ELASTO-PLASTIC ANALYSIS USING AN EFFICIENT FORMULATION OF THE FINITE ELEMENT METHOD

B. AAMODT, O. MO

A.S. Computas, Det Norske Veritas, Oslo 5, Norway

SUMMARY

The paper is concerned with an efficient computational procedure for elasto-plastic analysis.

Based on the flow theory of plasticity, the von Mises or the Tresca yield criterion and the isotropic hardening law, an incremental stiffness relationship can be established for a finite element model of the elasto-plastic structure. However, instead of including all degrees of freedom and all finite elements of the total model in a nonlinear solution process, a separation of elastic and plastic parts of the structure can be carried out. Such a separation can be obtained by identifying elastic parts of the structure as "elastic" superelements and elasto-plastic parts of the structure as "elasto-plastic" superelements. Also, it may be of advantage to use several levels of superelements in modelling the elastic parts of the structure.

For the "elasto-plastic" superelements the specific plastic computations such as updating of the incremental stiffness matrix and subsequent reduction (i.e. static condensation of all degrees of freedom being local to the superelements) have to be carried out repeatedly during the nonlinear solution process. This is in contrast to the "elastic" superelements for which no such calculations need to be performed. The only calculations that have to be made in connection with these superelements are the reductions of their stiffness matrices. This has to be done once only, and prior to the nonlinear solution process. A substantial saving of computer time is accomplished in this way.

The solution of the nonlinear equations is performed utilizing a combination of load incrementation and equilibrium iterations. In this connection, a comparative numerical study of the Newton-Raphson iteration scheme, the initial stress method, and modified Newton-Raphson iteration schemes is presented.

The present method of analysis is demonstrated for two larger examples of elasto-plastic analysis. Firstly, an elasto-plastic analysis of a plate with a central hole and subjected to tensile forces is carried out. The results are compared with experimental values. Secondly, a three dimensional analysis of a thick plate with a central through-crack subjected to tensile forces is considered. The variation through the plate thickness of the size of the plastic zones at the crack tip is studied.

The numerical examples show that the present method is a powerful and efficient tool in elasto-plastic analysis.

1. Introduction

The finite element method has proven to be a powerful tool in the analysis of elasto-plastic structural problems. The most common solution procedure for this type of problems is some combination of load incrementation and equilibrium iterations. Such processes are, however, known to be very expensive. In the present paper it will be shown how the efficiency of such analyses can be improved using various iteration algorithms and the superelement (substructure) technique.

There are basically three different approches to account for elasto-plastic material behaviour in the finite element method. These are the "initial strain" method, the "tangential stiffness" method and the "initial stress" method. Both the initial strain method and the initial stress method are based on the principle of accounting for plastic strain by introducing a set of equivalent body forces. By this technique the stiffness matrix of the structure is kept unchanged throughout the entire analysis. The tangential stiffness method requires that a new stiffness matrix be formulated for each load increment.

The present paper will concentrate on the tangential stiffness method, the initial stress method and some combination of these two, often denoted the modified Newton-Raphson method.

The superelement formulation of the finite element method has shown to be a very powerful tool for analyses of medium size and large,linear structural problems [1]. As regards the application of the superelement tèchnique in connection with elasto-plastic analyses, experience is relatively limited. However, recent studies [2] have indicated that very good economy can be gained by using this technique also for this type of analysis. In the present paper, the formulation of the superelement technique for elasto-plastic analysis will be discussed. Further, the efficiency of the technique will be demonstrated by numerical calculations.

2. Elasto-plastic formulation of the finite element method

In the present formulation, the flow theory of plasticity (Prandtl-Reuss), the von Mises or Tresca yield criterion and the isotropic hardening law are adopted. From these assumptions, the following incremental stress-stain relationship may be established [3]

$$d\boldsymbol{\sigma} = \mathbf{D}_I \, d\boldsymbol{\epsilon} \qquad (1)$$

where $d\boldsymbol{\sigma}$ and $d\boldsymbol{\epsilon}$ are increments in stress and strain. $\mathbf{D}_I$ represents the incremental form of the constitutive equations. Eq.(1) is valid between two consecutive plastic states. Before yielding has occurred or when unloading takes place the matrix $\mathbf{D}_I$ should be replaced by the elasticity matrix $\mathbf{D}$ representing Hooke's law.

