AXIAL CRACKS IN CYLINDRICAL SHELLS SUBJECTED TO THERMAL AND MECHANICAL LOADS

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SUMMARY

Cylindrical shells find frequent application in Nuclear Reactors as portions of containment vessels, supply skirts, coolant and flow inlet and outlet nozzles, fuel sleeves and moderators. In many of these usages such shells are subjected to thermal gradients as well as various mechanical loads. This paper intends to treat fracture of reactor vessels under such loadings as a complete cylindrical shell for the first time. Previous analyses owing to the use of Donnell's or Marguerre's Equations, effectively consider initially curved plates. In this presentation the governing shell equations are Morley's, formulated in terms of the radial displacement W and a stress function F.

$$\nabla^2(\nabla^2+1)\,W+\frac{a}{D}\frac{\partial^2 F}{\partial x^2}=0 \quad \text{here} \quad a = \text{shell radius} \qquad h = \text{shell thickness}$$

$$\nabla^2(\nabla^2+1)F-\frac{a}{A}\frac{\partial^2 W}{\partial x^2}=0 \qquad D=\frac{Eh^3}{12(1-\nu^2)} \qquad \nu = \text{Poisson ratio}$$

$$A=\frac{1}{Eh} \qquad E = \text{Young Modulus}$$

The equations are first solved for a complete shell, then stresses are relieved along the crack edges by superposing canceling edge loads along the crack so that the boundaries are stress-free.

This method results in a pair of coupled dual integral equations, which are converted into two coupled singular equations containing Cauchy-type singularities:

$$\int_{-1}^{1} a_{ij}u^j(\xi)\,\frac{d\xi}{\xi-x}+\int_{-1}^{1} k_{ij}(x,\xi)\,u^j(\xi)\,dt=f_i(x) \qquad |x|<1, \quad i=1,2$$

a_{ij} being constants and k_{ij} bounded functions.
The functions $u^j(\xi)$ are expanded in terms of Chebishev polynomials, and the stress intensity factors are evaluated. Griffith-Irwin fracture theory is used by averaging the stresses calculated over the thickness of the shell to obtain an appreciation of crack propagation criterion.

For the first time a complete cylindrical shell is considered including the effect of ring stiffness in the shell. In this treatment it is no longer specified the in-plane shear stress resultants are equal (i.e. $N_{\theta x}\neq N_{x\theta}$). Two particular types of loadings are included (1) internal pressure and (2) linearly distributed temperature variation throughout the shell thickness, extensive numerical results are presented and compared with earlier published works.

1. Introduction

The analysis of circular cylindrical shells finds frequent application in many engineering design problems particularly in design problems associated with nuclear reactors. Cracks may be induced in the cylindrical shells by design, manufacturing process, handling, installation or under normal usage. Whatever there origin, the presence of a crack is known to substantially reduce the load bearing capabilities of a structure. The conditions under which a cracked shell may no longer fulfill the structural requirements imposed upon it is the subject of the present paper.

Linear fracture mechanic has revealed itself to be a useful approach in the investigation cracked structures. The work of Griffith [1] implies that strength of the stress singularity is a determining factor in the prediction of crack extension, despite the implication of infinite stress at the crack vertex.

Previous investigation of an axial cracked cylindrical shell have linked the strength of the singularity with shell parameter λ defined in terms of the shell radius, the crack length, and material constants. These same works have largely centered on the Marguerre equations [2]. Folias [3] using these equations, developed a set of two coupled singular integral equation which were solved for small values of the shell parameter $\lambda < 1$. Copley and Sanders [4], using the identical model, developed a different method of solution based on the Fourier Integral Theorem. They similarily obtained two coupled singular integral differing from Folias which were solved numerically for values of λ up to 8. Duncan and Sanders [5] employing the Marguerre equations presented a solution for a cylindrical shell with an axial crack reinforced by circumferential ring stiffeners. Folias results were extended for values of λ up to 8 by Erdogan and Kibler [6] who observed that the fundamental function of the system was the weight of Tskebyscheff Polynomials of the second kind and, therefore, expanded the solution in terms of those polynomials.

