A STRESS CONCENTRATION THEORY OF FRACTURE AND ITS APPLICATION TO VARIOUS CRACK GEOMETRIES

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SUMMARY

By considering the elastic-plastic stress distribution occurring at the crack tip region, in real materials, such as steel, to be capable of being expressed in terms of a Hookean elastic distribution, it has been possible to use the Westergaard expression for the crack tip stress under elastic conditions in conjunction with the effective stress concept of Neuber to derive an exact expression for the onset of fast crack propagation in terms of two basic material parameters, i.e., the engineering Ultimate Tensile Strength (UTS) and a distance $S$ in front of the crack tip, over which the average stress at fracture is equal to the UTS.

It is shown that if application is restricted to stress levels of practical interest in engineering, i.e., between UTS and half the UTS, this concept and equation are, for practical purposes, equivalent to considering the load shed by the cracked material to be carried completely by the crack tip region $S$. This approximation provides a convenient physical concept which may be used for treating complex flaw geometries and stress distributions.

When the Neuber or effective stress concept is combined with the particular approximation to Westergaard’s expression used in Linear Elastic Fracture Mechanics (LEFM), an expression is obtained which is only valid at low stress levels. Comparison of this with the exact expression and its high stress range approximation, highlights the limited range of validity of LEFM.

The analysis is applied to several crack geometries and to the treatment of cases involving stress gradients and limited section size.

Failure data and overall behaviour of 6" thick tensile specimens, containing part circular surface flaws, and tested as part of the U.S.A.E.C.'s Heavy Section Steel Programme is shown to be in good agreement with predictions based on this stress concentration theory and material data derived from standard tensile and Charpy tests.
1. INTRODUCTION

The work of Irwin has led to the development of a fracture mechanics method which in its basic form characterises the stress conditions in a very small region local to the crack tip by a single parameter - the stress intensity factor \( K \). When the stress intensity factor increases until it reaches a critical value \( K_c \), which is considered to be a material property, fast crack propagation ensues. This is equivalent to the assumption that material failure takes place initially due to the maximum value of tensile stress normal to the crack occurring in a region local to the crack tip which is very small compared with the crack size. However, it is known that failure in real materials takes place over some distance rather than at a point and that for materials of engineering interest this distance may not be small compared with the crack size. There is also evidence that failure takes place at some distance inside the material in front of the crack tip. Obviously this behaviour is at variance with the assumptions mentioned above and thus it is to be expected that the single parameter \( K_c \) approach will only be reasonably valid at long crack lengths and low stresses.

Other workers, notably McClintock and Cottrell, apparently recognised the inadequacy of a single material parameter approach and to avoid the difficulty posed by the crack tip singularity, adopted the idea of taking into account average conditions over some distance in front of the crack in conjunction with a critical strain or stress parameter.

Neuber, had of course, successfully used such a two parameter approach for notches some time previously and although it had been referred to and compared to the stress intensity approach by Irwin, there does not appear to have been any attempt to apply it directly to the sharp crack case. Instead, Irwin attempted to remedy the deficiencies of the basic stress intensity approach by adding a plastic zone correction and thus an additional material parameter. However, there have been various theoretical and practical objections to this device.

The purpose of this report is to set out the basic stress concentration theory of fracture resulting from the direct application of Neuber's ideas in conjunction with Westergaard's elastic analysis of crack tip stresses and to show how this compares with Linear Elastic Fracture Mechanics in terms of accuracy and areas of application.

2. THE BASIC THEORY

2.1 The Stress Concentration Theory

The fundamental idea underlying this theory is that fracture will occur when the average value of stress over a certain critical distance in front of the crack tip reaches a certain value. (Ref. 1).

In real materials such as steel, yielding and plastic flow occur in this crack tip region and the exact distribution of stress and its deviation from an elastic distribution will depend on the ductility of the material, the yield stress and the level of the general field stress or gross stress. However, a particular elastic-plastic stress distribution may be considered to have been arrived at through a process of relaxation of a purely elastic distribution, (see Fig. 1), the area under the two curves remaining the same so as to continue to satisfy the requirement for equilibrium on the plane of the crack.

