A COMPUTATIONAL METHOD FOR DYNAMIC ANALYSIS OF STRUCTURES WITH LARGE SIZE STIFFNESS AND MASS MATRICES. APPLICATIONS TO SEISMIC ANALYSIS

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SUMMARY

The dynamic analysis of complex structures is conveniently performed by means of the finite element method. If modal analysis is considered, the most important computational problem is the solution of a generalized eigenvalue problem in which matrices are often large and sparse. In the discrete analysis the number of modes of vibration is equal to the finite number of degrees of freedom but, for practical purposes, only the first few modes of vibration must be computed.

Usually modal analysis is performed assuming that the damping matrix is a linear combination of stiffness and mass matrices, so that the eigenvalue problem is of the form $KX = \omega^2 MX$ where K and M are symmetric positive definite matrices.

Many numerical methods for such a problem are available, but to have acceptable running times, the choice of a convenient scheme is very important.

An efficient algorithm, in the case of vibration analysis, should have the following features:

(a) number of operations proportional to the number of eigenvalues and eigenvectors required,

(b) use of programming techniques which take into account the sparseness of matrices to minimize computer storage and arithmetic.

In the present paper the efficiency of an iterative method, based on the simultaneous iteration scheme, is evaluated. The simultaneous iteration scheme satisfies the requirement (a) and has a convergence rate for the ith eigenvector which depends on the ratio $\rho_i = \omega_i^2/\omega_{p+1}^2$ where $\omega_1 \leqslant \omega_2 \leqslant \ldots \leqslant \omega_p < \omega_{p+1} \leqslant \ldots \leqslant \omega_n$ and p is the number of modes required. More precisely the error in the eigenvector at the 1th iteration is $0(\rho_i^1)$. In the new procedure proposed here the convergence rate is improved by using Chebishev polynomials so that the error in the ith eigenvector at the 1th iteration is $0(\sigma_i^1)$ where $\sigma_i = \rho_i/(1+\sqrt{1-\rho_i})^2$.

The particular structure of this iteration scheme allows, like in the static case, the use of a "frontal technique" which assembles and solves symmetric positive definite equations taking into account the sparseness of the matrices.

This technique is more involved than the standard band-matrix algorithms but is more efficient for two and three-dimensional problems. A computer program, based on this numerical model and using isoparametric elements, has been coded to study dynamic problems for two-dimensional structures. The efficiency of the iterative scheme has been evaluated by means of numerical comparisons with the Householder-Givens method. Finally, illustrative results of earthquake response analysis are presented.

1. Introduction

This paper describes the preliminary achievements of a work aimed at setting up a finite element computer system to perform seismc analysis on two and three-dimensional structures by using the response spectrum method(*). This modular computer system is now in preparation and is at present available for dynamic analysis of plane stress and plane strain structures.

As is well known, the first step in the response spectrum method is to obtain the natural frequencies and modes of vibration, so that the primary computational problem is the solution of a generalized eigenvalue problem. If the damping matrix is assumed to be a linear combination of stiffness and mass matrices, the eigenvalue problem is of the form $KX = \omega^2 MX$ where the stiffness matrix K and mass matrix M are symmetric positive definite matrices.

The results here reported refer to the evaluation of some numerical methods to solve the above mentioned eigenvalue problem. For such a problem many numerical procedures are available which are normally based on the Householder transformation method and the Sturm sequence method. These algorithms are efficient only if the number of degress of freedom is not very large (≤ 200).

In the well-known Householder-Givens method [1], [2], transformations are carried out on matrices K and M to reduce the problem to a standard symmetric eigenproblem with a full matrix which must be further reduced to tridiagonal form. If the number of degrees of freedom is large, this method has two main drawbacks:

- the formation of the full matrix requires a large storage
- the reduction to tridiagonal form takes up most of the computing time so that this method soon becomes prohibitive to be used.

Gupta's Sturm sequence method [3], [4] , [5] is based on the computation of the leading principal minors of the matrix $(K - \omega^2 M)$ for many values of ω^2. This method takes into account the sparseness of K and M so that the storage requirements are not very high but most of the computing time will be spent in examining different values of ω .