2. Discussion of the Problem

For a thin cylindrical shell a through crack is modeled by a line discontinuity of length 2c across which all stress resultants are zero. Linearity assumption in the analysis allows the problem to be solved by the method of superposition. In effect two problems are solved. First, the uncracked shell is analyzed under a given loading condition, then the cracked shell is solved for edge loads along the line of the crack. The edge loads along the crack are taken to be equal in magnitude but opposite in sense to the stress and moment resultants at the crack location in the uncracked shell. Superposition of the two solutions loads to a resultant free crack. It should be noted that the systems of loads applied to the crack is self equilibrating and according to the expectations of the St. Venant principle the effect of the crack loading will decay rather rapidly.

3. Development of the Equations

Simmonds [7] developed a pair of coupled fourth order differential equation for the Morley [8] equations in terms of the radial deflection w and a stress function F, which will be used as the governing equations. In the case of no distributed load the equations become:

$$\nabla^2 (\nabla^2+1)\, w + \frac{R}{D} \frac{\partial^2 F}{\partial x^2} = 0$$

$$\nabla^2(\nabla^2+1)\, F - \frac{R}{Eh} \frac{\partial^2 w}{\partial x^2} = 0 \qquad (1)$$

In these and in subsequent expressions the following definitions are used

E ≡ Young's Modulus $\qquad D \equiv \frac{Eh^3}{12(1-\nu^2)}$ $\qquad x$ = non-dimensional axial distance

h ≡ Shell Thickness $\qquad$ θ = non-dimensional circumferential distance

r ≡ Poisson's Ratio $\qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}$

Both x and θ are nondimensionalized with respect to the shell radius R. The stress and moment resultants in terms of w and F are, for the symmetric case

$$M_{\theta\theta} = -\frac{D}{R^2}\left[\frac{\partial^2 w}{\partial \theta^2} + w + \nu\frac{\partial^2 w}{\partial x^2}\right] \tag{2a}$$

$$N_{\theta\theta} = \frac{1}{R^2}\frac{\partial^2 F}{\partial x^2} \tag{2b}$$

$$R_\theta = -\frac{D}{R^3}\left[\frac{\partial^3 w}{\partial \theta^3} + \frac{\partial w}{\partial \theta} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial \theta}\right] = 0 \tag{2c}$$

$$\frac{\partial S_\theta}{\partial x} = -\frac{1}{R^3}\frac{\partial^3 F}{\partial x^2 \partial \theta} = 0 \tag{2d}$$

Where S_θ and R_θ are the Kirchhoff edge resultants, necessitated by the use of eighth order shell theory.

4. Method of Solution

For the symmetric case w and F can be represented in the form

$$w(x,\theta) = \int_0^\infty \left\{W_1(\eta)\, e^{p_1|\theta|} + W_2(\eta)\, e^{p_1|\theta|} + W_3(\eta)\, e^{p_3|\theta|} + W_4(\eta)\, e^{p_4|\theta|}\right\} \cos\eta x\, d\eta$$

$$F(x,\theta) = i\sqrt{EhD}\int_0^\infty \left\{W_1(\eta) e^{p_1|\theta|} + W_2(\eta) e^{p_2|\theta|} + W_3(\eta) e^{p_3|\theta|} + W_4(\eta)\, e^{p_4|\theta|}\right\} \cos\eta x\, d\eta \tag{3}$$

where $\quad p_1 = -\sqrt{\eta^2 - \frac{1}{2} + \sqrt{\frac{1}{4} + 2iK^2\eta^2}}; \qquad p_2 = -\sqrt{\eta^2 - \frac{1}{2} + \sqrt{\frac{1}{4} - 2iK^2\eta^2}}$

$\quad p_3 = -\sqrt{\eta^2 - \frac{1}{2} - \sqrt{\frac{1}{4} + 2iK^2\eta^2}}; \qquad p_4 = -\sqrt{\eta^2 - \frac{1}{2} - \sqrt{\frac{1}{4} + 2iK^2\eta^2}}$

In addition w and F to insure the compatibility must be subject to the conditions

$$\lim_{y\to 0}\left\{\frac{\partial^n}{\partial\theta^n}(w^+, F^+) - \frac{\partial^n}{\partial\theta^n}(w^-, F^-)\right\} = 0 \quad \text{for } |x| > \frac{c}{R} \tag{4}$$

