A THEORY ON THE TWO-PHASE FLOW INDUCED VIBRATIONS IN PIPING SYSTEMS

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SUMMARY

This paper deals with a theoretical analysis of two-phase (gas + liquid) flow induced vibrations in piping system and newly concludes that the mechanism of exciting the vibrations is so called parametric excitation due to periodic changes of the mass of the system; centrifugal force and Coriolis' force of liquid pistons travelling in the pipe.

The equation of motion of the piping system conveying two-phase fluid is formulated from considering elastic restoring force; inertia force acting on the pipe and two-phase fluid; centrifugal force and Coriolis' force caused by travelling liquid pistons and the momentum change of two-phase flow.

The author represents the mass distribution along the pipe in the form of δ-function series for the case of "piston flow" which is one flow pattern of gas-liquid two-phase flow causing the most significant vibration in the piping system. After substituting the mass distribution into the equation of motion, the author applies modal analysis technique and obtains the coupled ordinary differential equations concerning the amplitudes of each mode with periodically time-variant coefficients. Here it is shown that the effect of momentum change of the two-phase flow is completely vanished in the translated equation of motion.

Employing the first mode approximation and replacing the periodically time-varying coefficients with the first term of their Fourier series expansion, the author obtains the Mathieu type equation for the first mode amplitude:

$$\ddot{X}_1 + \omega_0^2(1 - \varepsilon \cos pt)X_1 = 0.$$

Here ω_0 is the lowest eigen frequency of the piping system containing two-phase fluid; ε is a small parameter expressing the effect of mass of the two-phase fluid, centrifugal and Coriolis' forces; and p means an arrival frequency of liquid pistons. The theory on Mathieu's equation shows that the vibrational instability occurs when:

$$p/\omega_0 = 2, 1, \tfrac{2}{3}, \ldots$$

This result is verified by experimental study that gives the power spectral densities of vibrational strain signal and void signal of two-phase flow which means whether or not liquid exists at the point in question.

This paper shows the significant conclusion that in the case there is no effect of gravitation on the vibrational system, the two-phase flow induced vibration is dominantly caused by the parametric excitation due to the mass change of two-phase flow and that of centrifugal force and Coriolis' force.

1. Introduction

There have been many studies on the vibrations of reactor components such as fuel pins; control rods; piping systems and so on, which are caused by fluid flow in a nuclear reactor system. At the first conference on structural mechanics in reactor technology (Berlin,1971), S.S.Chen and M.W. Wambsganss[1] showed that the vibration of fuel rods in parallel single-phase flow was excited by the pressure fluctuation of turbulent boundary layer. G.P.Gau, P.Grillo and G.Testa[2] described experimentally that the amplitude of the vibrations of fuel bundle was linearly dependent on the flow momentum and K.D.Appelt, J.Kadlec and W.Kruger[3] insisted on the relation between the vibration and the fluctuation of flow pressure. In these three papers, it is shown that the vibrations of fuel rods in parallel flow are caused by the flow pressure fluctuation and are considered as forced vibration. On the other hand, P.G.Avanzini[4] mentioned the self-sustained vibration of rods in parallel flow which was first investigated by E.P.Quinn.[5]

When we turn our view point to the boiling water reactor, the vibrations induced by two-phase flow should be intensively investigated. R.W. Harris and P.G.Holland[6] studied the vibrations of cylindrical cantilever in the field of air-water two-phase flow and concluded that the excitation mechanism is impulsive reaction of water slugs on the cantilever. Concerning this problem, from the statistical view point, L.Cedolin, A.Hassid and T.Rossini[7] studied experimentally the relationship between the intensity of vibrations induced by parallel two-phase flow and the fluctuation of the flow.

Now it can be told that the attention should be paid to the vibrations of piping systems conveying fluid flow in the course of designing the channel type reactor or the piping systems. This problem was intensively studied by T.B.Benjamin,[8] R.W.Gregory[9] and S.S.Chen.[10] They dealt with the single phase flow induced vibrations in piping systems and showed the instability conditions of the vibrations with regard to the flow velocity.

Concerning the two-phase flow induced vibrations in piping systems, F.Hara, T.Shigeta and H.Shibata[11] studied on the relationship between vibrational characteristics and statistical properties of two-phase flow and F. Hara[12] showed the excitational mechanism of two-phase flow induced vibrations is periodic change of parameters of the vibration system.

This paper deals with a theoretical analysis of two-phase(gas + liquid) flow induced vibrations in a piping system. The equation of motion of the piping system conveying two-phase fluid is formulated from considering elastic restoring force; inertia force acting on the pipe and the two-phase fluid; centrifugal force and Coriolis' force caused by travelling liquid slugs and momentum change of two-phase flow, under the assumptions of neglecting the gravitation, axial forces due to flow-in and flow-out of two-phase fluid and friction force on the wall of the pipe. Representing the mass distribution of two-phase fluid along the pipe in the form of δ -function series, modal analysis technique can be applied to the original partial differential equation of motion. Then the author obtains the coupled ordinary differential equations for the amplitudes of each mode with periodic time-variant coefficients. The first mode approximation and the employment of the first terms of the Fourier series expansion of its coefficients lead to the Mathieu type equation. The theory on Mathieu equation shows that the vibrational instability occurs when the ratio of an arrival frequency of liquid slug to the lowest eigen frequency of the piping system containing two-phase fluid is 2, 1, 2/3,--- . This result is verified by the experimental study in which 1/2 subharmonic parametric excitation was observed.

