A GENERAL SHAKEDOWN THEOREM FOR ELASTIC/PLASTIC BODIES WITH WORK HARDENING

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SUMMARY

In recent years the design of metallic structures under variable loading has been assisted by the application of Melan's lower bound theorem for the shakedown of an elastic/perfectly plastic structure. The design codes for both portal frames and pressure vessels have taken account of such calculations.

The theory of shakedown suffers from two defects, geometry changes are ignored and the material behaviour is described by a perfectly plastic constitutive relationship which includes neither work hardening nor the Bauschinger effect. This paper is concerned with the latter problem. We derive a very general lower bound shakedown theorem for an arbitrary time-independent material in terms of functional properties of the constitutive relationship. The theorem is then applied to perfect, isotropic and kinematic hardening plasticity. It is shown that the result for all three constitutive relationships may be related to each other through certain extremal stress histories. In each case the theorem is expressed in terms of a stress history of the form

$$\sigma = \hat{\sigma} + \rho$$

where $\hat{\sigma}$ denotes an equilibrium stress field and ρ a time constant residual stress field. The state of hardening in each case is defined by an external stress history.

As well as providing a sufficient condition for shakedown, the theory also provides bounds of the deflection of the structure in the process of reaching the shakedown state. The bounds are discussed and derived for two simple beam problems. Both static and dynamic problems are considered.

The theory derived in this paper demonstrates that shakedown analysis may be extended to a wide range of material behaviour without increasing the complexity of the resulting calculation.

The classical theory of shakedown is concerned with the following problem: for a body of elastic/perfectly plastic material, for what histories of loading will the body suffer both a finite displacement and a finite plastic strain. A sufficient condition for this occurrence was first given in the quasi-static case by Melan [1], who proved that if there existed a time invariant residual stress field $\bar{\rho}_{ij}$ such that the history

$$\sigma_{ij}^{*} = \hat{\sigma}_{ij} + \bar{\rho}_{ij} \tag{1}$$

at no time violated the yield condition then the structure would eventually shakedown. Here $\hat{\sigma}_{ij}$ denotes the linear elastic solution corresponding to the loading history. The proof of Melan's theory was much simplified by Symonds [2], and the complete theory for quasi-static loading was reviewed by Koiter [3]. Since Koiter's review paper the advances in the basic theory of shakedown have not been extensive. The proof for dynamic loading has been given by Ho [4], and an extension to a broader class of plastic relationships was recently given by Maier and Vitiello [5].

Although the shakedown theorem of Melan yields a sufficient condition for shakedown, the accumulation of plastic strain and displacement which occurs during the history of loading is not known, and depends upon the details of the loading history (and not just the extreme values which usually define whether shakedown occurs), and also any initial residual stresses. A displacement bound was derived by Ponter [6] and a similar result was given by Capurso [7], which provides some indication of the accumulation of displacement to shakedown.

There remains a much broader problem which includes classical shakedown as a special case. For a body composed of an arbitrary time independent material, which suffers both elastic and inelastic strains, for what histories of loading will the displacement and energy dissipated tend asymptotically to a limiting value? The formulation of the solution to this problem has both a purely theoretical and a practical value. Perfect plasticity provides a poor approximation to observed behaviour under many circumstances, especially those that involve reversal of the sign of stress components, and, at the moment, no general theory exists for more appropriate material models.

In this paper a general sufficient condition for shakedown is formulated in terms of a functional property of the constitutive relationship. The theory is derived from general displacement and work bounds derived by the author [8]. A sufficient condition is derived and requires that a certain functional of the constitutive relationship should possess a finite upper bound for histories of stress of the form of eq. (1). When this condition is satisfied, shakedown will occur, and the theory yields both a sufficient condition for shakedown and an upper bound on the displacements induced in the body. The continuum problem may be broadly defined, both quasi-static and dynamic loading being allowed, and the body may possess initial residual stresses and initial velocities.