Equations (3) are substituted into the boundary conditions (2c, 2d) and $W_2(\eta)$ and $W_4(\eta)$ are eliminated. Then the compatability requirement becomes

$$u_1(x) = \int_0^\infty \left\{p_1 W_1(\eta) + p_3 W_3(\eta) + \frac{\sqrt{2iK^2\eta^2 + \frac{1}{4}}}{\eta^2}\right.\left[p_1 W_1(\eta) - p_3 W_3(\eta)\right] \cos x\eta\, d\eta$$

$$u_2(x) = \int_0^\infty \left\{p_1 W_1(\eta) + p_3 W_3(\eta) - \frac{\sqrt{2iK^2\eta^2 + \frac{1}{4}}}{\eta^2}\right.\left[p_1 W_1(\eta) - p_3 W_3(\eta)\right] \cos x\eta\, d\eta \tag{5}$$

For $x > \frac{c}{R}$ $\quad u_1(x) = u_2(x) = 0$

These expressions are inverted and the $W_i(\eta)$ are represented in terms of Fourier cosine integrals of u_1 and u_2, which are in turn applied to the remaining boundary equations (2a, 2b). The resulting equations are

$$R^2 N_{\theta\theta} = \frac{i\sqrt{EhD}}{\pi}\int_0^\infty \int_0^{c/R} \Bigg[u_1(\xi)\Bigg\{ \frac{\eta^2}{2}\left(\frac{e^{p_1(\eta)|\theta|}}{-p_1(\eta)} + \frac{e^{p_3(\eta)|\theta|}}{-p_3(\eta)}\right)$$

$$+ \frac{\eta^4}{2\sqrt{2iK^2\eta^2+\frac{1}{4}}}\left(\frac{e^{p_1(\eta)|\theta|}}{-p_1(\eta)} - \frac{e^{p_3(\eta)|\theta|}}{p_3(\eta)}\right) - \frac{\eta^2}{2}\left(\frac{e^{p_2(\eta)|\theta|}}{-p_2(\eta)} + \frac{e^{p_4(\eta)|\theta|}}{-p_4(\eta)}\right)$$

$$+ \left(\frac{\eta^4(-\frac{1}{2}-\nu)+\frac{\eta^2}{2}}{\sqrt{-2iK^2\eta^2+\frac{1}{4}}}\right)\left(\frac{e^{p_2(\eta)|\theta|}}{-p_2(\eta)} - \frac{e^{p_4(\eta)|\theta|}}{-p_4(\eta)}\right)\Bigg\} \tag{6}$$

$$+ u_2\Bigg\{\frac{\eta^2}{2}\left(\frac{e^{p_1(\eta)|\theta|}}{-p_1(\eta)} + \frac{e^{p_3(\eta)|\theta|}}{-p_3(\eta)}\right) - \frac{\eta^4}{2\sqrt{2iK^2\eta^2+\frac{1}{4}}}\left(\frac{e^{p_1(\eta)|\theta|}}{-p_1(\eta)} - \frac{e^{p_3(\eta)|\theta|}}{-p_3(\eta)}\right)$$

$$- \frac{\eta^2}{2}\left(\frac{e^{p_2(\eta)|\theta|}}{-p_2(\eta)} + \frac{e^{p_4(\eta)|\theta|}}{-p_4(\eta)}\right) + \left(\frac{\eta^4(\frac{3}{2}-\nu)+\frac{\eta^2}{2}}{\sqrt{-2iK^2\eta^2+\frac{1}{4}}}\right)\left(\frac{e^{p_2(\eta)|\theta|}}{-p_2(\eta)} - \frac{e^{p_4(\eta)|\theta|}}{-p_4(\eta)}\right)\Bigg\}\Bigg]$$

$$\cdot \cos\eta(x-\xi)\, d\xi\, d\eta$$

$$-\frac{M_{\theta\theta}\pi R^2}{D} = \int_0^\infty \int_0^{c/R} \Bigg[u_1\Bigg\{\frac{1}{2}(\eta^2(2-\nu)+\frac{1}{2})\left(\frac{e^{p_1(\eta)|\theta|}}{-p_1(\eta)} + \frac{e^{p_3(\eta)|\theta|}}{-p_3(\eta)}\right)$$

