A METHOD OF SOLUTION OF THE ELASTIC-PLASTIC THERMAL STRESS PROBLEM

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SUMMARY

The purpose of the work is an improvement of the numerical technique for calculating the thermal stress distribution in an elastic-plastic structural element. The work consists of two parts. In the first a new method of solution of the thermal stress problem for the elastic-plastic body is presented. In the second a particular numerical technique, based on the above method, for calculating the stress and strain fields is proposed.

A new mathematical approach consists in treating the stress and strain fields as mathematical objects defined in the space-time domain. The methods commonly applied use the stress and strain fields defined in the space domain and establish the relations between them at a given instant t. They reduce the problem to the system of ordinary differential equations with respect to time, which are usually solved with a step-by-step technique. The new method reduces the problem to the system of nonlinear algebraic equations. In this way we avoid the numerical problems connected with convergence and stability of solution, though we need larger computer capacity for calculation.

In the work the Hilbert space of admissible tensor fields is constructed. This space is the orthogonal sum of two subspaces: of statically admissible and kinematically admissible fields. Two alternative orthogonality conditions, which correspond to the equilibrium and compatibility equations with the appropriate boundary conditions, are derived. They directly imply two methods for finding the unknown stress field. Consequently, two sequences of finite-dimensional spaces where the approximate solution is to be found are introduced. Then the orthogonality condition can be presented in the form of a system of nonlinear algebraic equations.

The obtained system can be solved with a method of successive iterations. Then every iteration requires the calculation of the plastic component of strain field according to the plastic flow law.

The results of the work are to be used for construction of the computer program for calculation the stress and strain fields in the elastic-plastic body with a prescribed temperature field in the interior and appropriate displacement and force conditions on the boundary.
1. INTRODUCTION

The elastic-plastic model of material is frequently used for the stress analysis of the nuclear reactor components. The classical approach to the thermal stress problem in the elastic-plastic body consists in calculation of the strain and stress distribution for successive time values with a step-by-step technique.

In the present work the strain and stress distribution may be calculated in arbitrary subregion of the space-time domain. In every step of calculation all field values in the considered space-time domain are taken into account. This feature of the method enables us to develop a range of iterative numerical techniques.

The method is based on the orthogonality conditions in the Hilbert space. The concept of the Hilbert space approach to the residual stress problem, based on the variational approach presented by Koiter in [1], was given by Rafalski in [2]. The construction of the Hilbert space is similar to that given by Rafalski in [3]. The proposed numerical technique uses the finite element concept described by Zienkiewicz in [4].

2. NOMENCLATURE

\[ x = (x_1, x_2, x_3) \] Cartesian coordinates
\[ t \] time
\[ \Omega = V \times [0, \infty) \] space-time domain
\[ n = n_i = n_i(x) \] normal unit vector
\[ \Theta = \Theta(x, t) \] temperature field
\[ u = u_i = u_i(x, t) \] displacement field
\[ \varepsilon = \varepsilon_{ij} = \varepsilon_{ij}(x, t) \] strain field
\[ \sigma = \sigma_{ij} = \sigma_{ij}(x, t) \] stress field

Throughout the paper the index notation for the derivatives with respect to the Cartesian coordinates and the summation convention is used.

For example:
\[ \sigma_{ij}^{\prime \prime} = \frac{\partial \sigma_{ij}^{\prime \prime}}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_1} + \frac{\partial \sigma_{ij}}{\partial x_2} + \frac{\partial \sigma_{ij}}{\partial x_3} \]

3. FORMULATION OF THE PROBLEM

We shall consider a closed and bounded region \( \bar{V} \) in three-dimensional Euclidean space which has the interior \( V \) and sectionally differentiable boundary \( \partial V \) with the normal unit vector \( n \). The boundary \( \partial V \) consists of two parts: \( \partial V_k \) and \( \partial V_s \).

The domain of definition of fields appearing in this work is four-dimensional space-time \( \bar{\Omega} \) which is determined as the Cartesian product of the region \( \bar{V} \) and the time interval \([0, \infty)\). The three-dimensional
boundary $\partial \Omega$ of the space-time $\Omega$ consists of the region $V$ at the initial moment $t=0$ and the boundary $\partial \Omega$ composed of two Cartesian products: $B_k = \partial V_k \times [0, \infty)$ and $B_s = \partial V_s \times [0, \infty)$.