This paper shows the significant conclusion that the two-phase (gas + liquid) flow induced vibrations in piping systems are dominantly caused by the parametric excitation due to the periodic change of the mass of the vibrational system, of centrifugal force and of Coriolis' force.

2. The Equation of Motion

2.1 Assumptions

The author adopts the following assumptions in order to derive the equation of motion for the two-phase flow induced vibrations in piping system:

(1) There is no gravitation effect on the vibrational system;

(2) There is no axial force at both ends of the piping system; and

(3) There is no friction force on the inside wall of the pipe.

The first assumption means that the piping system is vertically constructed or the out-of-plane vibrations of a vertical L-shaped piping system are here considered. The second and third ones are employed for simplicity.

2.2 Notations and Geometry of Vibrational System

The author takes an axis x directed along the pipe in its unflexed state, both ends of the pipe being respectively $x = 0$ and $x = l$. The displacement of the pipe in a plane containing the x -axis is written $y(x,t)$, where t means time. The other notations for physical quantity are as follows:

E :Young's modulus;

I :Geometrical moment of inertia;

M :Mass of pipe for unit length;

m :Mass of two-phase fluid for unit length;

U :Uniform velocity of two-phase flow.

The geometry of the vibrational system is shown in Fig.1. The system is supposed to be in equilibrium when the pipe lies along a straight line (being x -axis). Fig.1 also shows the detail of the confugaration of the pipe when it takes a displacement y during its vibrations.

The two-phase fluid comes into the system at the bottom end and goes out at the top end as shown in Fig.1.

2.3 The Equation of Motion

For the purpose of derivation of the equation of motion, the author now needs to consider what kinds of force are acting on the small element of the pipe. As has already been assumed that the displacement y is confined in a plane containing the x -axis, it is sufficient only to take account of the forces perpendicular to the x -axis. Considering the small element of the pipe between the coordinate x and $x + dx$, it can be seen that the following three forces exist:

(1) Inertia force acting on the mass of the pipe;

$$- M \frac{\partial^2 y}{\partial t^2} dx ,$$

(2) Reaction of inertia force acting on the mass of two-phase fluid;

$$- \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left\{ m \left(\frac{\partial y}{\partial t} + U \frac{\partial y}{\partial x} \right) \right\} dx ,$$

(3) Restoring force caused by elasticity of the pipe;

$$EI \frac{\partial^4 y}{\partial x^4} dx .$$

Here it should be remarked that when the displacement y occurs in the pipe, it has the inclination $\partial y / \partial x$ and so that the time derivative of a certain physical quantity of two-phase flow has the form

$$\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}$$

for its perpendicular component to the x -axis. Therefore that component of the momentum of two-phase fluid can be written as

$$m \left(\frac{\partial y}{\partial t} + U \frac{\partial y}{\partial x} \right) .$$

And as the time derivative of this momentum is the force acting on the mass of two-phase fluid, its reaction acts on the pipe.

The balance of the elastic restoring force with the inertia force of the pipe and the reaction force caused by two-phase fluid leads to the equation:

$$EI \frac{\partial^4 y}{\partial x^4} dx = -M \frac{\partial^2 y}{\partial t^2} dx - \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left\{ m \left(\frac{\partial y}{\partial t} + U \frac{\partial y}{\partial t} \right) \right\} dx . \qquad (1)$$

Expanding the second term of the right hand side in eq.(1), eq.(2) is obtained

$$- \left\{ m \frac{\partial^2 y}{\partial t^2} + 2mU \frac{\partial^2 y}{\partial t \partial x} + mU^2 \frac{\partial^2 y}{\partial x^2} + \left(\frac{\partial m}{\partial t} + U \frac{\partial m}{\partial x} \right) \left(\frac{\partial y}{\partial t} + U \frac{\partial y}{\partial x} \right) \right\} dx . \qquad (2)$$

In eq.(2), the first term expresses the effect of the inertia force acting on the two-phase fluid without considering the fluid flow; the second means the Coriolis' force due to the flow velocity U and the angular velocity $\frac{\partial}{\partial t} \left(\frac{\partial y}{\partial x} \right)$; the third is the centrifugal force caused by the velocity U and the curveture rate $\frac{\partial^2 y}{\partial x^2}$; and the last term shows momentum change of two-phase flow due to the change of mass distribution in time and in space.

Finally, the equation of motion can be written in the form

$$EI \frac{\partial^4 y}{\partial x^4} + (M + m) \frac{\partial^2 y}{\partial t^2} + 2mU \frac{\partial^2 y}{\partial t \partial x}$$

$$+ \; mU^2\frac{\partial^2 y}{\partial x^2} \; + \; \left(\frac{\partial m}{\partial t} + U\frac{\partial m}{\partial x}\right)\left(\frac{\partial y}{\partial t} + U\frac{\partial y}{\partial x}\right) = 0 \,, \qquad (3)$$

which must be satisfied throughout the range $0 \leq x \leq l$ and over any time interval. Equation (3) agrees exactly with the result which T.B.Benjamin[8] found from the application of Hamiltonian principle except the last term which comes from the change of mass distribution of two-phase flow in space and in time. The author would like to draw attention to the fact that the mass distribution m of two-phase flow depends on space x and time t, namely, $m = m(x,t)$.