This idea may be used to express any real elastic-plastic stress distribution in front of the crack tip in terms of an equivalent elastic stress distribution.
From Westergaard's (Ref. 2) elastic analysis of the stress distribution for a through crack in an infinite plate, the stress normal to the plane of the crack in the region in front of the crack tip is:

\[ \sigma_y = \frac{\sigma_g}{\sqrt{1 - \frac{a^2}{x^2}}} \]  
\[ \sigma_y = \frac{\sigma_g x}{(x^2 - a^2)^{1/2}} \]  

where \( \sigma_g \) is the uniform gross stress or general field stress
\( \sigma_y \) is the local stress in front of the crack tip normal to and on the plane of the crack
\( a \) is the half crack length
and \( x \) is the distance in the \( x \) direction measured from the centre of the crack (see Fig. 2)

Utilizing Neuber's idea of effective or average stress (Ref. 1)

Effective stress
\[ \overline{\sigma} = \frac{1}{\varepsilon} \int_a^a + \varepsilon \sigma_y \, dx \]  

Substituting (2) into (3)

\[ \overline{\sigma} = \frac{1}{\varepsilon} \left[ \frac{\sigma_g x}{(x^2 - a^2)^{1/2}} \right]_{x = a + \varepsilon}^{x = a} \]
\[ = \sigma_g \left( \frac{2a}{\varepsilon} + 1 \right)^{1/2} \]  

at fracture when \( a \to 0 \) \( \sigma_g \to \) ultimate tensile strength \( \sigma_u \)

\[ \overline{\sigma} = \sigma_u = \sigma_g \left( \frac{2a}{S} + 1 \right)^{1/2} \]  

where \( S \) is the critical value of \( \varepsilon \) corresponding to particular material properties.

\[ \sigma_u \approx \sigma_g \left( \frac{2a}{S} + 1 \right) \text{ when } \frac{a}{S} \ll 1 \]  

This means that provided \( \frac{a}{S} \ll 1 \), the Neuber idea is, for practical purposes, equivalent to the concept that the load which would have been carried by the cracked material is carried in a region of size \( S \) in front of the crack tip (Ref. 3). As \( \frac{a}{S} \) increases above 1 this theory indicates that this concept should become increasingly approximate.

### 2.2 Linear Elastic Fracture Mechanics

Irwin (Ref. 4) showed how the Westergaard elastic stress analysis could be used to develop a single material parameter analysis.

From Eq. (1)

\[ \sigma_y = \frac{\sigma_g x}{(x - a) x} \]  

Writing \( r = x - a \) gives

\[ \sigma_y = \frac{\sigma_g (r + a)}{r (r + 2a)} \]

\[ \sigma_y = \sqrt{2ar (\frac{r}{2a} + 1)} \]
When \( \frac{x}{2a} \ll 1 \), \( (\frac{x}{2a} + 1) \approx \frac{x}{a} + 1 \)

and

\[
\sigma_y = \sigma_0 \sqrt{\frac{a}{2x}}
\]

Eq. (9)

\[
K = \sigma_0 \sqrt{\frac{a}{2x}}
\]

Eq. (10)

Where

\[
K = \sigma_0 \sqrt{\frac{a}{2x}} = \sqrt{\frac{\partial u}{\partial x}}
\]

Eq. (11)

and \( K \) is the stress intensity factor in the LEMFM approach.

\( G \) is the elastic strain energy release rate and \( E \) is the modulus of elasticity.

Now

\[
\bar{\sigma} = \frac{1}{E} \int_0^a \sigma_y \, dx
\]

Eq. (12)

Substituting Eq. (9) in Eq. (12) gives

\[
\bar{\sigma} = \sigma_0 \sqrt{\frac{a}{2x}} \int_0^a \sigma_y \, dx = \sigma_0 \sqrt{\frac{2a}{E}}
\]

Eq. (13)

At fracture by analogy with Eq. (5)

\[
\bar{\sigma} = \sigma_0 \sqrt{\frac{a}{S}}
\]

Eq. (14)

which approximates to Eq. (5) when \( \frac{a}{S} \gg 1 \). This is of course due to the approximation in Eq. (9).

2.3 Comparison of Stress Concentration Theory and LEMFM

Equations (5), (6) and (14) are compared over a range of \( \frac{S}{a} \) values from 0.1 to 10 in Fig. 3. The experimental evidence identifying \( \bar{\sigma} \) with \( \sigma_u \) is quite conclusive and thus in the stress range of interest, i.e., \( 1 < \frac{\sigma_u}{\sigma_0} < 2 \), Eq. (6) is a sufficiently close approximation to the exact solution of Eq. (5). On the other hand, Eq. (14) is very inaccurate in this region but improves in accuracy as the stress level decreases.