To provide continuity with known results, the classical Melan theorem is recovered for a perfectly plastic body. Possible applications of the more general result are demonstrated by particularizing the result for linear kinematic hardening. The resulting theorem appears to be easier to apply than the classical result, and examples are included in the full paper.

2. Displacement and Work Bounds

In a previous paper [8] a general bounding theory was described for a wide class of static and dynamic conditions and for an arbitrary inelastic material. In this section, the relevant parts of this theory are reviewed.

The material suffers both linear elastic strains e_{ij} and inelastic strains ϵ'_{ij}, and the total strain ϵ_{ij} is given by

$$\epsilon_{ij} = e_{ij} + \epsilon'_{ij} \quad , \quad e_{ij} = C_{ijk\ell}\,\sigma_{k\ell} \, ,$$

where $C_{ijk\ell}$ denotes the tensor of elastic constants which possesses the usual symmetry properties and gives rise to positive complementary strain energy

$$E(\sigma_{ij}) = \frac{1}{2} C_{ijk\ell}\,\sigma_{ij}\,\sigma_{k\ell} .$$

At time t = 0, inelastic strains are assigned the value zero and we assume that the strain at some subsequent time t is determinate in terms of the history of stress $\sigma_{ij}(\tau)$, for $0 \leq \tau \leq t$. The relationship between $\epsilon'_{ij}(t)$ and $\sigma_{ij}(\tau)$ need not be specified at this stage. The bounds are expressed in terms of certain known histories of stress $\sigma^*_{ij}(t)$, which are related to other possible histories of stress $\sigma_{ij}(t)$ through a functional W:

$$W(\sigma^*_{ij}(t), \sigma_{ij}(t)) = \int_0^T (\sigma^*_{ij} - \sigma_{ij})\,\dot{\epsilon}'_{ij}\,dt \, , \tag{2}$$

where $\dot{\epsilon}'_{ij}$ denotes the inelastic strain rate history resulting from σ_{ij}. We imagine σ^*_{ij} as a prescribed history and find that history $\sigma_{ij} = \bar{\sigma}_{ij}$ which provides a maximum value of W. If such a history exists, then for arbitrary $\sigma_{ij}(t)$,

$$W(\sigma^*_{ij}, \sigma_{ij}) \leq W(\sigma^*_{ij}, \bar{\sigma}_{ij}) = w(\sigma^*_{ij}(t)). \tag{3}$$

As $\bar{\sigma}_{ij}$ is an implicit function of σ^*_{ij} the maximum value of W will be a functional of $\sigma^*_{ij}(t)$ within the internal $0 \leq t \leq T$.

This argument is entirely formal and possesses inherent difficulties. There is no guarantee that W is even bounded and w may not exist. Even if W is bounded, construction of w may present considerable difficulty, but it is sufficient to bound w, i.e. to find some functional w which satisfies the inequality

$$W(\sigma^*_{ij} ; \sigma_{ij}) \leq w(\sigma^*_{ij}). \tag{4}$$

For perfect plasticity, then w may be constructed without difficulty. Consider the convex yield condition $f(\sigma_{ij}) = 0$ and the associated flow rule

$$\dot{\epsilon}'_{ij} = \mu \frac{\partial f}{\partial \sigma_{ij}} .$$

For any allowable state of stress σ^*_{ij} and σ_{ij}, the maximum work principle holds

$$(\sigma^*_{ij} - \sigma_{ij})\,\dot{\epsilon}'_{ij} \leq 0 .$$

Hence for any prescribed history $\sigma^*_{ij}(t)$ for which $f(\sigma^*_{ij}) < 0$ and any allowable history σ_{ij},

$$W \leq 0$$

and clearly $w = 0$. The maximizing history $\bar{\sigma}_{ij}$ is any history which satisfies $f(\bar{\sigma}_{ij}) < 0$ when $f(\sigma^*_{ij}) < 0$ and $\bar{\sigma}_{ij} = \sigma^*_{ij}$ when $f(\sigma^*_{ij}) = 0$.