$$+ \frac{1}{2}\left(\frac{\eta^4(1-\nu)+\frac{\eta^2}{2}}{\sqrt{2iK^2\eta^2+\frac{1}{4}}} + \sqrt{2iK^2\eta^2+\frac{1}{4}}\right)\left(\frac{e^{p_1(\eta)|\theta|}}{-p_1(\eta)} - \frac{e^{p_3(\eta)|\theta|}}{-p_3(\eta)}\right)$$

$$+ \frac{1}{2}\left(\eta^2(2-3\nu) - \frac{1}{2}\right)\left(\frac{e^{p_2(\eta)|\theta|}}{-p_2(\eta)} + \frac{e^{p_4(\eta)|\theta|}}{-p_4(\eta)}\right) \tag{7}$$

$$\left(\frac{\eta^4[2(1-\nu)^2-1+\nu]+\frac{\eta^2}{2}-\frac{1}{2}}{2\sqrt{-2iK^2\eta^2+\frac{1}{4}}} + \sqrt{-2iK^2\eta^2+\frac{1}{4}}\right)\left(\frac{e^{p_2(\eta)|\theta|}}{-p_2(\eta)} - \frac{e^{p_4(\eta)|\theta|}}{-p_4(\eta)}\right)\Bigg\} +$$

$$u_2\Bigg\{\frac{1}{2}(\eta^2\nu+\frac{1}{2})\left(\frac{e^{p_1(\eta)|\theta|}}{-p_1(\eta)} + \frac{e^{p_3(\eta)|\theta|}}{-p_3(\eta)}\right) + \frac{1}{2}\left(\frac{-\eta^4(1-\nu)-\frac{\eta^2}{2}}{\sqrt{2iK^2\eta^2+\frac{1}{4}}} + \sqrt{2iK^2\eta^2+\frac{1}{4}}\right)$$

$$\left(\frac{e^{p_1(\eta)|\theta|}}{-p_1(\eta)} - \frac{e^{p_3(\eta)|\theta|}}{-p_3(\eta)}\right) + \frac{1}{2}\left(\eta^2(4-3\nu) - \frac{1}{2}\right)\left(\frac{e^{p_2(\eta)|\theta|}}{-p_2(\eta)} + \frac{e^{p_4(\eta)|\theta|}}{-p_4(\eta)}\right)$$

$$\left(\frac{\eta^4[2(1-\nu)^2+(1-\nu)]+\frac{\eta^2}{2}-\frac{1}{2}}{2\sqrt{-2iK^2\eta^2+\frac{1}{4}}} + \sqrt{-2iK^2\eta^2+\frac{1}{4}}\right)\left(\frac{e^{p_2(\eta)|\theta|}}{-p_2(\eta)} - \frac{e^{p_4(\eta)|\theta|}}{-p_4(\eta)}\right)\Bigg\}\Bigg]$$

$\cdot \cos \eta (x-\xi)\, d\xi\, d\eta.$

In the above equations the order of integration is interchanged and the integration was carried out with respect to η. In order to effect this operation an approximation was made for the functions $p_i(\eta)$. A representative example is the case of $p_1(\eta)=[\eta^2 - \frac{1}{2} + (2iK^2\eta^2 + \frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}$ which was approximated by $p_1^*(\eta) = [-\frac{1}{2} + (2iK^2\eta^2 + \frac{1}{4})^{\frac{1}{2}}]^{\frac{1}{2}}$ for $\eta < .45/K^{1/3}$ and $p_1'(\eta) = [\eta^2 - \frac{1}{2} + \sqrt{2iK^2}\ \eta]^{\frac{1}{2}}$ for $\eta > .45/K^{1/3}$. For the evaluations of the integrals, recourse was made to Erdelyi [9]. The system of equations now has the form

$$f_i = \int_{-c/R}^{c/R} \sum_{j=1}^{2} k_{ij}(x,\xi;\lambda)\, u_j(\xi)\, d\xi; \qquad i = 1,2 \tag{8}$$

where $f_1 = -\dfrac{\sqrt{-1}\, N_{\theta\theta} R^2\, 2\pi}{\sqrt{EhD}}$ $\qquad \lambda = \left[\dfrac{12(1-\nu^2)\, c^4}{R^2 h^2}\right]^{1/4}$

$$f_2 = -\frac{M_{\theta\theta} R^2\, 2\pi}{D}$$

The range of integration in equation was doubled to render the equation more tractable. This is possible since u_1 and u_2 are symmetric for the symmetric problem. Integrating equations (8) once with respect to x leads to two coupled singular integral equations exhibiting Cauchy type singularities where u_1 & u_2 are bounded at the crack tips, this implies integral equations of index -1 [10].