2.4 Boundary Conditions

It is now necessary to consider what end conditions are imposed on the function $y(x,t)$ in addition to the conditions

$$y(0,t) = y(l,t) = 0 \qquad (4)$$

already specified in Fig.1. If the pipe is incased at $x = 0$ and $x = l$, which is the physical condition most easily realized,

$$y'(0,t) = y'(l,t) = 0 \qquad (5)$$

are employed; but alternatively the pipe may be assumed to be simply supported at $x = 0$ and $x = l$, which means that

$$y''(0,t) = y''(l,t) = 0 \qquad (6)$$

since the bending moment $EI\,y''$ must be zero.

Which end condition should be taken? The author employs the simply supported end condition for simplicity of the calculations to be shown later. If the conditions (4) and (5) are adopted, as well known, the normalized eigen functions for the bending vibration of a pipe have very complex form consisted of sine, cosine functions, hypersine and hypercosine functions, however the simply supported end condition is able to give us the very simple functions as the normalized eigen functions.

It should be noted here that: Even if any boundary conditions are taken into account, the essential part of the discussions to be described in this paper would not change; but if the normalized eigen functions have complex form, the calculational procedures for obtaining the final results would become very tedious. And the simply supported end condition has the advantage that the calculational procedures run analytically since the normalized eigen functions consist of only sine function.

3. Modal Analysis

3.1 Normalized Eigen Functions

The author has already employed the simply supported end condition, then for the straight pipe shown in Fig.1, the equation of bending vibration, in the case of not-containing two-phase fluid, is easily derived and written as follows:

$$EI\frac{\partial^4 y}{\partial x^4} + M\frac{\partial^2 y}{\partial t^2} = 0 \ . \tag{7}$$

If the author assumes the separability of space and time as in

$$y = e^{i\omega t}\,\psi(x) \ , \tag{8}$$

the function ψ satisfies

$$\frac{d^4\psi}{dx^4} - k^4\psi = 0 \ , \tag{9}$$

where

$$k^4 = \omega^2 \Big/ \frac{EI}{M} \ . \tag{10}$$

The general solution of eq.(9) has the form

$$\psi = A\sin kx + B\cos kx + C\sinh kx + D\cosh kx \tag{11}$$

which is easily derived by simple calculations. Applying the boundary conditions, eqs.(4) and (6), the fundamental solution ψ is obtained as

$$\psi_i = \sin\frac{i\pi x}{l} \qquad (i = 1,\ 2,\ 3,\ \cdots\ ) \ . \tag{12}$$

The normalization condition imposed to the functions

$$\int_0^l \psi_i\,\psi_j\,dx = \delta_{ij} \ , \tag{13}$$

where δ_{ij} is Cronecker's δ, gives the definite form to the functions:

$$\psi_i = \sqrt{\frac{2}{l}}\,\sin\frac{i\pi x}{l} \qquad (i = 1,\ 2,\ 3,\ \cdots\ ) \ . \tag{14}$$

3.2 Modal Analysis of Two-Phase Flow Induced Vibrations

By using the normalized eigen functions obtained in eq.(14), the displacement of the pipe conveying two-phase fluid is expanded in the form:

$$y = \sqrt{\frac{2}{l}}\sum_{i=1}^{\infty} Y_i(t)\,\sin\frac{i\pi x}{l} \ , \tag{15}$$

where $Y_i(t)$ means the amplitude of the i-th mode of vibration in the piping system.

Let us substitute the equation into eq.(3); which is the original equation of motion; and multiply $\sqrt{2/l}\,\sin j\pi x/l$ with each term. Then the author integrates the whole equation over the interval $[\,0,\ l\,]$ with respect to the space coordinate x. Each term in eq.(3) can be rewritten in the form:

$$\text{The first} = EI\left(\frac{j\pi}{l}\right)^4 Y_j(t) \ , \tag{16}$$

$$\text{the second} = M\ddot{Y}_j(t) + \sum_{i=1}^{\infty} m_{ji}(t)\,\ddot{Y}_i(t) \ , \tag{17}$$

$$\text{the third} = 2U\sum_{i=1}^{\infty}\left(\frac{i\pi}{l}\right) m_{ji}(t)\,\dot{Y}_i(t) \ , \tag{18}$$

$$\text{the fourth} = -U^2\sum_{i=1}^{\infty}\left(\frac{i\pi}{l}\right)^2 Y_i(t) \ , \tag{19}$$

and finally

$$\text{the fifth} = \sum_{i=1}^{\infty}\Big[\big\{m_{ji}^{(\circ)}(t) + U m_{ji}^{(\prime)}(t)\big\}\dot{Y}_i(t) + U\big\{m_{j\bar{i}}^{(\circ)}(t) + U m_{j\bar{i}}^{(\prime)}(t)\big\}Y_i(t)\Big] \quad (20)$$

