APPLICABILITY OF VARIOUS NON-HOMOGENEOUS ANISOTROPIC SHELL THEORIES IN PRESSURE VESSEL DESIGN

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SUMMARY

While many shell theories and solutions are available to the designer for conventional pressure vessel analysis, relatively little work exists for composite materials vessels. The fact that in composite construction the material properties have, in general, to be considered to be both anisotropic and non-homogeneous makes it difficult to derive shell theories by means of the classical method. This difficulty can, however, be overcome by using the method of asymptotic integration of elasticity equations.

In the first part of the paper, the asymptotic method is used to derive various first approximation theories, governing the unsymmetric deformation of non-homogeneous anisotropic cylindrical shells. The analysis is valid for materials which are non-homogeneous to the extent that their properties are allowed to vary with the thickness coordinate. The derivation of the shell theories is accomplished by first introducing the shell dimensions and as yet unspecified length scales via changes in the independent variables. Next, the stresses and displacements are expanded asymptotically in terms of a small geometric parameter. A choice of length scales is then made and corresponding to different combinations of these length scales, different sequences of systems of differential equations are obtained. Subsequent integration over the thickness and satisfaction of the boundary conditions yield the desired equations governing the construction of the shell stress states. For homogeneous isotropic material, the derived stress states are equivalent to those associated with the a) Donnell-Vlasov theory, b) simplified bending theory, c) membrane theory and d) semi-membrane theory. By including all terms necessary in the first approximation theories, a uniformly valid theory is also developed.

In the second part of the paper, the theories presented are extended to laminated shells. The most general case of a layered shell, one that consists of an arbitrary number of layers of different thicknesses having an arbitrary arrangement, is considered. Each layer is assumed to be homogeneous, but anisotropic with an arbitrary orientation of the elastic axes.

To illustrate the application of the derived theories, and to discuss their suitability for given loadings and shell dimensions, the problem of the laminated pressure vessel under combined loading is examined. The end conditions assumed are that the cylinder is clamped at both ends but free to rotate and extend or contract at one end. A comparison of the results obtained for a single isotropic layer, a single anisotropic layer, and for a four layer symmetric angle ply configuration (angle γ of elastic symmetry axes is oriented at $+\gamma$, $-\gamma$, $-\gamma$, $+\gamma$) clearly demonstrates the superiority of the angle ply configuration in providing additional stiffness with increase in angle of orientation. The presence of the edge effect and its change with angle of orientation is also demonstrated. The results show that the edge effect penetrates further into the shell for small orientation angles than for large ones.

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1. Introduction

In the following various theories governing the unsymmetric deformation of non-homogeneous, anisotropic cylindrical shells are derived by use of the method of asymptotic integration of the three-dimensional equations of elasticity theory[1-3]. To illustrate the application of the theories, the problem of the laminated pressure vessel under combined loading is examined.

2. Formulation as a Boundary Layer Problem

Consider a non-homogeneous, anisotropic volume element of a cylindrical body with longitudinal, circumferential(angular) and radial coordinates being noted as z, θ, r, respectively, and subjected to all possible stresses and strains. The cylinder occupies the space between $a \geq r \geq a+h$ and the edges are located at z=0 and z=l. Here, a is the inner radius, h the thickness and l the length. We assume that the deformations are sufficiently small so that linear elasticity theory is valid. A theory of shells is distinguished from the exact three-dimensional elasticity formulation of a problem by the fact that one of the coordinates is suppressed in the mathematical description. The procedure used here for obtaining the two-dimensional thin shell equations is that of the asymptotic integration of linear, three-dimensional equations describing the cylinder. As a first step towards integrating the elasticity equations, we non-dimensionalize the coordinates as follows:

$$x = z/L, \qquad y = (r-a)/h, \qquad \phi = \theta/\beta \tag{1}$$

where L and $\ell(=\beta a)$ are quantities which are to be determined later. The shell now occupies a fixed domain independent of the thickness h. Next the compliance matrix S_{ij}, the stresses σ_r, σ_θ, σ_z, τ_{rz},$\tau_{r\theta}$, $\tau_{\theta z}$ and the deformations u_r, u_θ, u_z are non-dimensionalized through the use of a representative stress level σ, a representative material property S, and the shell radius a, as follows:

$$S_{ij} = S\,\overline{S}_{ij}, \quad \sigma_z = \sigma t_z, \ \ldots\ , \tau_{\theta z} = \sigma t_{\theta z}, \quad u_r = \sigma a S V_r, \ u_\theta = \sigma a S V_\theta, \quad u_z = \sigma a S V_z \tag{2}$$

where the dimensionless displacements and stresses are functions of x, y and ϕ. These variables togather with their derivatives with respect to x, y and ϕ are assumed to be order unity. The parameters L and ℓ introduced in eq.(1) are thus seen to be characteristic legnth scales for changes of the stresses and displacements in the axial and circumferential directions, respectively. The S_{ij}'s(i,j=1,...,6) occuring in equs.(2) are the components of the compliance matrix and represent the directional properties of the material.

Complete anisotropy of the material is allowed for and there are thus 21 independent material constants. The non-homogeniety of the cylinder is restricted such that it occurs only in radial direction,

$$S_{ij} = S_{ij}(r) \tag{3}$$

The cylinder is considered to be free from surface tractions at its inner surface while the outer surface is subjected to a uniformly distributed tensile force,

$$\sigma_r = \tau_{r\theta} = \tau_{rz} = 0 \text{ at}(r=a), \quad \sigma_r = p(\theta,z), \ \tau_{r\theta} = \tau_{rz} = 0 \text{ at}(r=a+h) \tag{4}$$

In the analysis to follow, we will find it convenient to work with stress resultants rather than stresses themselves. They are defined as follows:

$$N_z = \int_a^{a+h} \sigma_z [1+ \frac{r-a-d}{a+d}] dr, \quad N_\theta = \int_a^{a+h} \sigma_\theta dr, \quad N_{\theta z} = \int_a^{a+h} \tau_{\theta z}\, dr, \quad N_{z\theta} = \int_a^{a+h} \tau_{\theta z} [1+ \frac{r-a-d}{a+d}] dr \tag{5}$$

$$M_z = \int_a^{a+h} \sigma_z [\frac{r-a-d}{a+d}] r dr, \quad M_\theta = \int_a^{a+h} \sigma_\theta (r-a-d) dr, \quad M_{\theta z} = \int_a^{a+h} \tau_{\theta z} (r-a-d) dr, \quad M_{z\theta} = \int_a^{a+h} \tau_{\theta z} [\frac{r-a-d}{a+d}] r dr$$