Consider a body divided into μ finite elements and using the incremental form of the principle of virtual work, the following incremental relationship between forces and displacements is obtained for the finite element model

$$\mathbf{K}_I \, \Delta\mathbf{r} = \Delta\mathbf{R} \qquad (2)$$

Here,

$\mathbf{K}_I$ is the incremental (tangential) stiffness matrix of the finite element model

$\Delta\mathbf{r}$ is an increment of displacement

$\Delta\mathbf{R}$ is an increment of load

$\mathbf{K}_I$ is given by

$$\mathbf{K}_I = \sum_{i=1}^{\mu} \mathbf{a}_i^T \mathbf{k}_{I_i} \mathbf{a}_i \tag{3}$$

where $\mathbf{k}_{I_i}$ is the incremental stiffness matrix of element no. i given by

$$\mathbf{k}_{I_i} = \int_{V_i} \mathbf{B}_i^T \mathbf{D}_{I_i} \mathbf{B}_i \, dV \tag{4}$$

and $\mathbf{a}_I$ is the connectivity matrix. V_i is the volume of element no. i, and the matrix $\mathbf{B}_i$ is defined by the following relationship

$$\boldsymbol{\varepsilon}_i = \mathbf{B}_i \mathbf{v}_i \tag{5}$$

where $\boldsymbol{\varepsilon}_i$ and $\mathbf{v}_i$ are strain and nodal displacement vectors. Further, $\Delta\mathbf{R}$ of eq. (2) is given by

$$\Delta\mathbf{R} = \sum_{i=1}^{\mu} \mathbf{a}_i^T (\Delta\mathbf{P}_{b_i} + \Delta\mathbf{P}_{s_i}) \tag{6}$$

where

$$\Delta\mathbf{P}_b = \int_{V_i} \mathbf{N}_i^T \Delta\mathbf{X}_i dV \tag{7}$$

$$\Delta\mathbf{P}_{s_i} = \int_{S_i} \mathbf{N}_i^T \Delta\mathbf{T}_i dS \tag{8}$$

Here, $\mathbf{N}_i$ is the displacement interpolation function for element no. i, and $\Delta\mathbf{X}_i$ and $\Delta\mathbf{T}_i$ are increments of body forces and surface forces. S_i is the part of the surface of element no. i where surface forces are given.

From the principle of virtual displacements the following equilibrium equation can be derived for the body

$$\mathbf{R} - \mathbf{R}_\sigma = \mathbf{0} \tag{9}$$

where $\mathbf{R}$ is the applied load and $\mathbf{R}_\sigma$ is given by

$$\mathbf{R}_\sigma = \sum_{i=1}^{\mu} (\mathbf{a}_i^T \int_{V_i} \mathbf{B}_i^T \boldsymbol{\sigma}_i dV) \tag{10}$$

where $\boldsymbol{\sigma}_i$ are the stresses in subelement no. i.

3. Numerical solution of the nonlinear equations

In accordance with the flow theory of plasticity, a solution procedure based on load incrementation should be used. However, since the load increments are finite, unbalanced (residual) forces will arise in the system, i.e. eq. (9) will no longer be satisfied. In order to restore equilibrium throughout the system, equilibrium iterations should be carried out after each load increment. The iterative process can be expressed as follows

$$\mathbf{K}_{I(j)} \Delta\mathbf{r}_{(j+1)} = \Delta\mathbf{R}_{(j)} \tag{11}$$

in which, according to eqs. (9) and (10), the unbalanced forces are found from

$$\Delta\mathbf{R}_{(j)} = \mathbf{R} - \sum_{i=1}^{\mu} (\mathbf{a}_i^T \int_{V_i} \mathbf{B}_i^T \boldsymbol{\sigma}_{i(j)} dV) \tag{12}$$

and the improved displacement vector is given by

$$\mathbf{r}_{(j+1)} = \mathbf{r}_{(j)} + \omega \, \Delta\mathbf{r}_{(j+1)} \tag{13}$$

Subscript j denotes the iteration cycle. If the matrix $\mathbf{K}_I$ of eq. (11) is set equal to the current incremental stiffness matrix at each iteration cycle, this will correspond to the Newton-Raphson iteration scheme. Modified Newton-Raphson schemes are obtained when the matrix $\bar{K}_I$ is not updated at each iteration cycle. The initial stress method corresponds to the case when $\bar{K}_I$ is kept equal to the original, elastic stiffness matrix K_0 during the entire solution process.