Following the work of Erdogan [11] the auxiliary $u_1(\xi)$ and $u_2(\xi)$ are expanded in terms of Tskhebyscheff Polyinomials multiplied by the weighting function (viz $u_i(y) = (1-y^2)^{\frac{1}{2}} \sum_{n=o}^{N} A_{in}\, U_n(y)$). The kernels $k_{ij}\,(x,\xi)$ are expanded in the Neumann series (positive power of the argument) for argument <5 [12] and in the Stokes series for values of the argument >5 [13]. The integral equations are then solved to determine the values of the A_{in}. For add n $A_{in} = 0$ owing to the symmetry of the problem.

The values of $W_i(\eta)$ may now be determined by the inversion of equation (5) and by noting [8]

$$\int_o^1 A_{in}(1-y^2)^{1/2}\, U_{2n}(y) \cos \eta y\, dx = A_{i(2n)} \frac{(-1)^n \pi J_{1+2n}(\eta)}{2\eta} \tag{9}$$

The values of $W_i(\eta)$ so determined are

$$W_i(\eta) = \frac{1}{-2p_1(\eta)} \sum_{n=o}^{N} \left\{ \left[\left(A_{1(2n)} + A_{2(2n)} \right) + \frac{\eta^2}{\sqrt{2iK^2\eta^2 + \frac{1}{4}}} \left(A_{1(2n)} - A_{2(2n)} \right) \right] \cdot \frac{(-1)^n J_{1+2n}(\eta)}{2\eta} \right\}$$

$$W_3(\eta) = -\frac{1}{-2p_3(\eta)} \sum_{n=o}^{N} \left\{ \left[\left(A_{1(2n)} + A_{2(2n)} \right) - \frac{\eta^2}{\sqrt{2iK^2\eta^2 + \frac{1}{4}}} \left(A_{1(2n)} - A_{2(2n)} \right) \right] \frac{(-1)^n J_{1+2n}(\eta)}{2\eta} \right\}$$

$$W_2(\eta) = \frac{1}{-p_2(\eta)} \left(\frac{\eta^2(1-\nu) - \frac{1}{2}}{\sqrt{-2iK^2\eta^2 + \frac{1}{4}}} + \frac{1}{2} \right) \sum_{n=o}^{N} \left\{ \left(A_{1(2n)} + A_{2(2n)} \right) \frac{(-1)^n J_{1+2n}(\eta)}{2\eta} \right\}$$

$$+ \frac{\eta^2}{p_2(\eta)\sqrt{-2iK^2\eta^2 + \frac{1}{4}}} \sum_{n=o}^{N} \left\{ \left(A_{1(2n)} - A_{2(2n)} \right) \frac{(-1)^n J_{1+2n}(\eta)}{2\eta} \right\} \qquad (10)$$

$$W_4(\eta) = \frac{1}{p_4(\eta)} \left(\frac{\eta^2(1-\nu) - \frac{1}{2}}{\sqrt{-2iK^2\eta^2 + \frac{1}{4}}} - \frac{1}{2} \right) \sum_{n=o}^{N} \left\{ \left(A_{1(2n)} + A_{2(2n)} \right) \frac{(-1)^n J_{1+2n}(\eta)}{2\eta} \right\}$$

$$+ \frac{\eta^2}{-p_4(\eta)\sqrt{-2iK^2\eta^2 + \frac{1}{4}}} \sum_{n=o}^{N} \left\{ \left(A_{1(2n)} - A_{2(2n)} \right) \frac{(-1)^n J_{1+2n}(\eta)}{2\eta} \right\}$$

The forms of the $W_i(\eta)$ have been determined and can be inserted into expression (3) to yield an integral representation of the normal displacement w (x, θ) and F (x, θ) or directly into the equation (2) to determine the stress and moment resultants.

The numerical computations leading to the values of A_{1n} and A_{2n} are presently being carried out on an IBM 370 digital computer. Owing to the temporal length of the computations involved and some difficulty in obtaining the required access to the IBM 370 numerical results take presently incomplete but will be presented at the conference.

4. References

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