INFLUENCE COEFFICIENTS FOR AXIALLY SYMMETRICAL THIN SHELLS

W.J. SUTCLIFFE
Department of Mechanical Engineering,
The University of Liverpool, Liverpool L69 3BX, United Kingdom

SUMMARY

Application of available tables of Influence Coefficients to practical examples of shell structures gives rise to some errors and difficulties, as owing to the discrete nature of tabulations, interpolation is often required. In addition, some well known tabulations (G.D. Galletly) for toroidal and spherical shells are limited to open crown cases, i.e. a shell covering a full 90° of meridional angle cannot be dealt with owing to singularities which arise at the crown ($\varnothing = 0°$ in the usual notation). (In addition, Galletly's formulation involved complex numbers.) Other tables (E.Y.W. Tsui), eliminate these singularities by the introduction of new variables.

A simplified presentation of the relevant equations has been produced with the aim of avoiding the use of complex numbers or 'auxiliary' variables, while retaining full capability for all meridional angles.

Equations for the general shell of revolution are first considered and then detailed application and results for toroidal and spherical shells are given. Comparisons with selected tabulated results from other sources are made.

The proposal is made that, as an alternative to the use of tables, a relatively simple computer program will produce, in a short running time, accurate values of Influence Coefficients, including an axial loading case, for any input of shell geometry and shell material properties.

Runge-Kutta numerical integration has been used throughout and details are given of the considerable extent to which segmentation of shell elements has been avoided. Consideration is given to the accuracy of the formula for critical shell length as proposed by A. Kalnins and H. Kraus and a detailed analysis of the calculations involved in solving the equations for the boundary conditions is given.

The well known tabulations of Influence Coefficients by G.D. Galletly cover cases of positive and negative curvature. Values are given corresponding to: loads perpendicular to the shell axis and edge moments, at both ends of the shell, and a pressure load case.

Here, in addition to these cases, a further pressure loading case and an overall axial load case have been included.

1. Introduction

The use of Influence Coefficients, for the stress analysis of compound shells, avoids the need to re-solve the basic differential equations for each analysis.

In this paper, the differential equations are solved by Runge Kutta, step by step, integration of a number of independent solutions, which are then superimposed. Consideration has to be given to:

(a) the correct cases to be superimposed,
(b) the equations for the boundary conditions at the shell edges,
(c) the calculation of the constants for each independent case,
(d) the addition of the cases to find the resultant deflections and/or stresses.

Once a computer program is available for obtaining Influence Coefficients, by the above procedure, it can be used to produce tabulated values for a range of shell geometries or to find the required Influence Coefficients for any exact geometry. The latter requires merely the feeding in of the dimensions and material properties of the shell as data. Given the edge Influence Coefficients, problems, involving compound shells, can be reduced to the solution of a set of algebraic equations.

2. Analysis of Compound Shells Using Influence Coefficients

Considering shells with axi-symmetrical load and edge forces, the resultant values of deflections and stresses will consist of the addition of, in general, the values from six possible independent solutions. Four of these correspond to the radial edge forces and edge moments, one to axial load and one to pressure load.

The magnitudes of the two latter are immediately given by the applied loading, and the four former are obtained from four equations for the boundary conditions.

For each determinate shell segment, there must be four boundary conditions from the conditions for radial deflection, δ, wall slope, β, radial force, H, and wall moment, M_ϕ, at the two edges.

For a compound shell, consisting of n segments, at the joints between each segment, there must be equality of the above four quantities, giving a total of 4(n-1) equations. The remaining four equations, required for the solution, correspond to four boundary conditions at the outer edges of the outer segments.

For a general axi-symmetric loading case for one segment, with axial load L and internal pressure p, the Influence Coefficients (deflections due to unit loading effects) total a possible 24 as shown in Table I.

Then, e.g., the resultant δ at ϕ_1

$$= H_1 \times I_1 + M_1 \times I_2 + H_2 \times I_3 + M_2 \times I_4 + pI_5 + LI_6 \qquad (1)$$

and the resultant β at ϕ_1 and δ and β at ϕ_2 give rise to similar equations. where H_1, M_1, H_2, M_2 are the resultant horizontal forces and moments at the edges. These four equations can then be solved for these edge forces and moments.

In many cases the maximum stresses will be at the joints between segments and so these stress resultants may be adequate for design purposes.

3. Calculation of Deflection and Stress Distributions From Internal Influence Coefficients

If, for example, the values of M_ϕ have been stored, during the integration process, at suitable intervals of ϕ, then the superposition of these values, multiplied by the appropriate constants of H_1, M_1, H_2, M_2, L and p, will give the resultant distributions of M_ϕ over the shell length. Similarly, distributions of all other displacements and stress resultants can be found.