3. Application

3.1 Areas of Application

It has previously been suggested (Refs. 5 and 6), that the energy approach which forms the basis of the LEMFM theory is a necessary but not always a sufficient condition and that it may be necessary to supplement this by a condition relating to the actual failure behaviour of the material at the crack tip. Neuber's concept provides such a statement regarding failure behaviour and as has been stated above the result is an expression for fast fracture (Eq. 5) which is approximated to by LEMFM at low stresses, where strictly speaking LEMFM is only valid anyway, and over-rides LEMFM at high stresses. However, because cracks usually develop in highly stressed regions in engineering structures, these are usually the points of most interest. For instance, in a pressure vessel constructed in a low alloy steel, stress levels of about 0.8 x UTS can occur at certain very local regions. What then is the value of Eq. 6? The foregoing theory clearly indicates that Eq. (6) is a useable approximation to Eq. (5) in the practical stress range, its main value being its simplicity, i.e., its association with the idea of load transfer (the use of this for complex crack geometries and non-uniform stress fields). Determination of the fracture parameter \( S \) can be carried out by performing a fracture test, just as \( K_0 \) would be determined.
The specimen size should be large enough to accommodate $S$ with a margin to spare so that its adequacy is obvious. Alternatively, a correlation with tensile and Charpy data may be used to obtain $S$. Ref. 3 gives the correlation between $S$ and small scale laboratory tests as

$$S = \alpha \left( \frac{\text{Charpy}}{\text{Yield stress}} \right)^{1/3} \text{ (Inches)} \quad \text{Eq. (15)}$$

where the Charpy energy is in ft.lbs, the yield stress is in Tons/in$^2$ and $\alpha$ is given by Eq. (16) in terms of per cent total elongation $\varepsilon_t$ measured on a specimen of gauge length equal to $\sqrt{S}$ Section area

$$\alpha = 0.00232 \varepsilon_t^2 \quad \text{Eq. (16)}$$

3.2 Effect of Crack Geometry

The treatment above has considerably idealised the problem. In practical cases the crack is usually a partial thickness crack and it may occur in a position where the stress is non-uniform. Such cases may be treated using the principles which have been described. Various types of partial thickness cracks are considered.

3.2.1. The 'Penny-shaped Crack'

The model used here is of a circular shaped crack of radius $a$, embedded in an infinite volume of metal and under uniform stress $\sigma_0$. Again by equating the load shed by the cracked material to the additional load carried in the crack tip region, an expression relating critical crack length, stress and $S$ may be obtained as in Fig. 4. In this case, because of symmetry, it is possible to take $\varepsilon$ as constant around the circumference of the crack.

3.2.2. The Long, Uniform Depth, Surface Crack

Since the crack is of uniform depth, the lines of load transfer may be easily drawn on the crack surface, see Fig. 5. This can be compared with similar lines on the 'penny-shaped' crack, Fig. 4. Also, because of the uniform depth and stress, $\varepsilon$ may be taken as constant along the crack tip. Thus the conditions for all elements such as $a$, $b$, $c$, $d$, are similar. Such an element may thus be regarded as being similar to a through thickness crack, where the depth of the crack for the case of a surface crack has the same significance as the half crack length $a$ of the through thickness crack. The totally embedded case is similar.

3.2.3. Nozzle Corner Crack (Fig. 6)

Here some method of mapping out the crack surface to determine the lines of load transference, and hence $\varepsilon$ is necessary. However, the results are not very sensitive to the method used since it is the longest element which gives the maximum $\varepsilon$ and controls the onset of fast fracture. In any of the above cases the crack may lie in a stress gradient and such a situation is depicted in Fig. 7. This problem is solved as in the case of uniform stress by taking the conditions for equilibrium on the plane of the crack.

3.2.4. Special Case of a Deep Partial Thickness Crack

In relatively low strength materials such as mild steel, the critical crack sizes associated with even the highest stress levels in, say, a pressure vessel, are usually too large to be accommodated in terms of one of the partial thickness models, i.e., for such a material the sum of critical depth $a$ and $S$ is greater than the plate thickness. In such a case, failure occurs at the crack tip when the remaining ligament has reached the UTS and the crack 'snaps-thro' to become a through thickness crack of the same length. If this
through crack is longer than the critical length, fast fracture will occur, while if it is less the worst result will be leakage. (Ref. 7). In contrast with this, in the case dealt with in paras. 3.2.4. - 3 where the sum of the crack size and S are less than the section thickness, the surface length is very unlikely to be less than twice the depth and consequently after initiation of fracture in the remaining ligament fast propagation is triggered off on the whole crack front and there is no possibility of there only being a snap-through phase.