In the above d is the distance from the inner surface to the reference surface where the stress resultants are defined. Substitution of dimensionless variables defined by (1) and (2) into the elasticity equations yields the following dimensionless equations:

Stress-Displacement Relations

$$v_{r,y} = \lambda[\overline{S}_{31} t_z + \overline{S}_{32} t_\theta + \overline{S}_{33} t_r + \overline{S}_{34} t_{r\theta} + \overline{S}_{35} t_{rz} + \overline{S}_{36} t_{\theta z}]$$

$$v_{z,y} + \lambda(a/L) v_{r,x} = \lambda[\overline{S}_{51} t_z + \ldots + \overline{S}_{56} t_{\theta z}]$$

$$\lambda v_{r,\phi} + (\ell/a)(1+\lambda y) v_{\theta,y} - (\ell/a)\lambda v_\theta = \lambda(\ell/a)(1+\lambda y)[\overline{S}_{41} t_z + \ldots + \overline{S}_{46} t_{\theta z}] \tag{6}$$

$$v_{z,x} = (L/a)[\overline{S}_{11} t_z + \ldots + \overline{S}_{16} t_{\theta z}], \quad (a/\ell) v_{\theta,\phi} + v_r = (1+\lambda y)[\overline{S}_{21} t_z + \ldots + \overline{S}_{26} t_{\theta z}]$$

$$(a/L)(1+\lambda y) v_{\theta,x} + (a/\ell) v_{z,\phi} = (1+\lambda y)[\overline{S}_{61} t_z + \ldots + \overline{S}_{66} t_{\theta z}]$$

Equilibrium Equations

$$[t_{rz}(1+\lambda y)]_{,y} + (\lambda a/\ell) t_{\theta z,\phi} + (\lambda a/L)(1+\lambda y) t_{z,x} = 0$$

$$[t_{r\theta}(1+\lambda y)]_{,y} + (\lambda a/\ell) t_{\theta,\phi} + t_{r\theta} + (\lambda a/L)(1+\lambda y) t_{\theta z,x} \tag{7}$$

$$[t_r(1+\lambda y)]_{,y} + (\lambda a/\ell) t_{r\theta,\phi} + (\lambda a/L)(1+\lambda y) t_{rz,x} - t_\theta = 0$$

where

$$\lambda = h/a \tag{8}$$

and a comma indicates partial differentiation with respect to the indicated coordinate. We restrict ourselves to the case where the thickness h is much less than the radius a,

$$\lambda >> 1 \tag{9}$$

The dimensionless coefficients $\overline{S}_{ij}$ of the compliance matrix, in general, are not all of the same order. We therefore assume that they can be expanded in terms of a finite sum as follows:

$$\overline{S}_{ij}(y;\lambda) = \sum_{n=0}^{M} S_{ij}^{(n)}(y)\lambda^{n/2} \tag{10}$$

where the $S_{ij}^{(n)}(y)$ are of order unity or vanish identically.

It is next assumed that each displacement component and each stress component can be expanded in terms of a power series in $\lambda^{1/2}$.

$$v(y,x,\phi;\lambda)=\sum_{m=0}^{M} v^{(m)}(y,x,\phi)\lambda^{m/2}, \quad v(y,x,\phi;\lambda)=\sum_{m=0}^{M} t^{(m)}(y,x,\phi)\lambda^{m/2} \tag{11}$$

where $v^{(m)}$ and $t^{(m)}$ are of order unity. Length scales L and ℓ are as yet arbitrary. Their choice, as will be seen in the section to follow, determines the type of shell theory to be obtained. Based in part on the results obtained for isotropic, homogeneous shells, shell equations corresponding to the following combinations of length scales will be derived:

$$a)L=a,\ell=a; \quad b)L=(ah)^{1/2},\ell=a; \quad c)L=a(ah)^{1/2},\ell=a; \quad d)L=(ah)^{1/2},\ell=(ah)^{1/2} \tag{12}$$

The last step in the procedure consists of substituting expansions (10) and (11) and one of the combinations of length scales into dimensionless elasticity equations (6) and (7). Upon equating terms of like powers in $\lambda^{1/2}$ on both sides of each equations and requiring that the resulting equations be a) integrable with respect to y in a step by step manner and b) be capable of yielding relations for all stresses and displacement components a sequence of systems of differential equations is obtained. In order to meet these two requirements, the leading term in expansions (10) and (11) may not be the term corresponding to m=0. The first system of equations thus obtained will be seen to yield the approximate "thin shell" equations and will be called the first approximation system. The other systems yield the higher order terms in the expansions and introduce thickness corrections. Thus a thick shell theory can be obtained in a systematic and consistant manner.

3. Theory Associated with Characteristic Length Scales a

We are interested here in deriving the shell theory associated with the case where the axial and circumferential length scales are both equal to the inner radius of the cylinder.

$$L = a, \qquad \ell = a \tag{13}$$

On substituting these length scales and the asymptotic expansions for the displacements and stresses into eqs. (6) and (7), the following equations representing the first approximation theory of the problem result upon use of the procedure outlined previously:

$$\begin{gathered}
v^{(0)}_{r,y} = 0,\ v^{(0)}_{z,y} = 0,\ v^{(0)}_{\theta,y} = 0;\ t^{(2)}_{rz,y}+t^{(0)}_{\theta z,\phi}+t^{(0)}_{z,x}=0,\ t^{(2)}_{r,y}-t^{(0)}_{\theta}=0,\ t^{(2)}_{rz,y}+t^{(0)}_{\theta,\phi}+t^{(0)}_{\theta z,x}=0 \\
v^{(0)}_{z,x} = S^{(0)}_{11}t^{(0)}_{z}+S^{(0)}_{12}t^{(0)}_{\theta}+S^{(0)}_{16}t^{(0)}_{\theta z},\quad v^{(0)}_{\theta,\phi}+v^{(0)}_{r}=S^{(0)}_{21}t^{(0)}_{z}+S^{(0)}_{22}t^{(0)}_{\theta}+S^{(0)}_{26}t^{(0)}_{\theta z} \\
v^{(0)}_{\theta,x}+v^{(0)}_{z,\phi} = S^{(0)}_{61}t^{(0)}_{z}+S^{(0)}_{62}t^{(0)}_{\theta}+S^{(0)}_{66}t^{(0)}_{\theta z}
\end{gathered} \tag{14}$$

The superscripts indicate the leading term in each of the expansions (11) and represent the relative order of magnitude of the displacements and stresses. Integrations of first three equations of (14) with respect to y yields,

$$v^{(0)}_{r}=V^{(0)}_{r}(x,\phi),\quad v^{(0)}_{z}=V^{(0)}_{z}(x,\phi),\quad v^{(0)}_{\theta}=V^{(0)}_{\theta}(x,\phi) \tag{15}$$

where $V^{(0)}_{r}$, $V^{(0)}_{z}$, $V^{(0)}_{\theta}$ are the displacements of the y=0(r=a) surface.