In the general case consider the following boundary value problem:

At time $t = 0$ a body with density η possesses initial stresses $\sigma_{ij}(0)$ and velocities $\dot{u}_i(0)$. For $t > 0$ a history of loading $P_i(t)$ operates on part of the surface S_T and displacements $u_i(t)$ are induced on the remainder of the surface S_u. Body forces $F_i(t)$ operate through the volume and thermal strains $d_{ij}(t)$ are induced due to changes in temperature.

We denote by $\hat{\sigma}_{ij}$ the linear elastic solution to the problem, but for all the given boundary conditions except that the initial velocities $\dot{u}_i(0)$ need not equal $\dot{u}(0)$. By $\bar{\rho}_{ij}$ we denote any static residual stress field which remains constant in time and gives rise to zero body forces and zero surface traction on S_T.

We now introduce a static stress field σ^T_{ij} which also remains constant in time and is in equilibrium with "dummy" surface tractions T_i on part of S_T. In terms of these quantities an upper bound on the displacement of the body is given by

$$\int_{S_T} T_i\,(u_i(T) - u_i(0))\,dS \;\leq\; B(0) + K(0) + \int_{S_T} T_i\,(\hat{u}_i(T) - \hat{u}_i(0))\,dS + \int_V w(\sigma^*_{ij}(t))\,dV, \tag{5}$$

where $$\sigma^*_{ij} = \hat{\sigma}_{ij} + \bar{\rho}_{ij} + \sigma^T_{ij}, \tag{6}$$

$$B(0) = \int_V E\,(\sigma^*_{ij}(0) - \sigma_{ij}(0))\,dV,$$

$$K(0) = \int_V \tfrac{1}{2}\eta\,(\dot{u}_i(0) - \dot{\hat{u}}_i(0))(\dot{u}_i(0) - \dot{\hat{u}}_i(0))\,dV.$$

The quantity $\hat{u}_i(t)$ denotes the linear elastic displacement of the body (i.e. the solution of the problem assuming $\epsilon'_{ij} = 0$). $B(0)$ and $K(0)$ are quantities which relate to the initial conditions at $t = 0$. The most important feature of this bound is that the inelastic properties of the material appear only through the functional w. The bound is finite provided w is finite everywhere.

We now pose the following problem. Suppose that the history of loading continues indefinitely and hence the terminal time $t = T$ may be made as large as we wish. Under what conditions will the upper displacement bound be finite for any point in the body, i.e. any dummy load T_i? We may make T_i as small as we wish and hence σ^*_{ij} may be made as close to $\hat{\sigma}_{ij} + \bar{\rho}_{ij}$ as we wish (assuming σ^T_{ij} possesses no singularities), and hence we arrive at the following result:

The displacement of the body will be finite for any history of loading for which there exists a finite value of $w(\hat{\sigma}_{ij} + \sigma^T_{ij} + \bar{\rho}_{ij})$ throughout the volume of the body for arbitrary small value of σ^T_{ij}.

It is also possible to show that the work done by the applied loads is also finite under the circumstance and hence this condition defines the situations under which shakedown occurs.

For a perfectly plastic material we immediately recover Melan's theorem. In this case $w = 0$ and is therefore finite provided

$$f(\hat{\sigma}_{ij} + \sigma^T_{ij} + \bar{\rho}_{ij}) \leq 0$$

As σ^T_{ij} may be made arbitrary small this result is equivalent to the condition

$$f(\hat{\sigma}_{ij} + \bar{\rho}_{ij}) < 0.$$

In the dynamic case $\hat{\sigma}_{ij}$ may result from any arbitrary finite initial velocity field $\dot{\hat{u}}_i(0)$, and not necessarily the initial velocities $\dot{u}(0)$ prescribed in the continuum problem.

The general result provides a prescription for a shakedown theorem for an arbitrary time independent material. To demonstrate its applicability to a wider range of behaviour than perfect plasticity, w is computed for linear kinematic hardening in the next section and a new shakedown theorem results.