The elastic-plastic material in the space-time $\Omega$ is described by the relation between the stress field $\sigma_{ij}(x,t)$, the elastic component of the strain field $\varepsilon_{ij}^{(e)}(x,t)$ and the temperature field $\Theta(x,t)$

$$\sigma_{ij} = A_{ijkl}^{-1} \left( \varepsilon_{kl}^{(e)} - \alpha_{kl} \Theta \right) \tag{1}$$

and the formula for calculating the plastic component of the strain field $\varepsilon_{ij}^{(p)}(x,t)$ at the point $x$ provided that the stress and temperature history is given

$$\varepsilon_{ij}^{(p)} = \varepsilon_{ij}^{(p)}(\sigma_{kl}, \Theta) \tag{2}$$

Here $A_{ijkl}$ is the symmetric tensor of elastic coefficients, described by Koiter in [1], and $\alpha_{ij}$ is the tensor of thermal expansion coefficients. We shall assume that the plastic strain field $\varepsilon_{ij}^{(p)}$ vanishes in the region $V$ at the initial moment. Hence every pair of fields $\varepsilon_{ij}^{(e)}$ and $\Theta$ determines uniquely the plastic strain field in the space-time $\Omega$.

The elastic-plastic problem of thermal stress is primarily formulated as follows:

Find the stress field $\sigma$ defined in the space-time $\Omega$, which satisfies the equation of equilibrium

$$\sigma_{ij} n_j = 0 \quad \text{in the domain } \Omega \tag{3}$$

and the force boundary condition

$$\sigma_{ij} n_j = F_i \quad \text{on the boundary } B_s \tag{4}$$

such that the total strain field $\varepsilon = \varepsilon^{(e)} + \varepsilon^{(p)}$ calculated from eqs. (1) and (2) can be derived from the displacement field $u$ with the compatibility equation

$$\varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \quad \text{in the domain } \Omega \tag{5}$$

where the displacement field $u$ satisfies the displacement boundary condition

$$u_i = U_i \quad \text{on the boundary } B_k \tag{6}$$

Here $\Theta$ is the temperature field prescribed in the space-time $\Omega$ and $F_i$ and $U_i$ are boundary tractions and boundary displacements prescribed.
on \( B_e \) and \( B_k \), respectively.

In the present work we shall make use of the solution \( 6^0 \) of the corresponding elastic problem. The field \( 6^0 \), defined in \( \Omega \), satisfies the equilibrium equation \( \sigma_{ij}^{0'} \cdot \beta = 0 \) in \( \Omega \) with the boundary condition \( \sigma_{ij}^{0'} n_j = F_i \) on \( B_e \) and the compatibility equation \( A_{ijkl} \sigma_{kl}^{0} + \alpha_{ij} \theta = \frac{1}{2} (u_{i,j}^{0} + u_{j,i}^{0}) \) in \( \Omega \) with the boundary condition \( u_i^{0} = U_i \) on \( B_k \).

4. SPACE OF ADMISSIBLE FIELDS

To solve the elastic-plastic problem of thermal stress we shall use the method of orthogonal projection in the Hilbert space. To construct the Hilbert space \( \mathcal{H} \) we introduce two scalar products

\[
(6, \tilde{6}) \triangleq \int_{\Omega} \sigma_{ij}(x,t) \cdot \tilde{\sigma}_{ij}(x,t) \, E_i(t) \, dx \, dt \tag{7}
\]

\[
(6, \tilde{6})_A \triangleq \int_{\Omega} A_{ijkl} \sigma_{ij}(x,t) \cdot \tilde{\sigma}_{kl}(x,t) \, E_i(t) \, dx \, dt \tag{8}
\]

which are primarily defined in the space \( \mathcal{F} \) of all smooth symmetric tensor fields defined in \( \Omega \). Here \( E_i(t) = \int_{-\infty}^{\infty} \frac{1}{2 \pi} e^{-zt} \, dz \) and the symbol \( \ast \) denotes the convolution operation

\[
\sigma_{ij}(x,t) \ast \tilde{\sigma}_{kl}(x,t) \triangleq \frac{d}{dt} \int_{0}^{t} \sigma_{ij}(x,t') \, \tilde{\sigma}_{kl}(x,t-t') \, dt' \tag{9}
\]

In the sequel we shall use the identity

\[
(6, \tilde{6})_A = \int_{B} \int_{\Omega} u_i(x,t) \ast \tilde{\sigma}_{ij}(x,t) \, n_j(x) \, E_i(t) \, dx \, dt \nonumber
\]

\[\quad - \int_{\Omega} \int_{S_2} u_i(x,t) \ast \tilde{\sigma}_{ij,j}(x,t) \, E_i(t) \, dx \, dt \tag{10}\]

which is valid in the space \( \mathcal{F} \) provided that the field \( 6 \) can be derived by means of

\[
A_{ijkl} \sigma_{kl} = \frac{1}{2} (u_{i,j} + u_{j,i}) \tag{11}
\]

from a smooth displacement field \( u \).