The middle three equations can be solved for the in-plane stresses as follows:

$$\begin{Bmatrix} t^{(0)}_{z} \\ t^{(0)}_{\theta} \\ t^{(0)}_{\theta z} \end{Bmatrix} = [C] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} \tag{16}$$

Here, C_{ij}(i,j=1,2,3) are the components of a symmetric matrix given by

$$[C] = \begin{bmatrix} S^{(0)}_{11} & S^{(0)}_{12} & S^{(0)}_{16} \\ S^{(0)}_{12} & S^{(0)}_{22} & S^{(0)}_{26} \\ S^{(0)}_{16} & S^{(0)}_{26} & S^{(0)}_{66} \end{bmatrix}^{-1} \tag{17}$$

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and ε_1, ε_2, ε_{12} are the in-plane strain components of the y=0 surface:

$$\varepsilon_1 = V^{(0)}_{z,x}, \quad \varepsilon_2 = V^{(0)}_{\theta,\phi} + V^{(0)}_{r}, \quad \varepsilon_{12} = V^{(0)}_{\theta,x} + V^{(0)}_{r,\phi} \tag{18}$$

On substituting the eqs.(16) into the last three equations of (14), integrating with respect to y and then satisfying boundary conditions(4) the following three differential equations for V_r, V_z and V_θ are obtained:

$$\underline{A}_{13}V_{z,x\phi}+\underline{A}_{23}(V_{\theta,\phi\phi}+V_{r,\phi})+\underline{A}_{33}(V_{\theta,x\phi}+V_{z,\phi\phi})+\underline{A}_{11}V_{z,xx}+\underline{A}_{12}(V_{\theta,x\phi}+V_{r,x})+\underline{A}_{13}(V_{\theta,xx}+V_{z,x\phi})=0$$

$$\underline{A}_{12}V_{z,x\phi}+\underline{A}_{22}(V_{\theta,\phi\phi}+V_{r,\phi})+\underline{A}_{23}(V_{\theta,x\phi}+V_{z,\phi\phi})+\underline{A}_{13}V_{z,xx}+\underline{A}_{23}(V_{\theta,\phi x}+V_{r,x})+\underline{A}_{33}(V_{\theta,xx}+V_{z,x\phi})=0$$

$$\underline{A}_{12}V_{z,x}+\underline{A}_{22}(V_{\theta,\phi}+V_{r})+\underline{A}_{23}(V_{\theta,x}+V_{z,\phi})=p^* \tag{19}$$

In the above equations

$$A_{ij}=\int C_{ij}\,dy, \quad \underline{A}_{ij}=A_{ij}(1), \quad p^*=p/(\sigma\lambda) \tag{20}$$

To obtain the appropriate expressions for the stress resultants we first non-dimensionalize those defined by (5) as follows:

$$\overline{N} = N/(\sigma\lambda a), \qquad \overline{M} = M/(\sigma\lambda^2 a^2) \tag{21}$$

Assuming it to be possible, we now asymptotically expand each of the dimensionless stress resultants in a power series in $\lambda^{1/2}$.

$$N=\sum_{m=0}^{M} N^{(m)}(x,\phi)\lambda^{m/2}, \quad M=\sum_{m=0}^{M} M^{(m)}(x,\phi)\lambda^{m/2} \tag{22}$$

where $N^{(m)}$ and $M^{(m)}$ are of the order unity. On substitution of (21), (22) and the results for in-plane stresses (16) into relations (5) and equating terms of like powers in $\lambda^{1/2}$ on each side of the equations, we obtain the following expressions for the first approximation stress resultants:

$$\left\{\begin{array}{c} N_i \\ \hline M_i \end{array}\right\} = \left[\begin{array}{c} \overline{A} \\ \hline \overline{B} \end{array}\right] \{\varepsilon_i\} \tag{23}$$

where $\{N_i\}$, $\{M_i\}$, $\{\varepsilon_i\}$ are column vector given by $\{N_i\} = \{N_z, N_\theta, N_{z\theta}, N_{\theta z}\}$, $\{M_i\} = \{M_z, M_\theta, M_{z\theta}, M_{\theta z}\}$, $\{\varepsilon_i\} = \{\varepsilon_1, \varepsilon_2, \varepsilon_{12}\}$ (where the superscripts zero have been omitted) and B_{ij} is defined as follows:

$$B_{ij} = \int C_{ij}\,y\,dy, \quad \underline{B}_{ij} = B_{ij}(1) \tag{24}$$

and the components of submatrices $\overline{A}$ and $\overline{B}$ are given by

$$\overline{A}_{ij}=[1/(1+d/a)]\underline{A}_{ij},\quad \overline{B}_{ij}=[1/(1+d/a)](-\frac{d}{h}\underline{A}_{ij}+\underline{B}_{ij}),\ (i=1,3;j=1,2,3)$$
$$\overline{A}_{ij}=\underline{A}_{ij},\quad \overline{B}_{ij}=-\frac{d}{h}\underline{A}_{ij}+\underline{B}_{ij},\ (i=2,4;\ j=1,2,3) \tag{25}$$

For an isotropic, homogeneous material, the C_{ij} are constants and $d/h=1/2$. This yields,

$$\underline{B}_{ij}=(1/2)C_{ij}=(1/2)\underline{A}_{ij} \tag{26}$$

On substituting this result into relations (25) it is seen that submatrix $\overline{B}$ is equal to zero and that relations (23) become those of the classical membrane theory of shell(zero moment resultants).