4. Differential Equations for any Shape of the Meridian

From the equations of Timoshenko [1], Chapter 16, four first order differential equations can be obtained, for a general, axi-symmetrically loaded, axi-symmetric shell, in terms of the independent variable S, corresponding to distance along the shell meridian.

$$\frac{d\delta}{ds} = \frac{\cos\phi}{D}(V\sin\phi - H\cos\phi) - \delta\frac{\mu\cos\phi}{r} + \beta\sin\phi \qquad (2)$$

$$\frac{d\beta}{ds} = \frac{M_\phi}{K} - \beta\frac{\mu\cos\phi}{r} \qquad (3)$$

$$N_\theta = \delta\frac{D}{r}(1-\mu^2) + \mu(V\sin\phi - H\cos\phi) \qquad (4)$$

$$M_\theta = D\left[\frac{\beta\cos\phi}{r} + \mu\frac{d\beta}{ds}\right] \qquad (5)$$

$$\frac{d(rH)}{ds} = - N_\theta - pr\sin\phi \qquad (6)$$

$$\frac{d(rM_\phi)}{ds} = M_\theta\cos\phi + rV\cos\phi + rH\sin\phi \qquad (7)$$

where $D = \frac{Eh}{1-\mu^2}$ $\quad K = \frac{Eh^3}{12(1-\mu^2)}$

E Young's Modulus, μ Poisson's Ratio

h shell thickness, p internal pressure loading

β the rotation of a tangent $= -\frac{(v + d)}{r_1}$

δ the outwards radial deflection $= v\cos\phi - w\sin\phi$

V and H are the force stress resultants along and perpendicular to the axis, V +ve away from a shell element at ϕ and H +ve outwards.

r the radius in a plane perpendicular to the axis, r_1 the radius of the meridian

The stress resultants M_ϕ, M_θ and N_θ and the displacements v and w are as defined by Timoshenko [1], d being $\frac{dw}{d\phi}$.

These equations can be integrated for shells with both straight and curved generators, for any range of ϕ, including 0° and 90°.

5. Differential Equations for Toroidal Shells

The actual equations used to integrate a number of cases for toroidal shells were:

$$\frac{dN_\phi}{d\phi} = \frac{\cos\phi}{R+\sin\phi}\left[N_\phi(\mu-1) + \frac{E}{T(R+\sin\phi)}(v\cos\phi - w\sin\phi)\right] + Q_\phi \qquad (8)$$

$$\frac{dM_\phi}{d\phi} = \frac{\cos\phi}{R+\sin\phi}\left[M_\phi(\mu-1) - \frac{Eb}{12T^3}\cdot\frac{\cos\phi}{(R+\sin\phi)}(v + d)\right] + bQ_\phi \qquad (9)$$

$$\frac{dQ_\phi}{d\phi} = bp - N_\phi\left(1 + \frac{\mu\sin\phi}{R+\sin\phi}\right) - \frac{1}{R+\sin\phi}\left[\frac{E\sin\phi}{T(R+\sin\phi)}(v\cos\phi - w\sin\phi) + Q_\phi\cos\phi\right] \qquad (10)$$

$$\frac{dv}{d\phi} = w + \frac{T}{E}(1-\mu^2) N_\phi - \frac{\mu}{R+\sin\phi}(v\cos\phi - w\sin\phi) \quad (11)$$

$$\frac{dw}{d\phi} = d \quad (12)$$

$$\frac{dd}{d\phi} = -w\left(\frac{\mu\sin\phi}{R+\sin\phi} + 1\right) - \frac{T}{E}(1-\mu^2) N_\phi - 12(1-\mu^2)\frac{T^3}{E}\frac{M_\phi}{b} - \frac{\mu\cos\phi . d}{R+\sin\phi} \quad (13)$$

These are in terms of the independent variable ϕ and stress resultants and displacements as defined by Timoshenko [1].

Here b is the radius of curvature of the meridian of a torus

a is the radius from the axis to the centre for b

R is the ratio a/b

T is the ratio b/h

h being the shell thickness.

To comply with the dimensionless form of the results in Ref. 2 and 3, all these equations, except 9, were multiplied by b and then equations 11, 12 and 13 by E to give six equations in the dependent variables bN_ϕ, M_ϕ, bQ_ϕ, Ebv, Ebw and Ebd.

Five integrations were then made for these equations corresponding to five cases, in each of which all the variables except one were initially set at zero. The non-zero values were bN_ϕ, M_ϕ, bQ_ϕ = 1, Ebv, Ebw = 10^4 in these five cases, respectively. A sixth case started with all variables = 0 and the pressure made $p = \frac{2}{b^2}$ for all ϕ.

Integration was by a Runge Kutta procedure, using a constant $\frac{1}{2}^\circ$ step length and the final values of all the variables were stored in the computer as an array of 36 numbers.

6. Boundary Conditions

ϕ_2 was taken as 90° and ϕ_1 varied for different shells. Then, with integrations starting at ϕ_1 and finishing at ϕ_2, the resultant solutions, corresponding to the various unit loading cases, for which the Influence Coefficients are required, is made up by superposition of the six independent solutions.