The Hilbert space of admissible fields \( \mathcal{H} \) consists of all symmetric tensor fields \( 6 \) for which the "energy" norm \( \|6\|_A \triangleq \int_{A} (6, \tilde{6})_A^{1/2} \) exists. Every admissible field \( 6 \) is uniquely represented by the set of real numbers \( (6, \tilde{6}) \) corresponding to all smooth fields \( 6 \) from \( \mathcal{F} \).
The field $\mathbf{e}$ from $H$ is called kinematically admissible if there exists the vector field $\mathbf{u}$ such that the equation

$$
(\mathbf{e}, \mathbf{\tilde{e}})_A = \int_{B_B} u_i(x,t) \star \mathbf{\tilde{\sigma}}_{ij}(x,t) n_j(x) E_i(t) \, dx \, dt
$$

$$
- \int_{\Omega} u_i(x,t) \star \mathbf{\tilde{\sigma}}_{ij} n_j(x,t) E_i(t) \, dx \, dt
$$

holds true for every smooth field $\mathbf{\tilde{e}}$. It follows from the identity (10) that the kinematically admissible field $\mathbf{e}$ satisfies the compatibility equation in the space-time $\Omega$ and the corresponding displacement field $\mathbf{u}$ vanishes on the boundary $B_B$. The subspace of all kinematically admissible fields from $H$ will be denoted by $K$.

Similarly we introduce the subspace $S$ of all statically admissible fields. The field $\mathbf{e}$ is called statically admissible if the equation

$$
(\mathbf{e}, \mathbf{\tilde{e}})^S_A = \int_{B_K} \tilde{u}_i(x,t) \star \mathbf{\tilde{\sigma}}_{ij}(x,t) n_j(x) E_i(t) \, dx \, dt
$$

holds true for every smooth displacement field $\tilde{\mathbf{u}}$, where the corresponding stress field $\mathbf{\tilde{e}}$ is determined with the equation (11). It follows from the identity (10) that the statically admissible field $\mathbf{e}$ satisfies the equilibrium equation in the space-time $\Omega$ and the corresponding force vector $\mathbf{\tilde{\sigma}}_{ij} n_j$ vanishes on the boundary $B_B$.

Making use of the identity (10) we can prove that the space of admissible fields $H$ is composed of the subspaces $K$ and $S$, which are $(,)_A$-orthogonal to each other. Hence every admissible field $\mathbf{e}$ can be decomposed in unique way into the sum of the kinematically admissible field $\mathbf{e}^{(k)}$ and the statically admissible field $\mathbf{e}^{(s)}$

$$
\mathbf{e} = \mathbf{e}^{(k)} + \mathbf{e}^{(s)} \quad \text{where} \quad \mathbf{e}^{(k)} \in K \quad \text{and} \quad \mathbf{e}^{(s)} \in S
$$

The components of the field $\mathbf{e}$ are to be obtained by the orthogonal projection of $\mathbf{e}$ onto the appropriate subspace

$$
\mathbf{e}^{(k)} = \mathbf{\Pi}^{(k)} \mathbf{e}
$$

$$
\mathbf{e}^{(s)} = \mathbf{\Pi}^{(s)} \mathbf{e}
$$

where $\mathbf{\Pi}^{(k)}$ and $\mathbf{\Pi}^{(s)}$ are the orthogonal projection operators onto the subspaces $K$ and $S$, respectively.
5. METHOD OF SOLUTION

Making use of the geometric properties of the space of admissible fields and using the solution \( \mathbf{6}^0 \) of the corresponding elastic problem we can present the elastic-plastic problem in the form:

Find in the space \( \mathbf{H} \) the stress field \( \mathbf{6} \) such that \( \mathbf{6} - \mathbf{6}^0 \) belongs to the subspace \( \mathbf{S} \) of statically admissible fields and \( \mathbf{6} + A^{-1} \mathbf{e}^{(p)}(\mathbf{6}, \mathbf{\varphi}) - \mathbf{6}^0 \) belongs to the subspace \( \mathbf{K} \) of kinematically admissible fields.