Here the quantities m_{ji} , $m_{j\bar{i}}$, $m_{ji}^{(\circ)}$, $m_{ji}^{(\prime)}$, $m_{j\bar{i}}^{(\circ)}$ and $m_{j\bar{i}}^{(\prime)}$ are defined as follows:

$$\left.\begin{aligned}
m_{ij}(t) &= \frac{2}{l}\int_0^l \sin\frac{j\pi x}{l}\, m(x,t)\sin\frac{i\pi x}{l}\,dx\,,\\
m_{j\bar{i}}(t) &= \frac{2}{l}\int_0^l \sin\frac{j\pi x}{l}\, m(x,t)\cos\frac{i\pi x}{l}\,dx\,,\\
m_{ji}^{(\circ)}(t) &= \frac{2}{l}\int_0^l \sin\frac{j\pi x}{l}\,\frac{\partial m(x,t)}{\partial t}\sin\frac{i\pi x}{l}\,dx\,,\\
m_{ji}^{(\prime)}(t) &= \frac{2}{l}\int_0^l \sin\frac{j\pi x}{l}\,\frac{\partial m(x,t)}{\partial x}\sin\frac{i\pi x}{l}\,dx\,,\\
m_{j\bar{i}}^{(\circ)}(t) &= \frac{2}{l}\int_0^l \sin\frac{j\pi x}{l}\,\frac{\partial m(x,t)}{\partial t}\cos\frac{i\pi x}{l}\,dx\,,\\
\text{and}\quad m_{j\bar{i}}^{(\prime)}(t) &= \frac{2}{l}\int_0^l \sin\frac{j\pi x}{l}\,\frac{\partial m(x,t)}{\partial x}\cos\frac{i\pi x}{l}\,dx\,.
\end{aligned}\right\} \quad (21)$$

Therefore the equation of motion in the modal form is written as follows:

$$EI\left(\frac{j\pi}{l}\right)^4 Y_j + M\ddot{Y}_j + \sum_{i=1}^{\infty} m_{ji}\ddot{Y}_i + 2U\sum_{i=1}^{\infty}\Big[\left(\frac{i\pi}{l}\right)m_{j\bar{i}} + \frac{1}{2}\big\{U^{-1}m_{ji}^{(\circ)} + m_{ji}^{(\prime)}\big\}\Big]\dot{Y}_i - U^2\sum_{i=1}^{\infty}\Big[\left(\frac{i\pi}{l}\right)^2 - \frac{i\pi}{l}\big\{U^{-1}m_{j\bar{i}}^{(\circ)} + m_{j\bar{i}}^{(\prime)}\big\}\Big]Y_i = 0 \,. \quad (22)$$

Equation (22) can be described in the form of vector and matrix if the author defines the mode vector $\{Y\}$, mass matrix $[M]$, damping matrix $[C]$ and stiffness matrix $[K]$ corresponding to eq.(22), namely,

$$[M]\{\ddot{Y}\} + [C]\{\dot{Y}\} + [K]\{Y\} = 0 \quad (23)$$

(the concrete definition of these vector and matrices being not written for the lack of space). Here it should be noted that the matrices $[M]$, $[C]$ and $[K]$ are time-variant.

Equation (23) consists of infinitely many elements, so it is impossible to obtain the analytical solution of this equation. But from the engineering point of view, the first mode of vibration is most important. Consequently, the author employs the first mode approximation to the equation (23):

$$\left\{ M + m_{11}(t) \right\} \ddot{Y}_1 + \frac{2\pi U}{l} \left[m_{1\bar{1}}(t) + \frac{l}{2\pi} \left\{ U^{-1} m_{11}^{(0)}(t) + m_{11}^{(1)}(t) \right\} \right] \dot{Y}_1$$

$$+ \left(\frac{\pi}{l}\right)^2 \left[EI \left(\frac{\pi}{l}\right)^2 - U^2 \left\{ m_{11}(t) - \frac{l}{\pi} \left\{ U^{-1} m_{1\bar{1}}^{(0)}(t) + m_{1\bar{1}}^{(1)}(t) \right\} \right\} \right] Y_1 = 0 . \quad (24)$$

In eq.(24), the first term means the inertia force caused by the masses of the pipe and the two-phase fluid; the latter is noted to be time-variant; the second is damping effect which comes from the Coriolis' force and the momentum change of the two-phase fluid; the third expresses the stiffness composed of elasticity of the pipe, the centrifugal force acting on the two-phase fluid and the momentum change.

4. Mathematical Representation of Piston Flow

In order to proceed to the detail discussions on the vibrational characteristics of the two-phase flow induced vibrations in the piping system, the author needs a concrete function of mass distribution $m(x,t)$ along the pipe. In experimental study, the author observed the flow patterns of two-phase flow by means of a high speed camera. The observation tells that in the piston flow, meaning that air and water slugs run through the pipe alternately, the arrival frequency of water slug is not fixed, but it randomly changes with a narrow band characteristics of frequency, so that it can be said that the frequency characteristics of the piston flow is of noisiness with narrow band having a dominant frequency $1/T$.