It is thus evident that the choice of a convenient scheme is crucial to have acceptable running times in the solution of large size eigenproblems such as those often arising from dynamic analysis of three-dimensional problems.

An efficient algorithm, in the case of large size vibration problems, should have the following features:

a) number of operations proportional to the number of eigenvalues and eigenvectors required
b) use of programming techniques which take into account the sparseness of matrices to minimize computer storage and arithmetic.

Since it meets the above requirements, the simultaneous iteration scheme [6], [7], [8], [9], [10], [11] has been considered and subsequently modified to enhance the convergence rate. The efficiency of this procedure has been evaluated by means of numerical comparison with the unmodified simultaneous iteration method and the Householder-Givens method.

As said before, at this early stage of development only two-dimensional finite element programs for seismic analysis have been completed. Therefore numerical experimentation has

(*) This work has been done in the frame work of ELFI-1 program supported by ENEL.

so far been restricted to two-dimensional problems regardless of whether or not such problems may be representative of reactor situations. Furthermore, the problems considered here turned out to be adequately treated with a small number of degrees of freedom (≃ 100). Nevertheless, and only for evaluation purposes, numerical experimentation was extended to cover finite element idealization with a large number of degrees of freedom (≃ 600).

2. Numerical method

2.1. Rutishauser simultaneous iteration scheme

The simultaneous iteration procedure, introduced by Jennings [6], [7], [8] and later improved by Rutishauser [9], [10], [11], can be used to compute the first p eigenvalues and eigenvectors of the problem :

$$KX_i = \omega_i^2 M X_i \quad (i = 1,n) \qquad \text{eq. (1)}$$

with the normalization condition $X_i^T KX_j = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{cases}$

where $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_p < \omega_{p+1} \leq \cdots \omega_n$ and K and M are symmetric non-negative definite n x n stiffness and mass matrices.

This algorithm leads to the following computing rule:

(a) Perform Cholesky decomposition of matrix K : $K = R^T R$

(b) Start with a n x p arbitrary orthogonal matrix $Y^{(1)}$

Start with j=1

(c) Perform m_j power steps by solving the systems:

$$Rz^{(k)} = y^{(k)}$$
$$R^T y^{(k+1)} = Mz^{(k)} \qquad (k=1,m_j \; ; \; y^{(1)} = Y^{(j)})$$

(d) Orthonormalize the columns of $y^{(m_j+1)}$ (Gram-Schmidt process):

$$y^{(m_j+1)} = U^{(j)} S^{(j)}$$

$U^{(j)}$ an nXp orthogonal matrix

$S^{(j)}$ a pXp upper triangular matrix

(e) Solve the p x p eigenvalue problem (Jacobi method):

$$S^{(j)} S^{(j)T} Q^{(j)} = Q^{(j)} D^{(j)}$$

for the eigenvalue matrix $D^{(j)}$ and the orthogonal eigenvector matrix $Q^{(j)}$

(f) Compute $Y^{(j+1)} = U^{(j)} Q^{(j)}$

(g) Test for termination

(h) Go to (c) if required

It has been proven [9] that:

$$\lim_{j \to \infty} \left[D_{ii}^{(j)} - \omega_i^{-4m_j} \right] = 0$$
$$\lim_{j \to \infty} \left\| Y_i^{(j+1)} - v_i \right\|_2 = 0 \qquad (i=1,p) \qquad \text{eq. (2)}$$

where $D_{ii}^{(j)}$ is the i^{th} element of the p x p diagonal matrix $D^{(j)}$, $Y_i^{(j+1)}$ is the i^{th} column of the n x p matrix $Y^{(j+1)}$ and:

$$v_i = R X_i \qquad \text{eq. (3)}$$

The convergence rate is as follows:

$$\left|\omega_i^{(j)} - \omega_i\right| = 0\,(\varrho_i^{2M_j}) \qquad (i=1,p) \qquad \text{eq. (4)}$$

$$\left\|Y_i^{(j+1)} - v_i\right\|_2 = 0\,(\varrho_i^{M_j})$$

where:

$$\omega_i^{(j)} = \left[D_{ii}^{(j)}\right]^{-1/4m_j} \qquad (i=1,p) \qquad \text{eq. (5)}$$

$$\varrho_i = \left(\frac{\omega_i}{\omega_{p+1}}\right)^2_j$$

$$M_j = \sum_{i=1} m_i$$

When convergence is reached, the eigenvectors of eq. (1) can be easily obtained by solving eqs. (3).