The rate of convergence during equilibrium iterations can be considerably improved by choosing the acceleration factor ω in eq. (13) larger than 1.

A criterion is required for the termination of the iteration process when sufficient convergence is obtained. For this purpose a maximum norm of the displacement increments or a criterion based on the accumulated plastic work may be used.

4. A superelement formulation of the finite element method for elasto-plastic analysis

It is a fact, that for elasto-plastic problems in which yielding is confined to a small part of the total structure, a substantial amount of computer time may be saved by reducing the size of the nonlinear problem. This can be done by considering the elastic part of the structure as coupled elastic boundary attached to the small zone in which the yielding occurs. Mathematically this can conveniently be carried out by means of a modified superelement formulation.

In the superelement technique, parts of the structure that are known to remain elastic throughout the analysis can be associated with "elastic superelements". These superelements need to be reduced only once (a partial Gaussian elimination in which all internal or local degrees of freedom are eliminated) [2]. This process is carried out prior to the start of the nonlinear analysis. Parts of the structure that may undergo yielding are included in "plastic superelements". These superelements are modified during load incrementation and equilibrium iterations. However, a reduction (formally: triangularization) of the equations has to be carried out only when a new incremental stiffness is formulated. Other computations associated with elasto-plastic analysis such as calculation of strains, stresses, etc. are also required for the "plastic superelements" only.

A symbolic formulation of the superelement technique will now be given. Assume that eq. (2) is valid for a plastic superelement, all its degrees of freedom being considered. Using partitioning, eq. (2) can be restated as follows

$$\begin{bmatrix} K_{I_{ss}} & K_{I_{si}} \\ K_{I_{is}} & K_{I_{ii}} \end{bmatrix} \begin{bmatrix} \Delta r_s \\ \Delta r_i \end{bmatrix} = \begin{bmatrix} \Delta R_s \\ \Delta R_i \end{bmatrix} \qquad (14)$$

where the subscript s refers to the "super" (retained) degrees of freedom and i indicates "internal" (local) degrees of freedom. A straightforward elimination of the internal degrees of freedom yields

$$K_{I_{red}} \Delta r_s = \Delta R_{red} \qquad (15)$$

$$K_{I_{red}} = K_{I_{ss}} - K_{I_{si}} K_{I_{ii}}^{-1} K_{I_{is}} \qquad (16)$$

and

$$\Delta R_{red} = \Delta R_s - K_{I_{si}} K_{I_{ii}}^{-1} \Delta R_i \qquad (17)$$

Here, $K_{I_{red}}$ is the reduced incremental stiffness matrix, and ΔR_{red} is the reduced incremental force or load vector. It should also be mentioned that unbalanced forces arising during the equlibrium iterations, see eq. (12), are treated according to eq. (17).

5. Numerical examples

Two computer programs for elasto-plastic analysis have been developed based on the theory outlined above. The first program is intended for analysis of two- and three-dimensional membrane structures. Alternatively, plane stress or plane strain conditions may be considered. Quadrilateral elements built up from four constant strain triangles are used in the program. The development of this program has been supported by the Swedish Ship Research Foundation. The second program is intended for analysis of three-dimensional solid structures using eight-node, isoparametric hexahedral elements with 2 by 2 by 2 Gaussian integration. This program has been developed in cooperation with the Norwegian Institute of Technology. Both programs have been included in the program system SESAM-69 [4] . The programming system NORSAM [5] has been used in both programs.

An extensive study of different solution algorithms has been carried out using the solid analysis program [2]. A thick-walled tube (inner radius a=10 and outer radius b=2a) with internal pressure was analysed considering a sector with 7 eight-node solid elements through the thickness. Elastic-perfectly plastic material behaviour was assumed.

Table I summarizes the results of the investigation. A maximum norm convergence criterion for the displacement increments was used during the equilibrium iterations. After load increment no. 4 yielding had occurred half-way through the wall thickness. $(u_r)_r$ denotes the radial displacement at a point with distance r from the centre of the tube. The computer times (CPU) indicated in the table refer to a UNIVAC 1108 computer.

It can be concluded from Table I that the solution algorithm $(K_I)_2$ is very beneficial. This algorithm represents a modified Newton-Raphson iteration scheme for which the incremental stiffness matrix is updated at the first iteration cycle at each load increment. It is also evident from Table I that the use of an acceleration factor, ω , larger than 1 can be very efficient. Concerning the calculated radial displacements, it is seen that they generally agree very well with the theoretical values.