3. Linear Kinematic Hardening Plasticity

At a state of plastic strain ε'_{ij}, the yield surface is given by

$$f(\sigma_{ij} - c\,\epsilon'_{ij}) = 0, \tag{7}$$

where c denotes a constant. This equation defines a convex surface in stress space which translates from an initial state $\varepsilon' = 0$. For a state of stress lying within the yield surface $f < 0$ and $f > 0$ in the remainder of stress space. The associated flow rule is given by

$$\dot{\epsilon}'_{ij} = \frac{1}{D}\left(\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl}\right)\frac{\partial f}{\partial \sigma_{ij}}, \tag{8}$$

where D is a hardening coefficient and the derivatives are taken at constant strain. From the self-evident expressions

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial \epsilon'_{ij}}\dot{\epsilon}'_{ij} = 0, \tag{9}$$

$$c\frac{\partial f}{\partial \sigma_{ij}} + \frac{\partial f}{\partial \epsilon'_{ij}} = 0,$$

the flow rule may be expressed in the form

$$c'\,\dot{\epsilon}'_{ij} = \left(\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl}\right)\frac{\partial f}{\partial \sigma_{ij}}, \tag{10}$$

where $$c' = c\left(\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}}\right) \tag{11}$$

Corresponding to a given yield surface f = 0, there exists an infinite number of yield functions which differ in their values (but not their sign) when $f \neq 0$. We assume that there exists a member which is a convex function and hence satisfies

$$f(\sigma^1_{kl}) - f(\sigma^2_{kl}) - \left(\frac{\partial f}{\partial \sigma_{ij}}\right)_{\sigma^2_{ij}}(\sigma^1_{ij} - \sigma^2_{ij}) \geqslant 0. \tag{12}$$

For example this conditions is satisfied by the Mises yield function

$$f = \sqrt{\sigma'_{ij}\,\sigma'_{ij}} - \sigma_0 \quad , \quad \sigma'_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\,\sigma_{kk}.$$

In fact this condition is satisfied by any yield function which is homogeneous of degree one in the components of σ_{ij} and defines a surface f = 0 which is convex in stress space.

From the convexity condition we may construct a bound on W. For a history of stress $\sigma^*_{ij}(t)$, suppose there exists a state of plastic strain $\varepsilon'_{ij} = \varepsilon^*_{ij}$ so that $f(\sigma^*_{ij} - c\,\varepsilon^*_{ij}) \leqslant 0$. Consider now any other history of stress and plastic strain $\sigma_{ij}(t)$ and ε'_{ij} which could occur in the material. When $\dot{\varepsilon}'_{ij} \neq 0$ then $f(\sigma_{ij} - c\,\varepsilon'_{ij}) = 0$.

Substituting $$\sigma^1_{ij} = \sigma^*_{ij} - c\,\epsilon^*_{ij} \quad , \quad f(\sigma^1_{ij}) \leqslant 0,$$

and $$\sigma^2_{ij} = \sigma_{ij} - c\,\epsilon_{ij} \quad , \quad f(\sigma^2_{ij}) = 0,$$

at such times into the convexity condition (12) we obtain

$$-\left(\frac{\partial f}{\partial \sigma_{ij}}\right)_{\sigma_{ij},\epsilon'_{ij}}\Big((\sigma^*_{ij} - \sigma_{ij}) - c\,(\epsilon^*_{ij} - \epsilon'_{ij})\Big) \geqslant 0 \tag{13}$$

Further $\frac{1}{c'}\frac{\partial f}{\partial \sigma_{k\ell}}\dot{\sigma}_{k\ell} \geqslant 0$ when $\dot{\varepsilon}'_{ij} \neq 0$ and we obtain from (10), (11) and (13),

$$(\sigma^*_{ij} - \sigma_{ij})\,\dot{\epsilon}'_{ij} \leqslant c\,(\epsilon^*_{ij} - \epsilon'_{ij})\,\dot{\epsilon}'_{ij}. \tag{14}$$

It is clear that inequality (14) remains valid for $\dot{\varepsilon}'_{ij} = 0$ and hence for an arbitrary history of stress σ_{ij}.