Rutishauser has described several stratagems which make the procedure more efficient and more automatic [9], [10]; in particular the following relevant facts can be mentioned:

- some information must be available on the eigenvalue spectrum because the algorithm does not converge if $\omega_{p+1} = \omega_p$;
- the sparseness of matrices K and M is not destroyed; more precisely the triangular factor R retains partially the sparseness of K while the sparseness of M is fully retained. Indeed it is not necessary that M be given as an array of numbers, but rather a rule is required for computing the vector Mz for an arbitrary vector z;
- different eigenvectors corresponding to the same eigenvalue can be exactly obtained. Such eigenvectors cannot be computed by using the Householder-Givens method or the Gupta method;
- in accordance with the convergence properties, in many cases higher efficiency is obtained if the iterations are performed with p vectors in order to compute $r < p$ eigenvectors. The number p must be chosen in view of two opposite circumstances: on one hand, the greater p the higher the convergence rate, on the other, increasing p means increasing the computing time;
- if, at the j^{th} iteration, the first t_j eigenvectors have satisfied the termination test, the first t_j columns of matrix $Y^{(j+1)}$ are frozen; i.e. they are no longer subject to iteration in step (c) and, in step (d), they are only used to orthogonalize the later $(p-t_j)$ columns of matrix $y^{(m_j+1)}$. In this way the computational work can be reduced;
- some caution must be taken in the choice of the number m_j of power steps. If m_j is large the convergence rate increases but, at the same time, the eigenvalues $D_{ii}^{(j)}$ of the reduced problem (e) become more and more spread. This effect tends to make the columns of $y^{(m_j+1)}$ parallel, so that numerical instabilities can arise. Rutishauser suggests increasing gradually m_j, if possible, as long as:

$$s^{(j)} = \frac{D_{ii}^{(j)}}{D_{pp}^{(j)}} < 100 \qquad \text{eq. (6)}$$

2.2. The accelerated procedure

The iterative procedure here proposed differs from the algorithm described in sect. 2.1. because, instead of the power step (c), the following three-term recurrence relation, based on the properties of the Chebyshev polynomials, is used

$$R z^{(j)} = Y^{(j)}$$

$$R^T \varphi^{(j)} = M z^{(j)} \qquad \text{eq. (7)}$$

$$y^{(j)} = \alpha_j \left(\frac{1}{\mu}\right)^{(j)} - Y^{(j)}) - \beta_j Y^{(j-1)} \cdot S^{(j-1)} \cdot Q^{(j-1)}$$

where $\alpha_1 = 1$, $\beta_1 = 0$, $\alpha_j = 2$, $\beta_j = 1$ for $j \geq 2$

and μ meets the condition:

$$\frac{1}{2\,\omega_{p+1}^2} \leq \mu < \frac{1}{2\,\omega_p^2} \qquad \text{eq. (8)}$$

The optimum value of parameter μ is:

$$\mu_{opt} = \frac{1}{2\,\omega_{p+1}^2} \qquad \text{eq. (9)}$$

but, ω_{p+1} being unknown, a practical choice of the parameter μ is the following:

$$\mu = \frac{1}{2\,\bar{\omega}_p^2} \qquad \text{eq. (10)}$$

where $\bar{\omega}_p$ is an upper bound of ω_p which is in general obtained by performing a few iterations with the non-accelerated power step (c).