4. Theory Associated with Longitudinal Length Scale $(ah)^{1/2}$ and Circumferential Length Scale a

In the previous section, it was shown that the theory corresponding to the length scales equal to the inner radius of the cylinder, a, yielded displacements which were constant over the thickness and moment stress resultants solely due to the variation of the material properties with respect to the thickness coordinate. In this section a shell theory with the same circumferential length scale but a shorter axial one, i.e.

$$L=(ah)^{1/2},\qquad \ell=a \tag{27}$$

is presented. As the derivation, the method being the same as in the previous section, is lengthy only the final results are presented. They are,

$$v_r^{(0)}=V_r^{(0)}(x,\phi);\quad v_z^{(1)}=V_z^{(1)}(x,\phi)-V_{r,x}^{(0)}y;\quad v_\theta^{(2)}=V_\theta^{(2)}(x,\phi)-V_{r,\phi}^{(0)}y \tag{28}$$

$$\begin{Bmatrix} t_z^{(0)} \\ t_\theta^{(0)} \\ t_{\theta z}^{(1)} \end{Bmatrix} = [C]\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} + [C]\begin{Bmatrix} K_1 \\ K_2 \\ K_{12} \end{Bmatrix} y \tag{29}$$

where [C] is given by (17) with C_{13}, C_{31}, C_{23}, C_{32} equal to zero and

$$\varepsilon_1=V_{z,x}^{(1)},\qquad \varepsilon_2=V_r^{(0)},\qquad \varepsilon_{12}=V_{z,\phi}^{(1)}+V_{\theta,x}^{(2)} \tag{30}$$

$$K_1=-V_{r,xx}^{(0)},\quad K_2=0,\quad K_{12}=-2V_{r,x\phi}^{(0)} \tag{31}$$

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$$\underline{A}_{11}V_{z,xx} + \underline{A}_{12}V_{r,x} - \underline{B}_{11}V_{r,xxx} = 0$$

$$\underline{A}_{12}V_{z,x\phi} - \underline{B}_{12}V_{r,xx\phi} + \underline{A}_{22}V_{r,\phi} + \underline{A}_{33}V_{z,x\phi} + 2\underline{B}_{33}V_{r,xx\phi} - \underline{A}_{33}V_{\theta,xx} = 0 \quad (32)$$

$$\underline{A}_{12}V_{z,x} - \underline{B}_{12}V_{r,xx} + \underline{A}_{22}V_{r} + \underline{D}_{11}V_{z,xxx} + \underline{D}_{12}V_{r,xx} - \underline{E}_{11}V_{r,xxxx} = p^*$$

where

$$D_{ij} = \int A_{ij}dy,\ \underline{D}_{ij} = D_{ij}(1);\quad E_{ij} = \int B_{ij}dy,\ \underline{E}_{ij} = E_{ij}(1) \quad (33)$$

The stress resultants for this theory are given by,

$$\left\{\begin{array}{c} N_i \\ \hline M_i \end{array}\right\} = \left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{B} & \overline{D} \end{array}\right]\left\{\begin{array}{c} \varepsilon_{id} \\ \hline K_i \end{array}\right\} \quad (34)$$

where $\{\varepsilon_{id}\}$, $\{K_i\}$ are column vectors given by $\{\varepsilon_{1d},\ \varepsilon_{2d},\ \varepsilon_{12d}\}$ and $\{K_1, K_2, K_{12}\}$, respectively. Here, ε_{1d}, ε_{2d}, ε_{12d} are the strain components of the $y = d/h$ surface, where the stress resultants are defined:

$$\varepsilon_{1d} = \varepsilon_1 + (d/h)K_1,\quad \varepsilon_{2d} = \varepsilon_2 + (d/h)K_2,\quad \varepsilon_{12d} = \varepsilon_{12} + (d/h)K_{12} \quad (35)$$

The stiffness matrix components $\overline{A}_{ij}$ and $\overline{B}_{ij}$ are given by eq.(25) except that $\overline{A}_{13}$, $\overline{A}_{23}$, $\overline{A}_{31}$, $\overline{A}_{32}$, $\overline{A}_{41}$, $\overline{A}_{42}$ and the like components of $\overline{B}_{ij}$ are equal to zero. The components of $\overline{D}_{ij}$ are given by

$$\overline{D}_{ij} = [1/(1+d/a)](\underline{F}_{ij} - 2\frac{d}{h}\underline{B}_{ij} + \frac{d^2}{h^2}\underline{A}_{ij}) \quad \text{for } (i=1,3;\ j=1,2,3)$$

$$\overline{D}_{ij} = \underline{F}_{ij} - 2\frac{d}{h}\underline{B}_{ij} + \frac{d^2}{h^2}\underline{A}_{ij} \quad \text{for } (i=2,4;\ j=1,2,3) \quad (36)$$

Here, $F_{ij} = \int C_{ij}y^2dy$, $\underline{F}_{ij} = F_{ij}(1)$ (37)

5. <u>Theory Associated with Longitudunal Length Scale $a(ah)^{1/2}$ and Circumferential Length a</u>

The shell theory developed in this section is somewhat unique as it is the only theory which is constructed with a shorter circumferential length scale than the axial one. The chosen length scales are:

$$L = a(ah)^{1/2},\quad \ell = a \quad (38)$$

It is also necessary to consider the second approximation equation to obtain the first approximation results. These are given by:

$$v_r^{(0)} = V_r^{(0)}(x,\phi); \quad v_z^{(1)} = V_z^{(1)}(x,\phi); \quad v_\theta^{(0)} = V_\theta^{(0)}(x,\phi) \tag{39}$$

$$V_{\theta,\phi}^{(0)} + V_r^{(0)} = 0\ ; \quad V_{\theta,x}^{(0)} + V_{z,\phi}^{(1)} = 0 \tag{40}$$

$$v_r^{(2)} = V_r^{(2)}(x,\phi); \quad v_z^{(3)} = V_z^{(3)}(x,\phi) - V_{r,x}^{(0)}y; \quad v_\theta^{(2)} = V_\theta^{(2)}(x,\phi)+(V_\theta^{(0)} - V_{r,\phi}^{(0)})y \tag{41}$$

$$\begin{Bmatrix} t_z^{(2)} \\ t_\theta^{(2)} \\ t_{\theta z}^{(3)} \end{Bmatrix} = [C]\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} + [C]\begin{Bmatrix} K_1 \\ K_2 \\ K_{12} \end{Bmatrix} y \tag{42}$$

where [C] is given as in Section 4 and $\varepsilon_1 = V_{z,x}^{(1)}$; $\varepsilon_2 = V_{\theta,\phi}^{(2)} + V_r^{(2)}$; $\varepsilon_{12} = V_{\theta,x}^{(2)} + V_{z,\phi}^{(3)}$ (43)

and $K_1 = 0$; $K_2 = V_{\theta,\phi}^{(0)} - V_{r,\phi\phi}^{(0)}$; $K_{12} = 2(V_{\theta,x}^{(0)} - V_{r,x\phi}^{(0)})$ (44)