The existence and uniqueness of the solution of the above problem depend on the properties of the nonlinear operator \( f_\varphi \) defined by

\[
f_\varphi(\mathbf{6}) \overset{df}{=} \mathbf{6} + A^{-1} \mathbf{e}^{(p)}(\mathbf{6}, \mathbf{\varphi})
\]

(17)

To assure the existence and uniqueness of the solution we shall assume that the operator \( f_\varphi \) maps the space \( \mathbf{H} \) into itself, is bounded and continuous and satisfies the inequality

\[
( f_\varphi(\mathbf{6}) - f_\varphi(\mathbf{\tilde{6}}), \mathbf{6} - \mathbf{\tilde{6}} )_A \geq \| \mathbf{6} - \mathbf{\tilde{6}} \|^2_A
\]

(18)

for every \( \mathbf{6} \) and \( \mathbf{\tilde{6}} \) from \( \mathbf{H} \). A wide class of operators \( f_\varphi \) describing the plastic flow complies with the above requirements. For those operators we propose two methods of solution of the elastic-plastic problem.

The first method consists in seeking the solution in the form \( \mathbf{6} = \mathbf{6}^0 + \psi \), where the unknown field \( \psi \) from the subspace \( \mathbf{S} \) is to be found from the equation

\[
\bigcap_{(\mathbf{S})} \left[ f_\varphi(\mathbf{6}^0 + \psi) - \mathbf{6}^0 \right] = 0
\]

(19)

The second method consists in seeking the solution in the form \( \mathbf{6} = f_\varphi^{-1}(\mathbf{6}^0 + \psi) \), where the unknown field \( \psi \) from the subspace \( \mathbf{K} \) is to be found from the equation

\[
\bigcap_{(\mathbf{K})} \left[ f_\varphi^{-1}(\mathbf{6}^0 + \psi) - \mathbf{6}^0 \right] = 0
\]

(20)

6. APPROXIMATE SOLUTION

Following the considerations in the previous section we propose two alternative forms of the approximate solution. The first form, corresponding to the first method, expresses the approximate stress field \( \mathbf{6}' \) as a linear combination of the prescribed statically admissible fields \( \psi_q \)

\[
\mathbf{6}' = \mathbf{6}^0 + \sum_{q=1}^{n} a_q \psi_q
\]

(21)
The set of fields $\psi_q$, $q=1,2,\ldots,n$ establishes the basis of $n$-dimensional subspace $S_n$ enclosed in subspace $S$. The orthogonality condition (19) for the subspace $S_n$ takes the form of the system of nonlinear algebraic equations

$$\left(\int_{V_0}(\sigma^0 + \sum_{q=1}^{n} a_q \psi_q) \cdot \psi_r\right)_A = \left(\sigma^0, \psi_r\right)_A \quad r=1,2,\ldots,n$$

(22)

with the unknown coefficients $a_q$, $q=1,2,\ldots,n$.

The second form of approximate solution, corresponding to the second method, expresses the stress field $\sigma''$ in terms of the prescribed kinematically admissible fields $\psi_q$

$$\sigma'' = \int_{V_0}^{-1}(\sigma^0 + \sum_{q=1}^{n} b_q \psi_q) \quad (23)$$

Now the set of fields $\psi_q$, $q=1,2,\ldots,n$ establishes the basis of $n$-dimensional subspace $K_n$ enclosed in $K$. The orthogonality condition (20) for the subspace $K_n$ takes the form of the system of nonlinear algebraic equations

$$\left(\int_{V_0}(\sigma^0 + \sum_{q=1}^{n} b_q \psi_q) \cdot \psi_r\right)_A = \left(\sigma^0, \psi_r\right)_A \quad r=1,2,\ldots,n$$

(24)

with the unknown coefficients $b_q$, $q=1,2,\ldots,n$.

7. NUMERICAL METHOD

The proposed method is intended to be used in conjunction with the standard finite element technique (described by Zienkiewicz in [4]) for calculating the thermal stress in elastic body. To construct the fields $\psi_q$, $q=1,2,\ldots,n$ for the first method or the fields $\psi_q$, $q=1,2,\ldots,n$ for the second method we can use the geometrical idealisation (division into elements) and the shape function already established for calculation of the stress $\sigma^0$. Then dividing the appropriate time interval into subintervals we obtain the mesh of space-time finite elements. Introducing the modified space-time shape function we can express the field $\psi$ or $\psi'$ in terms of its values $a_q$ or $b_q$, respectively, at the nodes $q=1,2,\ldots,n$.

The resulting system of nonlinear algebraic equations can be solved with an iteration method. One can start the iterative process from the elastic solution $\sigma^0$. If the number of nodes is too large for the computer capacity one can modify the nodal values in one iteration only in a chosen subspace of the considered space-time, while the rest of nodal values is kept constant. The particular choice of this subspace generates
the numerical technique.

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REFERENCES