It can be proven [12] that:

$$\lim_{j \to \infty} \left[D_{ii}^{(j)} - \left(\frac{T_j(\sigma_i)}{T_{j-1}(\sigma_i)} \right)^2 \right] = 0 \qquad \text{eq. (11)}$$

$$(i = 1,p)$$

$$\lim_{j \to \infty} \left\| Y^{(j+1)} - v_i \right\|_2 = 0$$

where

$$\sigma_i = \frac{1}{\mu\,\omega_i^2} - 1 \quad (i = 1,p) \qquad \text{eq. (12)}$$

and $T_j(\sigma_i)$ is the Chebyshev polynomial of degree j.

For the convergence rate one has:

$$\left| \omega_i^{(j)} - \omega_i \right| = 0\left((q_i^{(j)})^2 \right)$$

$$(i = 1,p) \qquad \text{eq. (13)}$$

$$\left\| Y_i^{(j+1)} - v_i \right\|_2 = 0\,(q_i^{(j)})$$

where:

$$\omega_i^{(j)} = \sqrt{\frac{2}{\mu}} \frac{\left[D_{ii}^{(j)}\right]^{1/4}}{1 + \left[D_{ii}^{(j)}\right]^{1/2}} \qquad (i = 1,p) \qquad \text{eq. (14)}$$

$$q_i^{(j)} = \frac{1}{T_j(\sigma_i)}$$

An accelerated procedure, based on the use of the Chebyshev polynomials, was also proposed by Rutishauser. This method uses, instead of eq. (7) the following equations

$$\begin{aligned} & y^{(1)} = Y^{(j)} \\ & \text{for } k = 1,m_j \\ & Rz^{(k)} = y^{(k)} \\ & R^T\varphi^{(k)} = Mz^{(k)} \\ & y^{(k+1)} = \alpha_k \left(\frac{1}{\mu}\varphi^{(k)} - y^{(k)}\right) - \beta_k y^{(k-1)} \end{aligned} \qquad \text{eq. (15)}$$

where α_k and β_k are defined as in (7) and μ is computed according to eqs. (9) or (10).

For this procedure it has been proven [10] that:

$$\begin{aligned} & \lim_{j\to\infty} \left[D_{ii}^{(j)} - T_{m_j}^2(\sigma_i)\right] = 0 \\ & \lim_{j\to\infty} \left\| Y_i^{(j+1)} - v_i \right\|_2 = 0 \end{aligned} \qquad \text{eq. (16)}$$

where σ_i is defined as in eq. (12), and $T_{m_j}(\sigma_i)$ is the Chebyshev polinomial od degree m_j.

For the convergence rate one has:

$$\begin{aligned} & \left| \omega_i^{(j)} - \omega_i \right| = O\left((\tau_i^{(j)})^2\right) \\ & \left\| Y_i^{(j+1)} - v_i \right\|_2 = O(\tau_i^{(j)}) \end{aligned} \qquad (i = 1,p) \qquad \text{eq. (17)}$$

where

$$\omega_i^{(j)} = \sqrt{\frac{2}{\mu}} \frac{\left[(D_{ii}^{(j)})^{1/2} + \sqrt{D_{ii}^{(j)} - 1}\right]^{1/2 m_j}}{1 + \left[(D_{ii}^{(j)})^{1/2} + \sqrt{D_{ii}^{(j)} - 1}\right]^{1/m_j}} \qquad (i = 1,p)$$

$$\tau_i^{(j)} = \prod_{k=1}^{j} \frac{1}{T_{m_k}(\sigma_i)} \qquad \text{eq. (18)}$$

This algorithm seems to be less convenient than procedure (7) for the following reasons:

- if the convergence rate of the two processes is compared for the same number of iterations j, one could think that procedure (15) is better than procedure (7), $q_i^{(j)}$ being greater than $\tau_i^{(j)}$. However, a fair comparison must take into account also the computing work which, at this stage, is not the same. Indeed, if matrices K and M are large, the most expensive contribution to the total computing time for the two procedures is the

computation of matrix $\varphi^{(k)}$ which is performed m_j times in scheme (15) and only once in scheme (7). Therefore it seems reasonable to compare the convergence properties of these numerical procedures not for the same number of iterations but for the same number of power steps. In this case $q_i^{(M_j)}$ is less than $\tau_i^{(j)}$ where M_j is defined in eq. (5).