As an example of using the membrane analysis program an elasto-plastic analysis of a plate with a central hole will be considered. The dimensions of the plate are indicated in fig. 1. The material properties are found from specimens under tension and are illustrated in fig. 2. The plate was loaded with uniaxial tensile forces (5 load increments) along the y-axis. The numerical results will be compared with results from an experimental study carried out by the Swedish Ship Research Foundation [6].

Due to symmetry conditions only one quarter of the plate and the attachment needed to be considered in the numerical study. The finite element idealization of the plate is shown in fig. 3. Only one level of superelements was used in the model. The total number of basic elements was 80 and the total number of degrees of freedom was 206.

The plastic zone was expected to remain in superelement type 1 during the first couple of load increments. During the subsequent load increments the plastic zone was expected to grow into superelement type 2 and 3. Superelement type 4 and 5 were expected to remain elastic throughout the whole solution process.

The calculations were carried out on a UNIVAC 1110 computer and the CAU-times for different solution algorithms and 4 load increments are shown in Table II. The number of equilibrium iteration cycles required at each load increment and the plastic work per unit of volume, W_p, in the lower left element in superelement type 1 (see fig. 3) are given in the table. The highest value of W_p corresponds to the best agreement between computed and measured strains. The convergence criterion that was used in the iterations was the "change of plastic work criterion", $\Delta W_p/W_p < 0.02$.

It is concluded from Table II that as long as the plastic zone is relatively small, the modified Newton-Raphson solution algorithms identified by $(K_I)_1$ and $(K_I)_2$ are very economical. However, when the plastic zone grows larger, pure Newton-Raphson iteration schemes are preferable.

Fig. 4 shows the calculated boundaries of the plastic zones after various load increments. In figs. 5a and 5b the calculated and measured total axial strains at two different points in the plate are illustrated, see fig. 4. The nearly horizontal parts of the curves correspond to the state when complete yielding has occurred across the plate. It is seen that the calculated strains are somewhat smaller than the measured. However, full yielding across the plate was computed to occur at the same load level as found by the experiments. A reason for the difference between computed and measured strain may be the rather coarse element mesh used in the plastic zone. Other calculations also show that small changes in the material properties (lowering the horisontal part of the material curve by 4%) give much better correspondance between computed and measured results. Further, it should be mentioned that some convergence problems were experienced at full yielding due to the discontinuity in the slope of the experimental stress-strain curve in fig. 2.

Strain-load curves for points outside the plastic enclave showed complete agreement between calculations and experiments.

As a last example, the analysis of a thick plate with a central crack will be considered. The plate with all dimensions and material properties is shown in fig. 6. The plate was subjected to a uniform, uniaxial tension σ_0. It was of particular interest to study the state of stress in the vicinity of the crack tip for increasing loads on the plate.

Due to symmetry about three coordinate planes, it was necessary to consider only one eighth of the plate in the analysis. The finite element idealization is shown in fig. 7. A total of 296 eight-node, isoparametric elements and 1416 nodal degrees of freedom were used in the model.

Table III gives an outline of the various steps in the solution procedure for the elasto-plastic analysis. The incremental stiffness matrix was updated only for the first iteration cycle at every new load increment. In addition, an acceleration factor of $\omega = 1.4$ was used, see eq.(13). It is seen from Table III that only superelement (1,1), had to be treated as being elasto-plastic during the first five load increments. However, during the last two load increments superelement (1,2) also became elasto-plastic. An elastic analysis of this problem required 9 minutes (CPU) on a UNIVAC 1108 computer and the requirement for the elasto-plastic analysis was 27 minutes. It is expected that the time requirement for a corresponding solution in which no distinction between elasto-plastic superelements were made, would run about two or three times higher than for the present method.

Figs. 8a and 8b show the extensions of the plastic zones at the mid-plane (z=0) and at the plane z/B=0.49, obtained for different load levels. B is the plate thickness. The plastic zones shown in the figures were drawn so that the element integration points that had yielded were enclosed by the boundary lines. It is seen that all plastic zones except for zone no. 4 are larger at the free surface than at the mid plane. This is due to the transition from a state of plane stress at the free surface to a three-dimensional stress state in the interior of the plate. However, when the plastic zone grows larger, as is zone no. 4, almost pure plane stress conditions will prevail through the entire thickness of the plate, and the plastic zone will become nearly uniform through the thickness.