4. **PREDICTION OF FAILURE OF 6° THICK TENSILE SPECIMENS CONTAINING CIRCULAR SURFACE FLAWS**

Grierson (Ref. 8) has reported failure data from tensile tests on 6° thick specimens of A533 Grade B Class I plate containing part circular surface flaws. This crack geometry is treated in Fig. 8. Test data, tensile data from remnants of the test pieces and failure predictions using the stress concentration theory and the analysis of Fig. 8 are summarised in Table I and shown graphically in Fig. 9. When the sum of the flaw size and S is compared with specimen thickness, it is seen that only in two of the tests - 3 and 5 - is the specimen thick enough relative to the flaw size. In the other five tests, S in Eq. (17) must be replaced by the distance between the back face and the tip of the flaw. The high rigidity of the ends of the specimen (at grips) suggests that the conditions at these ends will be closer to constant displacement, across width. of specimen, rather than constant stress. Consequently, because of increased compliance of the specimen along a line at mid-width, at right angles to the plane of the defect, most of the deflection being taken up by crack opening displacement, there will be a stress and strain depression at mid width. This will result in the specimen fracturing at a higher load than it would under uniform stress conditions. The greater the strain at maximum load the greater this effect; however, a quantitative analysis of this is not yet available. There is close agreement in the case of specimens 3, 5, 6, 7 and 1 (within 5%), the first 4 of which have low values of strain at maximum load. The prediction on specimen 2 is 10% low and in the case of specimen 4, it is 20% low. Specimen 4 has the highest recorded strain at maximum load. Thus, the prediction of overall behaviour and numerical values of failure stresses are in good agreement with experimental observations.

5. **REFERENCES**

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<th>Specimen</th>
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<th>$\alpha$</th>
<th>ft.lbs</th>
<th>$S$ (ins)</th>
<th>flaw depth $a$ (ins)</th>
<th>S + a (ins)</th>
<th>(SCT Pred) $\sigma_y$ (ksi)</th>
<th>$\sigma_y$ (exp)</th>
<th>Average Strain at Max load %</th>
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* $(S + a) \geq t$  
$R = 5.0$ inches
FIG. 1. RELAXATION OF ELASTIC STRESS DISTRIBUTION.

FIG. 2. AVERAGE OR EFFECTIVE STRESS IN CRACK TIP REGION.

FIG. 3. COMPARISON OF STRESS CONCENTRATION AND L.E.F.M. BASED EQUATIONS.
(i) Load originally carried by crack and region of stress perturbation = \( \pi \sigma_0' (a + S)^2 \)

(ii) Load carried by region of stress perturbation (at failure) = \( \pi \sigma_u \left[ (a + S)^2 - a^2 \right] \)

Equating (i) and (ii) gives:

\[ \sigma_0 = \sigma_u \frac{2aS + S^2}{(a + S)^2} \]

**Fig. 4** The Penny-shaped crack

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**Fig. 5.** The Long, Uniform depth, surface crack
FIG. 6 NOZZLE CORNER CRACK

FIG. 7 CRITICAL CONDITIONS FOR A CRACK IN A STRESS GRADIENT
Considering Equilibrium of Forces on Longest Element

\[ \frac{6}{2}(R + s)^2 - \frac{6}{2}(R - a)^2 = \frac{6}{2}(R + s)^2 - \frac{6}{2}R^2 \]

\[ \frac{\sigma_y}{\sigma_y} = 1 + \frac{2aR - s^2}{2R - s^2} \ldots \text{Eqn (17)} \]

When \((S + a) \gg t\) (Where \(t\) = thickness)

The size of the region carrying the load from the crack is restricted to \(t - a\), instead of \(S\). Hence:-

\[ \frac{\sigma_y}{\sigma_y} = 1 + \frac{2aR - s^2}{2R(t - a) + (t - a)^2} \ldots \text{Eqn (18)} \]

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**FIG. 8** THE PART CIRCULAR SURFACE CRACK
\[ S + a > t \quad (\text{Eqn. 18, Fig. 8}) \quad \oplus \]
\[ S + a < t \quad (\text{Eqn. 17, Fig. 8}) \quad \ominus \]

**Fig. 9.** Comparison of failure data and S.C.T predictions for 6" thick tensile specimens with part circular surface cracks.