Assuming $\varepsilon'_{ij}(0) = 0$ integrating (14) over $0 \leqslant t \leqslant T$ yields

$$W = \int_0^T (\sigma^*_{ij} - \sigma_{ij})\,\dot{\epsilon}'_{ij}\,dt \leqslant c\,\epsilon^*_{ij}\,\epsilon'_{ij}(T) - \tfrac{1}{2}\,c\,\epsilon'_{ij}(T)\,\epsilon'_{ij}(T) \tag{15}$$

As c > 0 we may write

$$\tfrac{1}{2}\,c\,\big(\epsilon^*_{ij} - \epsilon'_{ij}(T)\big)\big(\epsilon^*_{ij} - \epsilon'_{ij}(T)\big) \geqslant 0 \tag{16}$$

and hence combining (15) and (16) we obtain the bound

$$W = \int_0^T (\sigma_{ij}^* - \sigma_{ij})\, \dot{\epsilon}_{ij}'\, dt \leq \frac{1}{2} c\, \epsilon_{ij}^* \epsilon_{ij}^* \tag{17}$$

Hence

$$W(\sigma_{ij}^*) \leq \frac{1}{2} c\, \epsilon_{ij}^* \epsilon_{ij}^* \tag{18}$$

as the history σ_{ij} is arbitrary.

Equality will hold in (18) when there exists some σ_{ij} for which equality holds in (17), but this question is somewhat complex and will not be discussed here.

We may use this result to find the particular form of the shakedown theorem for kinematic hardening. Hence:

Shakedown will occur when there exists some plastic strain distribution ε_{ij}^* such that

$$f(\sigma_{ij}^* - c\,\epsilon_{ij}^*) < 0 \quad , \quad \sigma_{ij}^* = \hat{\sigma}_{ij} + \bar{\rho}_{ij}$$

throughout V from some elastic solution $\hat{\sigma}_{ij}$ corresponding to initial velocities $\dot{\hat{u}}_i$ and some residual stress field $\bar{\rho}_{ij}$.

As there is no limit to the values of ε_{ij}^*, this result may be simplified further on noting that ε_{ij}^* and $\bar{\rho}_{ij}$ remain constant in time, provided $\bar{\rho}_{ij}$ and ε_{ij}^* are both finite throughout the volume. Hence:

Shakedown will occur provided there exists a distribution P_{ij} throughout the volume so that

$$f(\hat{\sigma}_{ij}(t) - P_{ij}) < 0$$

for some elastic solution corresponding to initial velocities $\dot{\hat{u}}_i(0)$.

The vector quantity $P_{ij} = c\,\varepsilon_{ij}^* + \bar{\rho}_{ij}$ need not satisfy any field equation and need not be continuous. This result merely places a restriction on the variation of $\hat{\sigma}_{ij}$ and not on its maximum values. Of course, to evaluate the bound (5) it is necessary to define $\bar{\rho}_{ij}$ and hence ε_{ij}^* separately. This result is much simpler to apply than the Melan theorem for perfect plasticity as the variation of $\hat{\sigma}_{ij}$ may be investigated at each point in the body separately. The generation of residual stress fields $\bar{\rho}_{ij}$ and plastic strains ε_{ij}^* only becomes necessary if the displacement bound (5) is required.

Finally it is worth noting that if a shakedown condition exists assuming perfect plasticity and a yield surface $f(\sigma_{ij}) = 0$, then shakedown will certainly occur if kinematic hardening is assumed with $f(\sigma_{ij} - c\varepsilon'_{ij}) = 0$.

Linear kinematic hardening remains a somewhat inadequate description of metallic behaviour, but it would appear that the general procedures described in this paper would allow the generation of a shakedown result for any stable plastic constitutive relationship.

In the complete version of this papaper, some simple examples of both static and dynamic loading are included.

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