The author will proceed to the mathematical representation of piston flow by using this dominant arrival period of water slug T in this section. The illustration of transition process of the mass distribution $m(x,t)$ is shown in Fig.2. Generally speaking, l/UT ,which means the number of water slugs contained in the pipe, is not natural number, the following is defined:

$$N = \left[\frac{l}{UT} \right] + 1 , \quad (25)$$

where [ ] is Gaussian symbol meaning the natural number generated by cutting off the decimal part.

In reality, a particular water slug locates on the interval $[x, x+l_w]$ (l_w being the length of the water slug) at a certain time t , but the author idealizes the location of the water slug at a point x at t and expresses the water slug by δ -function as follows:

$$m_0 \delta(x,t)$$

where m_0 is the total mass of the water slug. It is easily seen that there exsits a simple relationship between x and t concerning the location of a water slug, namely, the location of the water slug is $Ut - x$. Then it can be expressed as

$$m_0 \delta (Ut - x) \quad .$$

When the author takes account of the ordering of water slug series along the pipe, the mass distribution of the two-phase flow is represented in the series of δ -functions:

$$m(x,t) = m_0 \sum_{i=1}^{N-1} \delta\{U(i-1)T + Ut - x\} + m_0\, \delta\{U(N-1)T + Ut - x\}\, h(\tfrac{t}{\beta T}),\ (0 \le t \le T), \tag{26}$$

where $0 \le \beta \le 1$; and the function h is defined as:

$$h(\tau) = \begin{cases} 0 & (\tau < 0), \\ 1 & (0 \le \tau \le 1), \\ 0 & (\tau > 1). \end{cases} \tag{27}$$

The illustration in Fig. 2 shows that $m(x,t)$ is a general periodic function with a period T .

The concrete form of the mass distribution of the two-phase flow along the pipe has been obtained in eq.(26), then if eq.(26) is substituted into eq.(21) and the integration is done over the interval [0 , l] with respect to x , m_{ji} , $m_{j\bar{i}}$, ---- , and $m_{j\bar{i}}^{(r)}$ are explicitly obtained. For eq.(24), only the case i = j = 1 is necessary. Defining A , B , C and D as follows:

$$\left.\begin{aligned} A &= \sum_{i=1}^{N-1} \cos \frac{2(i-1)\pi UT}{l}, \\ B &= \sum_{i=1}^{N-1} \sin \frac{2(i-1)\pi UT}{l}, \\ C &= \cos \frac{2(N-1)\pi UT}{l}, \\ D &= \sin \frac{2(N-1)\pi UT}{l}, \end{aligned}\right\} \tag{28}$$

the following time-variant quantities are needed for eq.(24):

$$\left.\begin{aligned} P(t) &= \{A + C h(\tfrac{t}{\beta T})\} \cos \frac{2\pi Ut}{l} \\ &\quad - \{B + D h(\tfrac{t}{\beta T})\} \sin \frac{2\pi Ut}{l}, \\ Q(t) &= \{A + C h(\tfrac{t}{\beta T})\} \sin \frac{2\pi U t}{l} \\ &\quad + \{B + D h(\tfrac{t}{\beta T})\} \cos \frac{2\pi Ut}{l}. \end{aligned}\right\} \tag{29}$$

Then m_{11} , $m_{1\bar{1}}$, $m_{11}^{(0)}$, $m_{11}^{(r)}$, $m_{1\bar{1}}^{(0)}$ and $m_{1\bar{1}}^{(r)}$ are obtained:

$$m_{11}(t) = \frac{m_0}{l}\{N - 1 + h(\tfrac{t}{\beta T}) - P(t)\},$$

$$\left.\begin{aligned}
m_{1\bar{1}}(t) &= \frac{m_0}{l}\,Q(t)\,,\\
m_{11}^{(0)}(t) &= -\frac{2\pi U}{l^2}\,m_0\,Q(t)\,,\\
m_{11}^{(1)}(t) &= \frac{2\pi}{l^2}\,m_0\,Q(t)\,,\\
m_{1\bar{1}}^{(0)}(t) &= -\frac{2\pi U}{l^2}\,m_0\,P(t)\,,
\end{aligned}\right\}\qquad(30)$$

and

$$m_{1\bar{1}}^{(1)}(t) = \frac{2\pi}{l^2}\,m_0\,P(t)\,,\qquad (0 \le t \le T)\,.$$