$$V_{z,x}^{(1)}\underline{A}_{12} + (V_{\theta,\phi}^{(2)} + V_r^{(2)})\underline{A}_{22} + (V_{\theta,\phi}^{(0)} - V_{r,\phi\phi}^{(0)})\underline{B}_{22} = p^* \tag{45}$$

$$\begin{aligned} V_{\theta,x\phi}^{(2)} + V_{z,\phi\phi}^{(3)} = {} & (\underline{A}_{12}/\underline{A}_{22}\underline{A}_{33})[V_{z,xx}^{(1)}\underline{A}_{12} + (V_{\theta,\phi x}^{(0)} - V_{r,\phi\phi x}^{(0)})\underline{B}_{22} - p^*_{,x}] \\ & - (2\underline{B}_{33}/\underline{A}_{33})(V_{,x}^{(0)} - V_{r,x}^{(0)}) - (\underline{A}_{11}/\underline{A}_{33})V_{z,xx}^{(1)} \\ & - (\underline{B}_{12}/\underline{A}_{33})(V_{\theta,x\phi}^{(0)} - V_{r,x\phi\phi}^{(0)}) \end{aligned}$$

$$\begin{aligned} & [\underline{D}_{12} - (\underline{D}_{22}\underline{A}_{12}/\underline{A}_{22})]V_{z,x\phi\phi} + [\underline{A}_{11} - (\underline{A}_{12}^2/\underline{A}_{22})]V_{z,xxx} + [\underline{D}_{12} - (\underline{D}_{22}\underline{A}_{12}/\underline{A}_{22})]V_{z,x\phi\phi\phi\phi} \\ & + [\underline{E}_{22} - (\underline{B}_{22}\underline{D}_{22}/\underline{A}_{22})]V_{\theta,\phi\phi\phi} + [\underline{B}_{12} - (\underline{A}_{12}\underline{B}_{22}/\underline{A}_{22})]V_{\theta,xx\phi} + [\underline{E}_{22} - (\underline{B}_{22}\underline{D}_{22}/\underline{A}_{22})]V_{\theta,\phi\phi\phi\phi\phi} \\ & + [-\underline{B}_{12} + (\underline{A}_{12}\underline{B}_{22}/\underline{A}_{22})]V_{r,xx\phi} - [\underline{E}_{22} - (\underline{B}_{22}\underline{D}_{22}/\underline{A}_{22})]V_{r,\phi\phi\phi\phi} \\ & - [\underline{E}_{22} - (\underline{B}_{22}\underline{D}_{22}/\underline{A}_{22})]V_{r,\phi\phi\phi\phi\phi\phi} + (\underline{A}_{12}/\underline{A}_{22})p^*_{,xx} = 0 \end{aligned}$$

where $p^* = p/(\sigma\lambda^2)$ (46)

The theory developed here corresponds to the semi-membrane theory for homogeneous, isotropic shells.

6. Theory Associated with Length Scales $(ah)^{1/2}$

The theories developed in the previous chapters all were associated with the circumferential characteristic length scale a. As one might expect more rapid variation in this

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direction for certain types of loading, a theory with a short circumferential length scale is now developed, i.e.,

$$L = (ah)^{1/2}, \qquad \ell = (ah)^{1/2} \tag{47}$$

The results for this theory are:

$$V_r^{(0)}=V_r^{(0)}(x,\phi),\quad V_z^{(1)}=V_z^{(1)}(x,\phi)-V_{r,x}^{(0)}y,\quad V_\theta^{(1)}=V_\theta^{(1)}(x,\phi)-V_{r,\theta}^{(0)}y \tag{48}$$

$$\begin{Bmatrix} t_z^{(0)} \\ t_\theta^{(0)} \\ t_{\theta z}^{(0)} \end{Bmatrix} = [C]\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} + [C]\begin{Bmatrix} K_1 \\ K_2 \\ K_{12} \end{Bmatrix} y \tag{49}$$

$$\varepsilon_1=V_{z,x}^{(1)},\quad \varepsilon_2=V_r^{(0)}+V_{\theta,\phi}^{(1)},\quad \varepsilon_{12}=V_{\theta,x}^{(1)}+V_{z,\phi}^{(1)},\quad K_1=-V_{r,xx}^{(0)},\quad K_2=-V_{r,\phi\phi}^{(0)},\quad K_{12}=-2V_{r,x\phi}^{(0)} \tag{50}$$

$$\begin{aligned} &\underline{A}_{1j}\varepsilon_{j,x} + \underline{A}_{3j}\varepsilon_{j,\phi} + \underline{B}_{1j}K_{j,x} + \underline{B}_{3j}K_{j,\phi} = 0 \\ &\underline{A}_{3j}\varepsilon_{j,x} + \underline{A}_{2j}\varepsilon_{j,\phi} + \underline{B}_{3j}K_{j,x} + \underline{B}_{2j}K_{j,\phi} = 0 \\ &\underline{A}_{2j}\varepsilon_j + \underline{D}_{1j}\varepsilon_{j,xx} + 2\underline{D}_{3j}\varepsilon_{j,x} + \underline{D}_{2j}\varepsilon_{j,\phi} + \underline{B}_{2j}K_j + \underline{E}_{1j}K_{j,xx} + 2\underline{E}_{3j}K_{j,x\phi} \\ &\qquad + \underline{E}_{2j}K_{j,\phi\phi} = -p^* \end{aligned} \tag{51}$$

where p^* is given by (20) and [C] is given by (17).

The stress resultants are of the form (34) where $[\overline{A}]$ and $[\overline{B}]$ are given by (25) and

$$\begin{aligned} &\overline{D}_{ij}=[1/(1+d/a)](\underline{B}_{ij}-\underline{E}_{ij}-2\,\frac{d}{h}\underline{B}_{ij} + \frac{d^2}{h^2}\underline{A}_{ij}) \quad (i=1,3;\ j=1,2,3) \\ &\overline{D}_{ij}= \underline{B}_{ij}-\underline{E}_{ij}-2\,\frac{d}{h}\,\underline{B}_{ij} + \frac{d^2}{h^2}\,\underline{A}_{ij} \qquad (i=2,4;\ j=1,2,3) \end{aligned} \tag{52}$$