Seven loading cases were considered:

Line 1 $bH_{90}=1$ Edge force

Line 2 $M_{90} = 1$ Edge moment

Line 3 $bH_{\phi 1} = 1$ Edge force

Line 4 $M_{\phi 1} = 1$ Edge moment

Line 5 $\frac{pb^2}{2} = 1$ with membrane support forces at ϕ_1 and ϕ_2

Line 6 $\frac{pb^2}{2} = 1$ with membrane support force at 90° only

Line 7 $bN_{90} = 1$ End load

For Lines 5, 6 and 7 it was necessary to make the boundary condition equations comply with axial equilibrium conditions, which are always determinate (axial forces are zero for Lines 1 to 4).

7. Displacements

The required Influence Coefficients being the radial deflections δ_1 and δ_2 and the rotations β_1 and β_2, at ϕ_1 and ϕ_2, for the loads corresponding to Lines 1 - 7, then the resultant values of these Influence Coefficients can be found by superimposing the six integration cases in proportions given by constants, K_1 - K_6, found from equations for the boundary conditions corresponding to Lines 1 - 7.

The input data for the production of 7 Lines of Influence Coefficients, i.e. the 24 values of Table I plus 4 for the second pressure case of Line 6, are T, R, ϕ_1 and μ.

8. Results

Seven examples have been solved for representative shells, including a negative curvature case, a case of a full 90° segment and a hemisphere. All results, where comparison was possible, gave excellent agreement with the values given in Refs. 2 and 3. e.g. for T = 30, R = 7, $\phi_1 = 75^\circ$, $\mu = 0.3$

Line 4 ($M_{75} = 1$)

Constants: $K_1 = K_3 = 0$, $K_2 = 1.0$, $K_4 = 133.327$, $K_5 = 18.2612$

Influence Coefficients	Sutcliffe	Galletly
$Eb\delta_1$	168686	168685
$Eb^2\beta_1$	1333270	1333270
$Eb\delta_2$	- 169138	- 169140
$Eb^2\beta_2$	1292940	1292940

For a spherical shell, for which T = 90.5, R = 0, $\phi_1 = 10.5^\circ$, identical results were obtained compared to an independent solution by Fayed [4].

9. Length of Integration Possible Without Segmentation

The example of the sphere cited above has a length of meridian very much greater than the 'critical length' as defined by other workers [5] and [6]. For a toroidal shell this reduces to ϕ critical (degrees) = $\frac{134}{\sqrt{T}}$. For the above sphere ϕ critical = 14°, whereas satisfactory results were obtained over a ϕ range of 79.5°.

Experience with other examples has shown that the critical length as suggested is very pessimistic and therefore it would appear that much of the segmentation proposed is unnecessary, if a computer of adequate accuracy is used.

10. Conclusions

Influence Coefficients for toroidal shells have been produced from equations, which avoid the use of complex numbers or auxiliary variables and an end load case has been included, extra to previous results.

Rather than use tabulated values, it is proposed that interpolation difficulties can be avoided by evaluating the Influence Coefficients for any exact geometry. This requires approximately two minutes computer time and, if required, complete stress distributions can be obtained. This is not very practicable by the use of tabulations [7].

The outline of a set of equations applicable to a wider variety of shells has been given. The avoidance of segmentation, which has been shown to be possible, makes this type of shell solution more attractive in comparison with other approaches, such as finite element methods.

G 1/

References

[1] TIMOSHENKO AND WOINOWSKY-KREIGER, "Theory of Plates and Shells," McGraw Hill, New York (1959).

[2] GALLETLY, G. D., "Edge influence coefficients for toroidal shells of positive Gaussian curvature," Trans. ASME, 60-68 (1960).

[3] GALLETLY, G. D., "Edge influence coefficients for toroidal shells of negative Gaussian curvature," Trans. ASME, 82B, 69-77 (1960).

[4] FAYED, M. D., Ph.D. Thesis (1968), Department of Mechanical Engineering, University of Liverpool.

[5] KALNINS, A., "Analysis of shells of revolution subjected to symmetrical and non-symmetrical load," J. Applied. Mech.31,467-476 (1964).

[6] KRAUS, H., "Thin Elastic Shells," Wiley (1967).

[7] GALLETLY, G. D., "Edge and Interior Influence Coefficients for Toroidal Shells," (Ten volumes in libraries of ASME, I.Mech.E.)

Table I

Possible Influence Coefficients for a shell with edges at ϕ_1 and ϕ_2 (ϕ being the angle that a perpendicular, to the meridian, makes with the axis)

Loading	Influence Coefficients			
	At ϕ_1		At ϕ_2	
	δ	β	δ	β
unit H at ϕ_1	I_1	I_7	I_{13}	I_{19}
unit M_ϕ at ϕ_1	I_2	I_8	I_{14}	I_{20}
unit H at ϕ_2	I_3	I_9	I_{15}	I_{21}
unit M_ϕ at ϕ_2	I_4	I_{10}	I_{16}	I_{22}
unit pressure	I_5	I_{11}	I_{17}	I_{23}
unit end load	I_6	I_{12}	I_{18}	I_{24}