The equation of motion for the first mode of the two-phase flow induced vibrations in the pipe; eq.(24) contains the terms $U^{-1}m_{11}^{(0)}(t) + m_{11}^{(1)}(t)$ and $U^{-1}m_{1\bar{1}}^{(0)}(t) + m_{1\bar{1}}^{(1)}(t)$, but these become zero

$$U^{-1}m_{11}^{(0)} + m_{11}^{(1)} = U^{-1}m_{1\bar{1}}^{(0)} + m_{1\bar{1}}^{(1)} = 0 \qquad (31)$$

which is easily derived by using the relations in eq.(30). Therefore the equation of motion for the first mode (24) can be described as follows:

$$\left[M + \frac{N-1}{l}m_0 + \frac{m_0}{l}\left\{h\left(\frac{t}{\beta T}\right) - P(t)\right\}\right]\ddot{Y}_1 + \frac{2\pi U}{l^2}m_0\,Q(t)\,\dot{Y}_1$$

$$+\left(\frac{\pi}{l}\right)^2\left[EI\left(\frac{\pi}{l}\right)^2 - \frac{U^2 m_0}{l}(N-1) - \frac{U^2 m_0}{l}\left\{h\left(\frac{t}{\beta T}\right) - P(t)\right\}\right]Y_1 = 0,$$

$$(0 \le t \le T)\,. \qquad (32)$$

In the first term of eq.(32), $\frac{(N-1)m_0}{l}$ means the average mass of two-phase fluid contained in the pipe and $m_0(h-P)/l$ is the time-variant deviation of the mass of two-phase fluid; the second is caused by the Coriolis' force due to two-phase flow; and in the third term, $U^2(N-1)m_0/l$ expresses the average centrifugal force due to two-phase flow and $U^2m_0(h-P)/l$ is its time-variant derivation.

5. Mathieu's Equation

5.1 Transformation into Mathieu's Equation

In eq.(32), the time-variant coefficients are general periodic function with the period T, so it is rather difficult to investigate the stability of the first mode amplitude. In order to overcome this difficulty, the author expands the general periodic functions $h(t/\beta T)-P(t)$ and $Q(t)$ into Fourier series, and employs only their first term. As the arrival period of water slug is T, there exist $l/UT(=K)$ water slugs in the pipe, so that the fundamental frequency in the Fourier series is $2\pi UK/l$. Therefore

$$h\left(\frac{t}{\beta T}\right) - P(t) = -\gamma_1\cos\frac{2\pi UKt}{l}\,,\qquad Q(t) = \gamma_{\bar{1}}\sin\frac{2\pi UKt}{l}\,. \qquad (33)$$

Here the coefficients γ_1 and $\gamma_{\bar 1}$ are determined as

$$\left.\begin{aligned} \gamma_1 &= \frac{2}{T}\int_0^T \left\{ P(t) - h\left(\frac{t}{\beta T}\right)\right\} \cos\frac{2\pi U K t}{l}\, dt\,, \\ \gamma_{\bar 1} &= \frac{2}{T}\int_0^T Q(t) \sin\frac{2\pi U K t}{l}\, dt\,. \end{aligned}\right\} \tag{34}$$

In eq.(33), it should be taken account of the phase angle in each Fourier expansion, but for simplicity, the author has neglected it. However the generality of the discussions to be mentioned later would not be lost.

Substituting eq.(33) into eq.(32) and performing a slightly tedious calculation, the following ordinary differential equation is obtained:

$$\begin{aligned} \left(1 - \epsilon_1 \cos\frac{2\pi U K t}{l}\right)\ddot{Y}_1 &+ \frac{2\pi U}{l}\epsilon_{\bar 1} \sin\frac{2\pi U K t}{l}\,\dot{Y}_1 \\ &+ \omega_0^2\left\{1 + \left(\frac{\pi U}{l\omega_0}\right)^2 \epsilon \cos\frac{2\pi U K t}{l}\right\} Y_1 = 0\,, \end{aligned} \tag{35}$$

where

$$\begin{aligned} \epsilon_1 &= \frac{m_0\gamma_1}{l}\Big/\left\{M + \frac{(N-1)m_0}{l}\right\}, \quad \epsilon_{\bar 1} = \frac{m_0\gamma_{\bar 1}}{l}\Big/\left\{M + \frac{(N-1)m_0}{l}\right\}, \\ \omega_0^2 &= \frac{\left(\frac{\pi}{l}\right)^2\left\{EI\left(\frac{\pi}{l}\right)^2 - \frac{(N-1)m_0}{l}U^2\right\}}{\left\{M + \frac{(N-1)m_0}{l}\right\}}\,. \end{aligned} \tag{36}$$

ϵ_1 is smaller than 1, which is easily seen from eq.(36), so that eq.(35) can be divided by the coefficient $(1 - \epsilon_1 \cos 2\pi U K t/l)$. And taking account of ϵ_1 and $\epsilon_{\bar 1}$ being sufficiently small and considering the order $O(\epsilon_1)$ or $O(\epsilon_{\bar 1})$, eq.(35) can be rewritten in the form:

$$\begin{aligned} \ddot{Y}_1 &+ \frac{2\pi U}{l}\epsilon_{\bar 1}\sin\frac{2\pi U K t}{l}\,\dot{Y}_1 \\ &+ \omega_0^2\left[1 + \epsilon_1\left\{1 + \left(\frac{\pi U}{l\omega_0}\right)^2\right\}\cos\frac{2\pi U K t}{l}\right] Y_1 = 0. \end{aligned} \tag{37}$$

If the variable Y_1 is transformed into X_1 by the relation

$$Y_1 = \exp\left\{-\int^t \frac{2\pi U}{l}\epsilon_{\bar 1}\sin\frac{2\pi U K t}{l}\,dt\right\} X_1\,, \tag{38}$$

eq.(37) can be easily changed into Mathieu's equation:

$$\ddot{X} + \omega_0^2\left(1 - \epsilon\cos\frac{2\pi U K t}{l}\right) X = 0\,, \tag{39}$$

where the term of $O(\epsilon_1^2)$ or $O(\epsilon_{\bar 1}^2)$ is neglected and ϵ is defined as

$$\epsilon = \epsilon_1\left\{1 + \left(\frac{\pi U}{l\omega_0}\right)^2\right\} + \epsilon_{\bar 1}\left(\frac{2\pi U}{l\omega_0}\right)^2 K\,. \tag{40}$$

In the definition of ϵ, eq.(40), the first term shows the influence of the mass change of the vibrational system due to two-phase flow and of the centrifugal force, and the second is the effect of the Coriolis' force.

