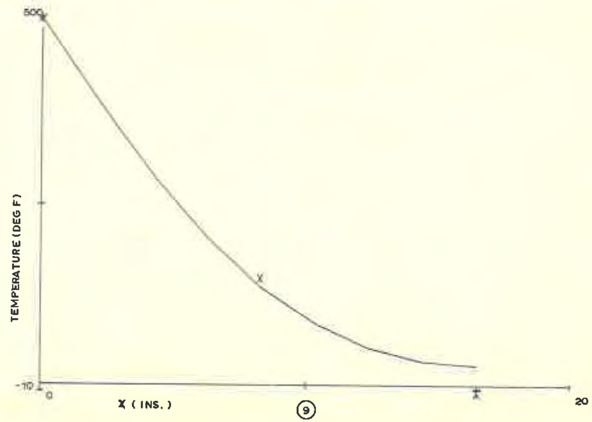
(6) Transient temperature in bar after 8.01 hours.
x = 3 degrees of freedom ; + = 6 degrees of freedom



(7) The bar heat conduction problem represented as a 20 node solid brick element.

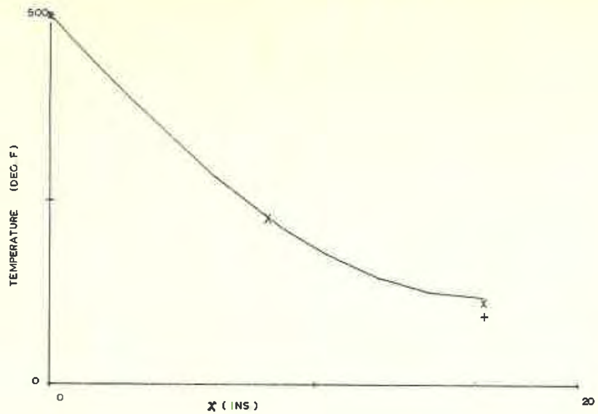


(8) Basic temperature distributions used in matrix B for three-dimensional thermal transient problem.

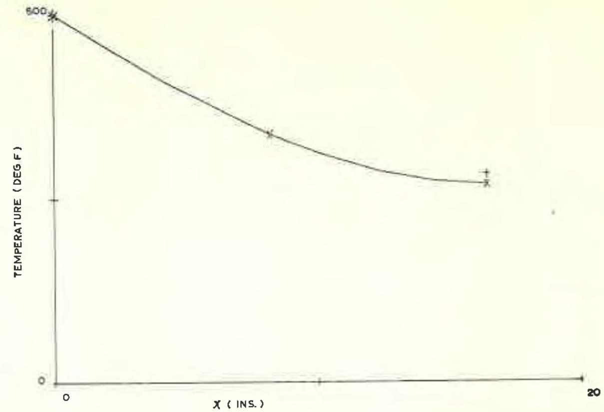


(9) Transient temperature in three-dimensional element after 0.5 hours.
x = 3 degrees of freedom ; + = 2 degrees of freedom

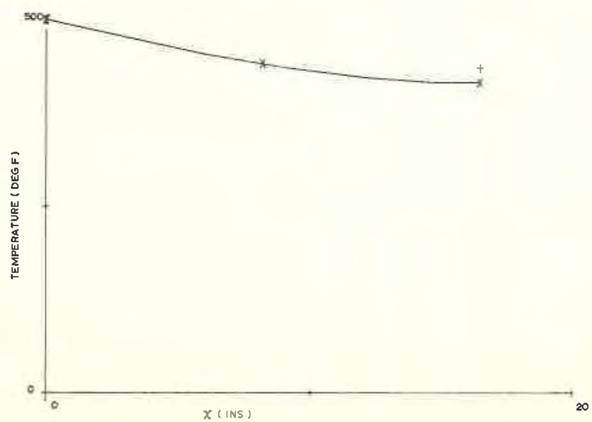
L 1/2



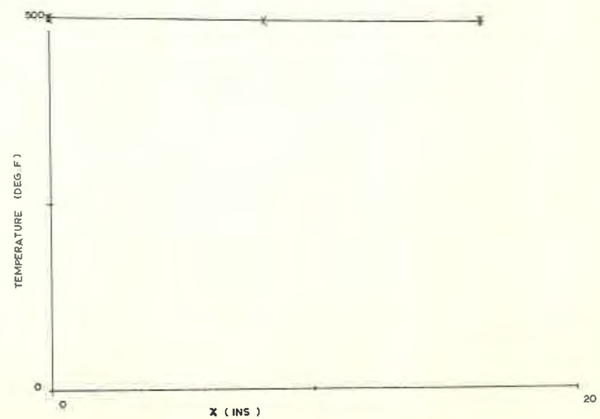
(10) Transient temperature in three-dimensional element after 1.0 hours.
x = 3 degrees of freedom ;   + = 2 degrees of freedom



(11) Transient temperature in three-dimensional element after 2.0 hours.
x = 3 degrees of freedom ;   + = 2 degrees of freedom



(12) Transient temperature in three-dimensional element after 4.0 hours.
x = 3 degrees of freedom ;   + = 2 degrees of freedom



(13) Transient temperature in three-dimensional element after 8.0 hours.
x = 3 degrees of freedom ;   + = 2 degrees of freedom

L 1/2