- The eigenvalues $D_{ii}^{(j)}$ obtained in the procedure here proposed are less spread than those computed by means of the accelerated Rutishauser scheme. Indeed, the asymptotic behaviour of the ratio $s^{(j)}$, defined in eq. (6) is:

$$s_1^{(j)} \cong \left(\frac{T_j(\sigma_1)}{T_{j-1}(\sigma_1)} \right)^2 \qquad \text{for the procedure which uses eq. (7)}$$

and

$$s_2^{(j)} \cong T_{m_j}^2(\sigma_1) \qquad \text{for the procedure which uses eqs. (15)}$$

In this way, being $s_1^{(j)} < s_2^{(j)}$, the risk of numerical instabilities is reduced.

2.3. Application of frontal technique.

The frontal techniques provides the solution of the linear system

$$Ax = b \qquad \text{eq. (19)}$$

where A is a definite positive matrix which, as met in finite element applications, is the sum of several contributions from different elements [13]. This algorithm assembles the rows and columns of matrix A following an order which depends on the numbering of the elements and, as soon as the j^{th} row is fully summed, it eliminates the j^{th} unknown. In this way eq. (19) is solved by means of Gaussian elimination, with symmetric row-column pivoting, in which the interchange strategy is determined by the ordering of the elements.

The main steps in this process are:

(i) Perform the Cholesky decomposition of the positive definite matrix $\tilde{A} = PAP^T$, where P is a permutation matrix which takes into account the row interchanges:

$$PAP^T = \tilde{R}^T\tilde{R}$$

(ii) Solve the linear systems:

$$\tilde{R}^T z = Pb$$
$$\tilde{R}\tilde{x} = z$$

(iii) Compute the solution:

$$x = P^T\tilde{x}$$

The efficiency of the frontal technique obviously depends not only on the pivoting strategy, but also on many programming artifices which take into account the sparseness of matrix $\tilde{R}$ to minimize storage and arithmetic.

The afore-mentioned features of the frontal solution can be employed in the simultaneous iteration scheme if, instead of eq. (1), the following problem is considered:

$$PKP^T\tilde{X}_i = \omega_i^2 PMP^T\tilde{X}_i$$

where P is defined in (i) and

$$PX_i = \tilde{X}_i$$

In this case the frontal technique provides the Cholesky decomposition of matrix PKP^T:

$$PKP^T = \tilde{R}^T\tilde{R}$$

and power step (c) is so modified:

$$\tilde{R}z(k) = y(k)$$

$$\tilde{R}^T y(k+1) = PMP^T z(k)$$

Of course similar formulae can be employed in eq. (7) or in eq. (15).

3. The computer program

The computer program consists of four links whose main functions are described in this section.

Link 1: Mesh generation

Manual and/or automatic input can be processed. Suitable subroutines can easily be introduced to prescribe element geometry for many types of structures. At present, only two-dimensional rectangular isoparametric elements with linear, quadratic or cubic shape functions are available.

Link 2: Computations of stiffness and mass matrices

For each class of structures a separate subroutine provides for the computation of the element stiffness and mass matrices. Rigid and/or elastic constraints and additional lumped masses are accounted for by suitably modifying stiffness and mass matrices.

Link 3: Eigenproblem solution

In this link the undamped natural frequencies and modes are computed. As the program is still in preparation, for evaluation purposes, the following numerical methods can be used on option:

Procedure I - Householder - Givens method with banded stiffness and mass matrices.

Procedure II - Non-accelerated Rutishauser simultaneous iteration method as described in sect. 2.1. with banded stiffness matrix.

Procedure III - Accelerated simultaneous iteration method as described in sect. 2.2. (eq.(7)) with banded stiffness matrix.

Procedure IV - Procedure II with "frontal technique" as described in sect. 2.3.