5.2 Stability

The theory[15] on Mathieu's equation tells us that the instability in the solution occurs when:

$$\frac{2\pi UK}{l\omega_0} = 2,\ 1,\ 2/3,\ \cdots \quad . \tag{41}$$

This relation means that when the arrival frequency of water slug in the two-phase flow makes its ratio to the eigen frequency of the piping system 2, 1, or 2/3, ---, the vibration can be induced in the piping system and becomes more and more intensive in time.

5.3 Approximate Solution

The two-phase flow has, in nature, randomness in its flow characteristics, so it is quite difficult to establish the condition (41) and as seen in later section, the vibrational strain signal is of beats. Now let us obtain the approximate solution of eq.(39) when $2\omega_0$ is slightly different from the arrival frequency of water slug in the piston flow.

Defining the difference between two frequencies as

$$\Delta = 2\omega_0 - \frac{2\pi UK}{l}, \tag{42}$$

The Mathieu's equation (39) is rewritten in the form:

$$\ddot{X}_1 + \omega_0^2\left\{1 - \epsilon\cos(2\omega_0 - \Delta)t\right\}X_1 = 0 \quad . \tag{43}$$

If X_1 is expanded into the power series of ϵ as

$$X_1 = X_1^{(0)} + \epsilon X_1^{(1)} + \cdots , \tag{44}$$

eq.(43) becomes equivalent to the following system:

$$\left.\begin{aligned} \ddot{X}_1^{(0)} + \omega_0^2 X_1^{(0)} &= 0 , \\ \ddot{X}_1^{(1)} + \omega_0^2 X_1^{(1)} &= \omega_0^2\cos(2\omega_0 - \Delta)t\, X_1^{(0)}, \\ &\vdots \end{aligned}\right\} \tag{45}$$

From the first equation of eq.(45), $X_1^{(0)}$ has two fundamental solutions

$$X_1^{(0)} = \cos\omega_0 t \ , \quad \sin\omega_0 t \quad . \tag{46}$$

For the case $X_1^{(0)} = \cos\omega_0 t$, $X_1^{(1)}$ can be easily obtained from the second equation in eq.(45), and applying the initial conditions

$$X_1(0) = \dot{X}_1(0) = 0 \tag{47}$$

to X_1 , X_1 is obtained:

$$X_1(t) = -\frac{\omega_0\epsilon}{2\Delta}\sin\frac{\Delta t}{2}\sin(\omega_0 - \frac{\Delta}{2})t \ . \tag{48}$$

Similarily for the case $X_1^{(0)} = \sin\omega_0 t$,

$$X(t) = \frac{\omega_0\epsilon}{2\Delta}\sin\frac{\Delta t}{2}\cos(\omega_0 - \frac{\Delta}{2})t \ . \tag{49}$$

Here the terms $O(\Delta/\omega_0)$ are neglected.

In the transformation (38), the parameter $2\pi U\epsilon_i/l$ is so small, then it can be said that

$$Y_1 \approx X_1 \quad .$$

Therefore the strain signal of the pipe in vibration induced by the piston flow takes the form of beats.

6. Comparison with Experimental Results

6.1 Outline of Experimental Study

Figure 3 shows a schematic diagram of the two-phase (gas + liquid) flow loop from which the void signal, meaning whether or not gas/ liquid exists, and the strain signal of the pipe in out-of-plane vibration were obtained. The two-phase flow conditions for the measurement of these signals were specified in terms of flow rates of air and water. These two signals were simultaneously recorded on magnetic tape and were translated into digital form by an analog-digital data converter. Then their power spectral density was calculated. In this statistical calculation, the size of data was 1500 digits and the sampling interval was 0.01 sec..

6.2 Stability

In Fig.4, the typical power spectral densities of the void signal(empty dot o being for void signal) of the piston flow and the strain signal (solid dot • being for strain signal) of the pipe in vibration are shown. The void signal has a dominant peak at 6.5 hz in its power spectral density and the strain signal has a sharp peak at 3.5 hz in its one. 6.5 hz for the void signal is considered as a dominant arrival frequency of water slug in the two-phase flow, on the other hand, 3.5 hz means the lowest eigen frequency of the L-shaped piping system containing the two-phase fluid.

The ratio of these two frequencies is

$$\frac{2\pi UK}{2\omega_0} \doteqdot 1.86 \quad . \tag{51}$$

From the condition (41) and eq.(51), for the case shown in Fig.4, the two-phase flow induced vibration in the L-shaped piping system is concluded as 1/2 subharmonic parametric oscillation.