7. Uniformly Valid Theory

In this section the results for a uniformly valid theory are presented. This theory can be obtained by including all terms found necessary in each of the proceding theories. The starting equations are given by

$$u_{r,y}=0,\ u_{z,y}+\lambda a u_{r,z}=0,\quad \lambda u_{r,\theta}+(1+\lambda y)u_{\theta,y}-\lambda u_\theta = 0,\quad [(1+\lambda y)\tau_{r\theta}]_{,y}+\lambda\sigma_{\theta,\theta}+a\lambda\tau_{\theta z,z}+\lambda\tau_{r\theta} = 0$$

$$u_{z,z} = S_{11}\sigma_z + S_{12}\sigma_\theta + S_{16}\tau_{\theta z}, \qquad \tau_{rz,y}+\lambda\tau_{\theta z,\theta}+a\lambda\sigma_{z,z} = 0 \tag{53}$$

$$(1/a)(u_{\theta,\theta}+u_r) = S_{21}\sigma_z + S_{22}\sigma_\theta + S_{26}\tau_{\theta z}, \quad [(1+\lambda y)\sigma_r]_{,y}+\lambda\tau_{r\theta,\theta} + a\lambda\tau_{rz,z} - \lambda\sigma_\theta = 0$$

$$(1+\lambda y)u_{\theta,z}+(1/a)u_{z,\theta} = S_{61}\sigma_z + S_{62}\sigma_\theta + S_{66}\tau_{\theta z} \qquad (53)\ \text{Cont'd}$$

Note that the only dimensionless quantity appearing in the above is the radial coordinate y. The first six equations of (53) yield the following results:

$$u_r=U_r(z,\theta),\ u_z=U_z(z,\theta)-a\lambda U_{r,z}y,\ u_\theta=U_\theta(1+\lambda y)-U_{r,\theta}y$$

$$\begin{Bmatrix}\sigma_z\\ \sigma_\theta\\ \tau_{\theta z}\end{Bmatrix} = [C]\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_{12}\end{Bmatrix} + [C]\begin{Bmatrix}K_1\\ K_2\\ K_{12}\end{Bmatrix}\lambda a y \qquad (54)$$

where

$$[C] = \begin{bmatrix} S_{11} & S_{12} & S_{16}\\ S_{12} & S_{22} & S_{26}\\ S_{16} & S_{26} & S_{66}\end{bmatrix}^{-1}$$

$$\varepsilon_1=U_{z,z};\ \varepsilon_2=(1/a)(U_r+U_{\theta,\theta});\ \varepsilon_{12}=U_{\theta,z}+(1/a)U_{z,\theta};\ K_1=-U_{r,zz};\ K_2=-(1/a^2)(U_{r,\theta\theta}-U_{\theta,\theta});$$

$$K_{12}=-(2/a)(U_{r,\theta z}-U_{\theta,z}) \qquad (55)$$

The last three equations of (53) yield, upon satisfaction of the boundary conditions,

$$\underline{A}_{11}aU_{z,zz}+2\underline{A}_{13}U_{z,\theta z}+\underline{A}_{33}(1/a)U_{z,\theta\theta}+\underline{A}_{13}aU_{\theta,zz}+(\underline{A}_{12}+\underline{A}_{33})U_{\theta,\theta z}+\underline{A}_{23}(1/a)U_{\theta,\theta\theta}+\underline{A}_{12}U_{r,z}+$$
$$\underline{A}_{23}(1/a)U_{r,\theta}-a\lambda[\underline{B}_{11}aU_{r,zzz}+3\underline{B}_{13}U_{r,zz\theta}+(1/a)(\underline{B}_{12}+2\underline{B}_{33})U_{r,z\theta\theta}+\underline{B}_{23}(1/a^2)U_{r,\theta\theta\theta}+$$
$$+(1/a)(\underline{B}_{12}+2\underline{B}_{33})U_{r,z}-2\underline{B}_{13}U_{\theta,zz}] = 0 \qquad (56)$$

$$\underline{A}_{13}aU_{z,zz}+(\underline{A}_{12}+\underline{A}_{33})U_{z,\theta z}+\underline{A}_{23}(1/a)U_{z,\theta\theta}+\underline{A}_{22}(1/a)U_{\theta,\theta\theta}+\underline{A}_{33}aU_{\theta,zz}+2\underline{A}_{23}U_{\theta,\theta z}+$$
$$\underline{A}_{23}U_{r,z}+\underline{A}_{22}(1/a)U_{r,\theta}-a\lambda[\underline{B}_{13}aU_{r,zzz}+(\underline{B}_{12}+2\underline{B}_{33})U_{r,\theta zz}+3\underline{B}_{23}(1/a)U_{r,\theta\theta z}+$$
$$\underline{B}_{22}(1/a^2)U_{r,\theta\theta\theta}+2\underline{B}_{23}(1/a)U_{r,z}+\underline{B}_{22}(1/a^2)U_{r,\theta}] - a\lambda^2[2\underline{F}_{23}(1/a)U_{r,\theta\theta z}+ \qquad (57)$$
$$\underline{F}_{22}(1/a^2)U_{r,\theta\theta\theta}-2\underline{F}_{23}(1/a)U_{\theta,\theta z}-\underline{F}_{22}(1/a\ )U_{\theta,\theta\theta}] = 0$$

$$\underline{A}_{12}U_{z,z}+\underline{A}_{22}(1/a)(U_r+U_{\theta,\theta})+\underline{A}_{23}(U_{\theta,z}+U_{z,\theta}/a) - a\lambda[\underline{B}_{12}U_{r,zz}+\underline{B}_{22}(U_{r,\theta\theta}+U_r)/(a^2)+$$
$$2\underline{B}_{13}(U_{r,\theta z}-U_{\theta,z})/a] + a\lambda[a\underline{D}_{11}U_{z,zzz}+3\underline{D}_{13}U_{z,zz\theta}+(\underline{D}_{12}+2\underline{D}_{33})U_{z,\theta\theta z}/a +$$
$$(1/a^2)\underline{D}_{23}U_{z,\theta\theta\theta}+a\underline{D}_{13}U_{\theta,zzz}+(\underline{D}_{12}+2\underline{D}_{33})U_{\theta,zz\theta}+(3/a)\underline{D}_{23}U_{\theta,\theta\theta z}+(1/a^2)\underline{D}_{22}U_{\theta,\theta\theta\theta}+$$
$$\underline{D}_{12}U_{r,zz}+2\underline{D}_{23}U_{r,\theta z}/a+\underline{D}_{22}U_{r,\theta\theta}/(a^2)] - a^2\lambda^2[\underline{E}_{13}U_{r,\theta zzz}+\underline{E}_{23}U_{r,\theta\theta\theta z}/(a^2)+ \tag{58}$$
$$2\underline{E}_{33}U_{r,\theta\theta zz}/a] - a^3\lambda^2[\underline{E}_{11}U_{r,zzzz}+\underline{E}_{12}U_{r,\theta\theta zz}/a^2+2\underline{E}_{13}U_{r,\theta zzz}/a] -$$
$$a\lambda^2[\underline{E}_{12}U_{r,\theta\theta zz}+\underline{E}_{22}(U_{r,\theta\theta\theta\theta}+U_{r,\theta\theta})/a^2+2\underline{E}_{23}U_{r,\theta\theta\theta z}/a] -$$
$$a^2\lambda^2[\underline{E}_{13}U_{r,zzz\theta}+\underline{E}_{23}U_{r,\theta\theta\theta z}/a^2+2\underline{E}_{33}U_{r,\theta\theta zz}/a] = p$$