6. Conclusions

A numerical technique for elasto-plastic analysis based on a superelement formulation of the finite element method has been presented. By this technique, entirely elastic parts of the structure can be eliminated from the nonlinear solution process, thus drastically reducing the computer time requirements.

The solution of elasto-plastic problems requires that a combination of load incrementation and equilibrium iterations be used. For this purpose, various iteration schemes have been studied. It was found that modified Newton-Raphson schemes were very efficient as long as the plastic zone was not too large. However, when the plastic zone grew very large, and the material behaviour was perfectly plastic or nearly perfectly plastic, pure Newton-Raphson iteration schemes appeared to be preferable.

The results of an analysis of a plate with a circular hole were compared with experimental values. It was found that the computed loading at full yielding across the plate was of the same size as obtained by the experiments. However, there was some deviation between the calculated and the experimental strain-loading curves for points in the plastic enclave. Further a three-dimensional analysis of a thick plate with a central crack was discussed. Of particular interest was to study the variation through the plate thickness of the size of the plastic zone at the crack tip.

In conclusion, the numerical calculations showed that elasto-plastic finite element analyses of two- and three-dimensional structures can be carried out at reasonable cost when the present technique is used.

References

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Table I Different solution algorithms for thick-walled tube

Increment no.		1	2	3	4	Total CPU-time (sec.)
Theoretical	$(u_r)_{r=1.525a}$		0.013540		0.019498	
	$(u_r)_{r=2.0a}$		0.011 727		0.016887	
	No. of iterations	6	8	11	18	
	(u_r) r=1.525a		0.013529		0.019380	140
K_0	(u_r) r=2.0a		0.011724		0.016794	
ω=1.0						
	No. of iterations	6	7	9	13	
$(K_I)_1$	(u_r) r=1.525a		0.013537		0.019400	107
ω=1.0	(u_r) r=2.0a		0.011731		0.016812	
	No. of iterations	4	6	8	13	
$(K_I)_2$	(u_r) r=1.525a		0.013538		0.019409	97
ω=1.0	(u_r) r=2.0a		0.011732		0.016820	
	No. of iterations	4	6	8	12	
$(K_I)_3$	(u_r) r=1.525a		0.013538		0.019406	180
ω=1.0	(u_r) r=2.0a		0.011732		0.016817	
	No. of iterations	2	4	5	9	
$(K_I)_2$	(u_r) r=1.525a		0.013544		0.019424	84
ω=1.5	(u_r) r=2.0a		0.011737		0.016833	
	No. of iterations	8	3	4	8	
$(K_I)_2$	(u_r) r=1.525a		0.013548		0.019432	83
ω=1.7	(u_r) r=2.0a		0.011470		0.016840	
	No. of iterations	No				
$(K_I)_2$	(u_r) r=1.525a	con-				
ω=1.9	(u_r) r=2.0a	verg- ence				

K_0 : Constant stiffness method (initial stress method).

$(K_I)_1$: Stiffness matrix updated only at start of each increment.

$(K_I)_2$: Stiffness matrix updated only at first iteration cycle for each increment.

$(K_I)_3$: Stiffness matrix updated for every increment and every iteration cycle (Newton-Raphson).

ω : Acceleration factor.

Table II Different solution algorithms for plate with hole

Code	Number of iterations at each load increment				Wp(N/mm^2)	Total CAU-time (sec.)
Increment no.	1	2	3	4		
$(K_I)_1$	2	3	6	15 *)	1.24	112
$(K_I)_2$	2	3	6	6	1.06	82
$(K_I)_3$	2	3	3	4	1.06	68
$(K_I)_4$	2	3	3	4	1.05	67

$(K_I)_1$: Stiffness matrix updated only at the start of each increment.

$(K_I)_2$: Stiffness matrix updated at every other iteration cycle at each increment.

$(K_I)_3$: Stiffness matrix updated for every increment and every iteration cycle (Newton-Raphson).

$(K_I)_4$: Stiffness matrix updated at the start of each of the two first increments and at every iteration cycle at the two last increments.

*) : The iteration process was terminated after 15 iteration cycles although the convergence criterion was not satisfied $(\frac{\Delta Wp}{Wp} > 0.02)$.