6.3 Wave Form of the Vibration

Figure 5 shows the typical void signal form in the piston flow and the one of out-of-plane vibration in the L-shaped pipe for the case of the piston flow. The void signal shows the slight randomness in the arrival frequency of water slug and the vibrational strain signal is of beats with the beat period of some 4 second. On the other hand, from theoretical investigation, Fig.4 tells us the beat period is

$$\frac{4\pi}{\Delta} \doteqdot 4 \quad . \tag{52}$$

This result agrees with the experimental one.

From the theoretical and experimental studies, the author can conclude that the two-phase flow induced vibration in piping system is most intensive in the case of "piston flow" and is excited by the parametric change of the mass of the vibration system; centrifugal force and the Coriolis' force due to water slugs travelling in the pipe.

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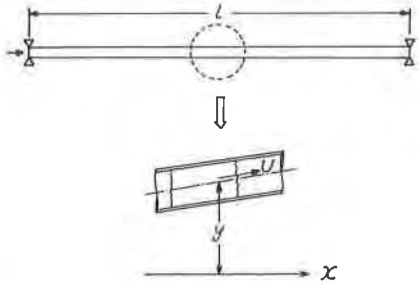

Fig.1 Illustrations of a vibrational piping system conveying two-phase flow

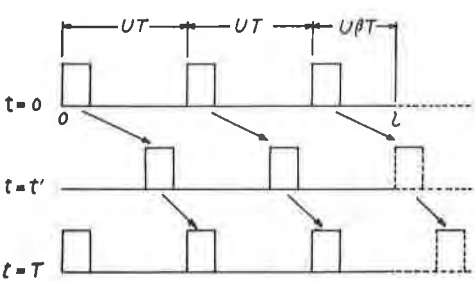

Fig.2 Illustration of the transition process of water slugs in two-phase flow

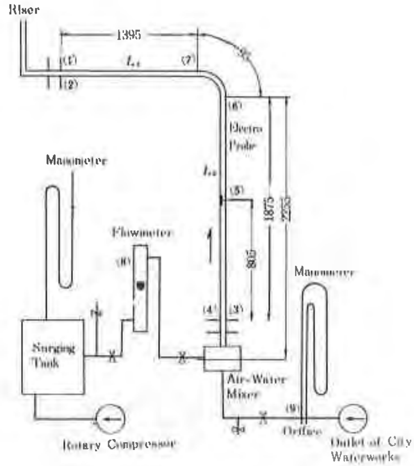

Fig.3 Schematic diagram of two-phase (air + water) flow induced vibration system

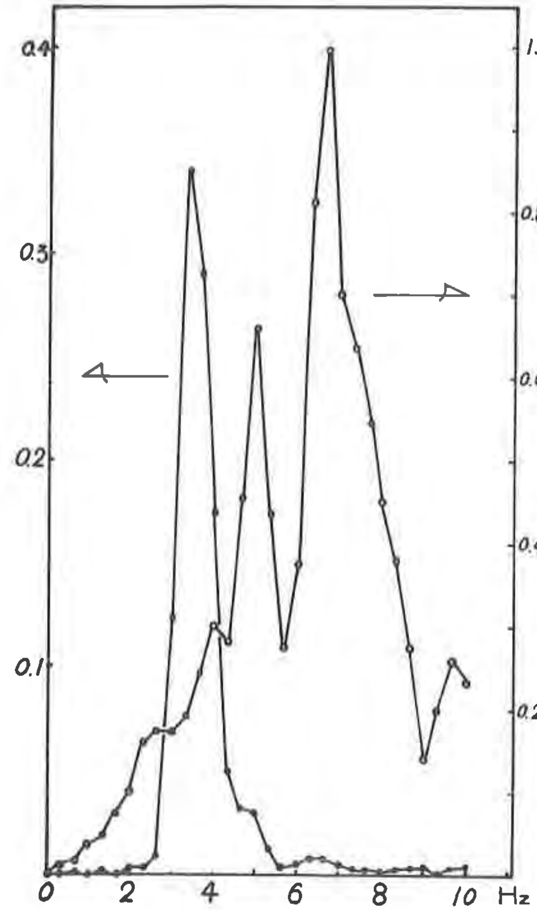

Fig.4 Typical power spectral densities of the void signal and the vibrational strain signal in piston flow

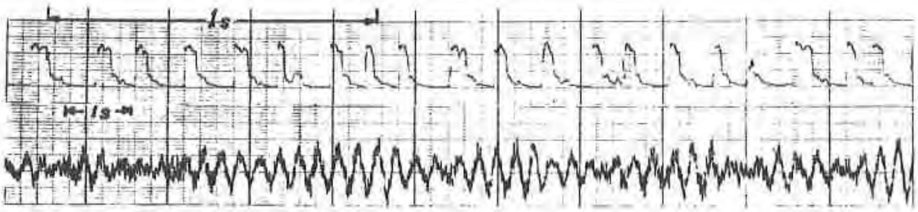

Fig.5 Typical wave forms of the void signal and the vibrational strain signal in piston flow