In derivation of (56-58) we have omitted all terms which the preceding analysis showed to be of higher order. The stress resultants for the uniformly valid theory are given by eq.(34) where

$$\varepsilon_{1d}=\varepsilon_1+dK_1;\quad \varepsilon_{2d}=\varepsilon_2+dK_2;\quad \varepsilon_{12d}=\varepsilon_{12}+dK_{12} \tag{59}$$

where d is the distance at which the stress resultants are defined and

$$\bar{A}_{ij}=\frac{\lambda a}{1+d/a}\underline{A}_{ij};\quad \bar{B}_{ij}=\frac{\lambda a}{1+d/a}(\lambda a\underline{B}_{ij}-d\underline{A}_{ij});\quad \bar{D}_{ij}=\frac{\lambda^2 a^2}{1+d/a}(a\underline{F}_{ij}-2d\underline{B}_{ij}+\frac{d^2}{\lambda a}\underline{A}_{ij})\quad (i=1,3;j=1,2,3)$$
$$\bar{A}_{ij}=a\underline{A}_{ij};\quad \bar{B}_{ij}=a(a\underline{B}_{ij}-d\underline{A}_{ij});\quad \bar{D}_{ij}=\lambda^2a^2(\lambda a\underline{F}_{ij}-2d\underline{B}_{ij}+\frac{d^2}{\lambda a}\underline{A}_{ij})\quad (i=2,4;j=1,2,3) \tag{60}$$

8. Application and Discussion

For the purpose of examining the suitability and applicability of the derived theories, the problem of a laminated cylinder under internal pressure and edge loadings is considered. If we assume that each layer is made from a homogeneous anisotropic material, then the material constants appearing in the shell equations can be written as follows:

$$\underline{A}_{ij}=\sum_{n=1}^{N}(s_{n+1}-s_n)\,C_{ij}^{(n)},\quad \underline{B}_{ij}=\sum_{n=1}^{N}(1/2)(s_{n+1}^2-s_n^2)\,C_{ij}^{(n)},$$
$$\underline{F}_{ij}=\sum_{n=1}^{N}(1/3)(s_{n+1}^3+s_n^3)\,C_{ij}^{(n)},\quad \underline{D}_{ij}=\underline{A}_{ij}-\underline{B}_{ij},\quad \underline{E}_{ij}=\underline{B}_{ij}-\underline{F}_{ij} \tag{61}$$

Here, we are considering a shell of N layers where the s_n signify the dimensionless distances measured from the inner surface of the shell($s_1=0$, ..., $s_{N+1}=1.$). Let the cylinder be subjected to an internal pressure p, an axial force per unit circumferential length N and a torque T. The axial force is taken to be applied at r=a+H such that a moment equal to N(H-d) is produced about the reference surface r=a+d. The cylinder is taken to be clamped at both ends but free to rotate and extend axially at one end. If dimensionless external forces and moments are introduced by

$$N^* = N/(\sigma\lambda a), \quad M^*=N(H-d)/(\sigma\lambda^2 a), \quad T^*=T/[2\pi\sigma\lambda^2 a^3(1+d/a)] \tag{62}$$

then the end conditions can be stated as follows:

$$\begin{aligned} &v_r=V_{r,x}=V_z=V_\theta = 0 \qquad (z=0,\ y=d/h) \\ &v_r = 0,\ \ \overline{N}_z=N^*,\ \overline{M}_z= M^*,\ \ (1+d/a)\overline{N}_{z\theta} + \overline{M}_{z\theta} = \lambda\overline{T} \quad (z=L,\ y=d/h) \end{aligned} \tag{63}$$

where L is the actual length of the cylinder. In the theories developed in the above, the distance d at which the stress resultants were defined was left arbitrary. We now choose it such that there exists no coupling between N_z and K_1 and M_z and ε_{1d}. This can be achieved by setting the first component of submatrix $[\overline{B}]$ equal to zero,

$$\overline{B}_{11} = 0 \tag{64}$$

This yields

$$d/h = \underline{B}_{11}/\ \underline{A}_{11} \tag{65}$$

Numerical results were carried out for the case of a four layer symmetric angle ply configuration. For this configuration the angle of the elastic symmetry axis γ is oriented at $+\gamma$, $-\gamma$, $-\gamma$, $+\gamma$ in the four layers and the layers are of equal thickness (see Fig.2). Each layer is taken to be made from a boron/epoxy material system having the following elastic properties:

$$E_1=35\text{x}10^6\text{psi},\ E_2=2.75\text{x}10^6\text{psi},\ \nu_{12}=.25,\ G_{12}=.75\text{x}10^6\text{psi} \tag{66}$$

The representative stress level introduced in the non-dimensionalization process is chosen as

$$\sigma = p/\lambda \tag{67}$$

Shown in Figs. 3-5 is the variation of the dimensionless radial displacement vs. length along the cylinder for the following dimensions: a=4in., L=12in., and h=.1in.. Elastic angles chosen were $\gamma= 0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ$. Fig. 3 shows the results for the length scale a theory while Figs. 4 and 5 show the results for $L=(ah)^{1/2}$, ℓ=a and $L = \ell = (ah)^{1/2}$ theories, respectively. The reference surface for each of the shells is at d/h=1/2. In each case it is seen that the displacement first increases with increase in

G 1/

angle γ (being largest at $\gamma=30^\circ$) and then decreases. The dash-dot line in Fig. 3 represents an isotropic solution ($E = 30\times10^6$ psi, $\nu=.3$). The presence of an edge effect is clearly shown in Figs. 4 and 5. It is seen that this edge effect penetrates further into the shell for small elastic angle γ than for large ones. Figs. 4 and 5 also show that there exists very little difference between the two theories shown there for the problem being considered. A comparison between Figs. 3 and 4 (or 5) shows that bending action needs to be considered in the analysis. While the ASME Boiler and Pressure Vessel Code for fiberglass-reinforced plastic pressure vessels [4] gives design guides for membrane action (Appendix 1-2) no explicit guides are given for bending and shearing action. It does however state (Article D-1) that the geometry and wall thickness of a vessel be such that "the vessel be capable of withstanding the bending and shearing stresses resulting from any expected combination of loads ... ". To accomplish this theories like the ones developed in the above need to be made use of.