Table III Elasto-plastic solution procedure for plate with crack

Increment no.	$\Delta\sigma_0/\sigma_Y$	σ_0/σ_Y	Number of iterations	Superelement being plastic (level, type)
Initial yielding		0.190		(1,1)
1	0.113	0.303	2	(1,1)
2	0.030	0.333	2	(1,1)
3	0.031	0.364	4	(1,1)
4	0.040	0.404	2	(1,1)
5	0.041	0.445	2	(1,1)
6	0.040	0.485	2	(1,1) + (1,2)
7	0.040	0.525	2	(1,1) + (1,2)

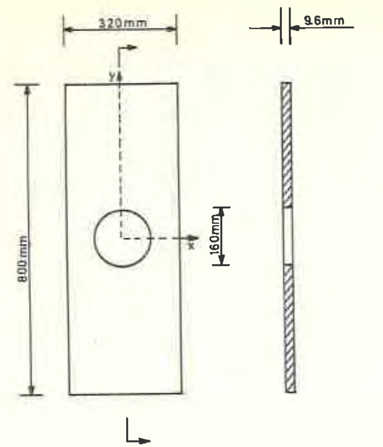

Fig. 1: Plate with a circular hole

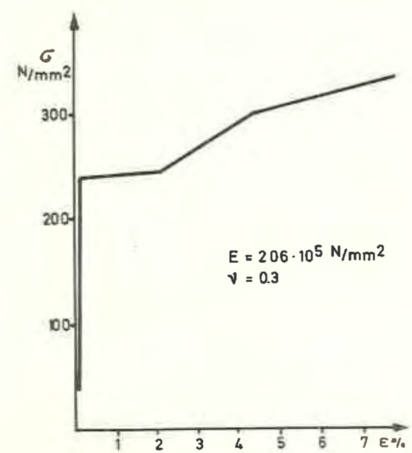

Fig. 2: Material properties

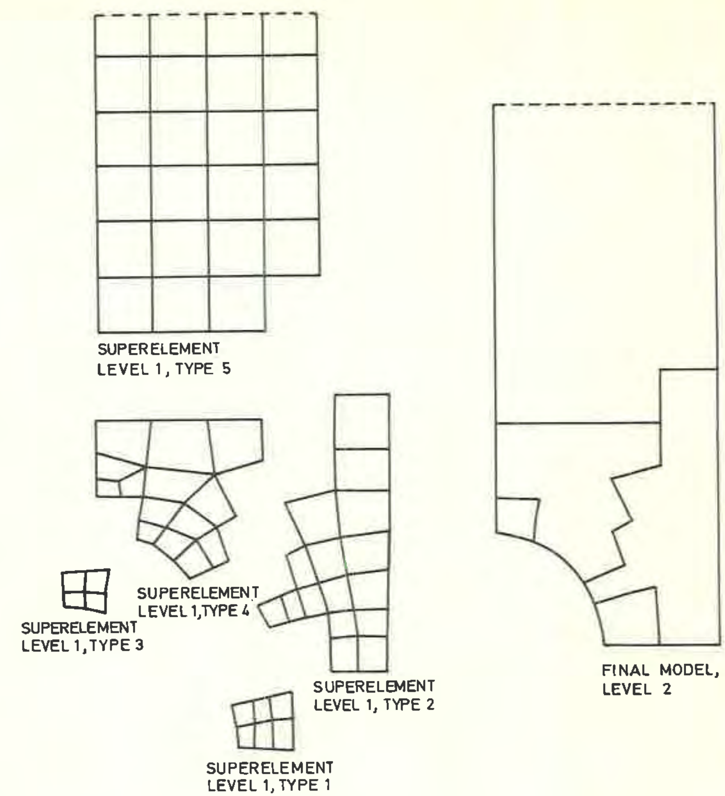

Fig. 3: Finite element idealization of one quarter of the plate

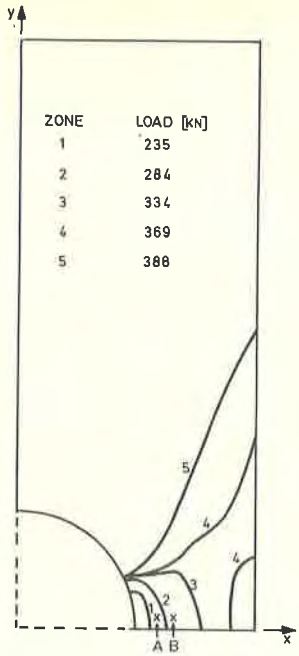

Fig. 4: Plastic zones at various load increments

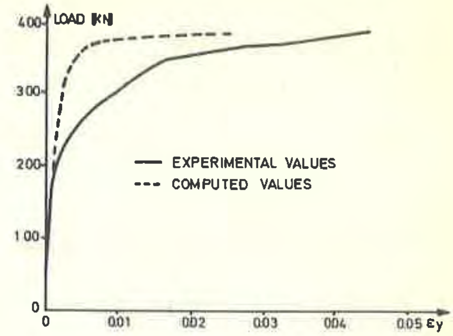

Fig. 5a: Comparison of strain along the y-axis at point A (see fig. 4)

Dimensions:

W = 200 mm

L = 8/3 W = 533.33 mm

B = W/5 = 40 mm

C = W/6 = 33.33 mm

Material properties:

Young's modulus:

$E = 7.06\ 10^4 N/mm^2$

Poisson's ratio: $\nu = 0.34$

Uniaxial yield stress:

$\sigma_y = 29.4\ N/mm^2$

Strain hardening rate:

$H' = d\bar{\sigma}/d\bar{\varepsilon}^P = 670\ N/mm$

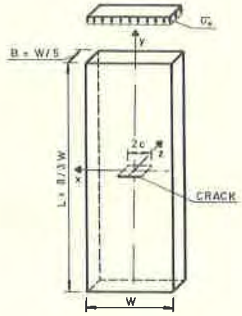

Fig. 6: Thick plate with a central crack

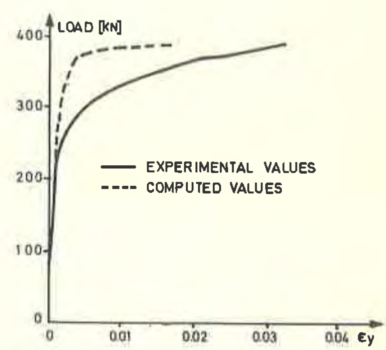

Fig. 5b: Comparison of strain along the y-axis at point B (see fig. 4)

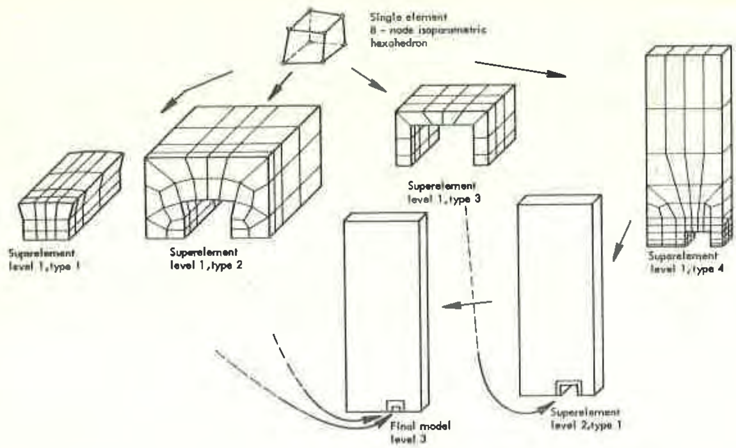

Fig. 7: Finite element idealization of
one eighth of the plate

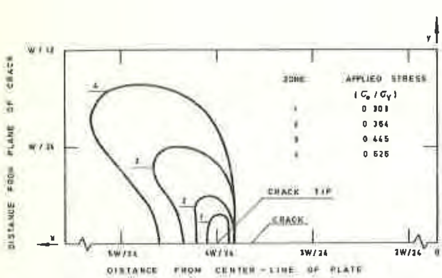

Fig. 8a: Plastic zone sizes at the mid-plane
(z = 0) at various load levels

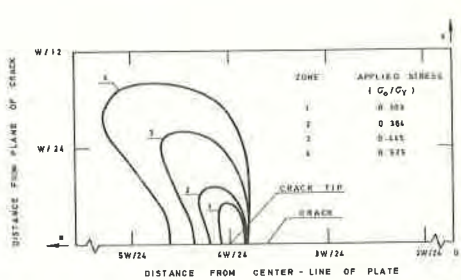

Fig. 8b: Plastic zone sizes at the plane
z/B = 0.49 at various load levels