REFERENCES

[1] REISSNER, E., "On some problems in Shell Theory", in Proc. of the First Symposium on Naval Structural Mechanics, Pergamon Press, New York (1960).

[2] JOHNSON, M. W., WIDERA, O. E., "An Asymptotic Dynamic Theory for Cylindrical Shells", Stud. Appl. Math. , 48, 205(1969).

[3] GOLDENVEIZER, A. L., "Methods for Justifying and Refining the Theory of Shells", PMM 32, 684(1968).

[4] ASME Boiler and Pressure Vessel Code, Section X, 1971 ed., American Society of Mechanical Engineers, New York (1971).

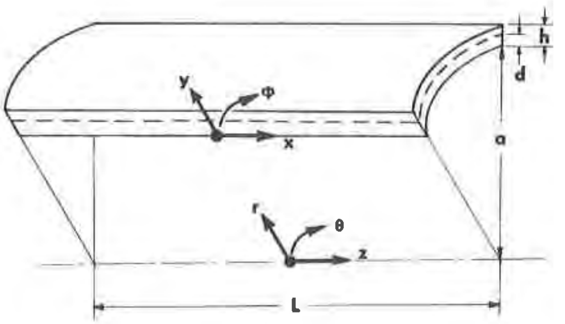

Figure 1: CYLINDRICAL SHELL

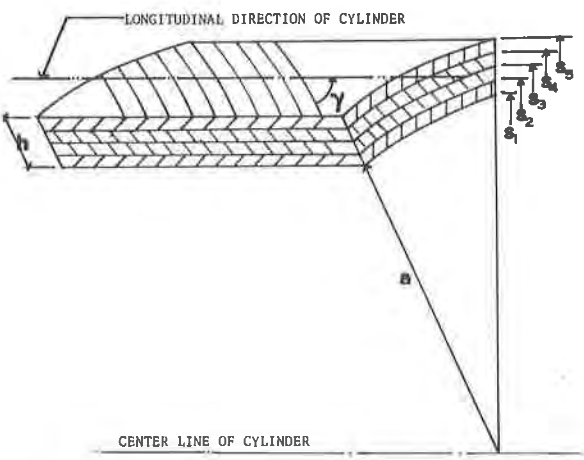

Figure 2: LAMINATED CYLINDER

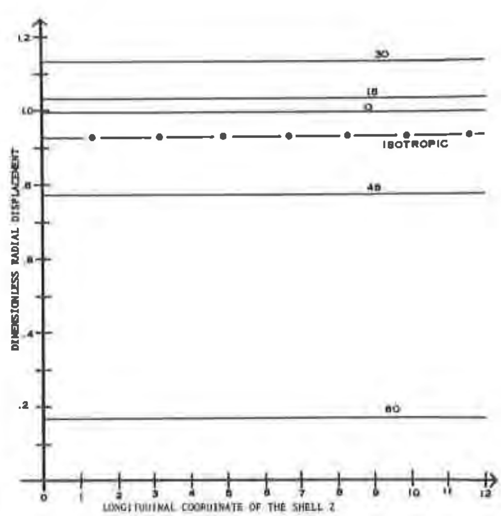

Figure 3: RADIAL DISPLACEMENT OF THE THEORY ASSOCIATED WITH LENGTH SCALES a

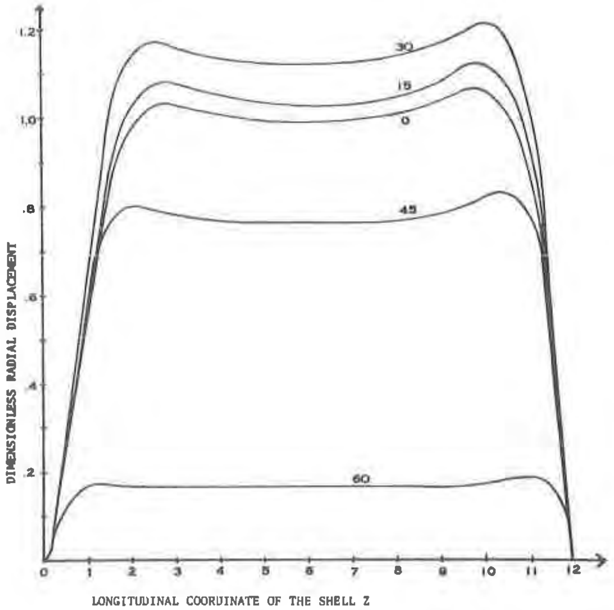

Figure 4: RADIAL DISPLACEMENT OF THE THEORY ASSOCIATED WITH LONGITUDINAL LENGTH SCALE $(ah)^{1/2}$ AND CIRCUMFERENTIAL LENGTH SCALE a

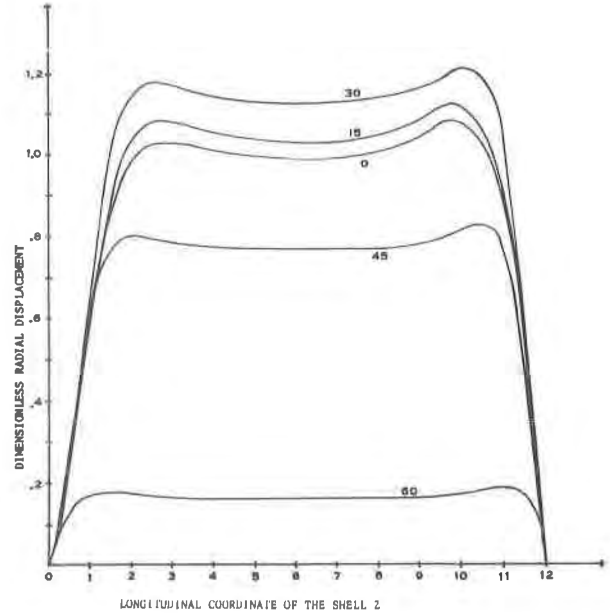

Figure 5: RADIAL DISPLACEMENT OF THE THEORY ASSOCIATED WITH LENGTH SCALES $(ah)^{1/2}$