If procedure I is chosen, link 2 provides also for the assembling of the element stiffness and mass matrices in banded matrices relevant to the whole structure. For procedures II and III anly the assembling of the stiffness matrix is performed. These computations are not required if procedure IV is used.

Link 4: Seismc analysis (response spectrum method)

The response of a structure to earthquake forces is computed by means of the response spectrum modal analysis [14], [15]. According to this procedure the vector of nodal displacements has the form:

$$\delta(t) = \sum_{i=1}^{r} p_i(t)\, X_i \qquad \text{eq. (20)}$$

where X_i are the vibration modes defined by the eigenproblem (1) and r is the number of modal contributions in which the essential dynamic response is contained. The coefficients

$p_i(t)$ are solutions of the decoupled equations:

$$\frac{d^2 p_i}{dt^2} + \beta_i \frac{d\, p_i}{dt} + \omega_i^2\, p_i = - a_o\, g_i \qquad \text{eq. (21)}$$

where

$$\beta_i = \alpha\, \omega_i^2$$

$$g_i = \omega_i^2\, X_i^T\, M\, e \qquad (i = 1,r) \qquad \text{eq. (22)}$$

In the above equations ω_i are the undamped frequencies, M is the mass matrix, a_o is the earthquake acceleration applied at the base of the structure and "e" is a vector geometrically connecting the acceleration at nodes and a_o. The constant α is a proportionality coefficient between the stiffness matrix K and damping matrix $C = \alpha K$.

The response spectrum method supplies the maximum values of the coefficients $p_i(t)$ which are then combined by root-mean-square or other approximations to give an estimate of the total maximum response.

The earthquake spectrum used in the program is constructed in accordance with the procedure pubblished by Newmark and Hall [16].

The following responses are currently available for plane stress and plane strain systems:

- nodal displacement components and stresses (σ_x, σ_y, τ_{xy}) computed as the root-mean-square of the modal maxima;
- maximum shear stress $\tau_{max} = \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}$ where $\left|\frac{\sigma_x - \sigma_y}{2}\right|$ and $|\tau_{xy}|$ are computed as the sum of the modal maxima.

The input information to the program consists of the geometric description of the structure, the material properties (elastic properties and unit weight of each element, internal viscous damping coefficient α assumed equal for all the materials of the structure) the load data (the components of the maximum ground velocity and acceleration).

The output of the program includes:

- vibration frequencies and modes,
- an estimate of the maximum nodal displacement components and stresses.

Auxiliary routines are available to edit a plot of the output data.

4. Numerical experiments

This section reports the results of numerical experiments aimed at evaluating the numerical methods mentioned in sect. 3. (procedures I, II, III, IV). Finally an example of the type of results relevant to seismc analysis which may be obtained by the program is presented.

Since at this stage the program permits calculating plane stress or plane strain systems only, and our primary purpose is to carry out a numerical experimentation rather than solve a specific problem, we restrict ourselves to considering an idealized two-dimensional structure. One such structure, that also presents an interest of its own, is a concrete gravity dam which can be approximated by a plane strain system when the analysis is performed on its cross section. For the purpose of this work it is adequate to ignore, in the earthquake response, both the dam-water interaction and the elasticity of foundations. The

geometry and the material properties of the dam considered are shown in fig. 1.

To evaluate the efficiency of the numerical procedures described, when varying the number of degrees of freedom, we have considered three finite element idealizations (see fig. 1).

Case A : 28 linear elements with 84 degrees of freedom

Case B : 28 cubic elements with 360 degrees of freedom

Case C : 52 cubic elements with 624 degrees of freedom

4.1. Evaluation of numerical methods

The estimate of the computational efficiency of the procedures I, II and III is based on the number of arithmetic operations required in each scheme, according to the convention to count only multiplications and divisions.

The comparison between procedures IV and II, aimed at evaluating the efficiency of the "frontal technique", is based on the actual computing time.

The number of operations for procedure I is:

$$N_I = \frac{2}{3} n^3 + \frac{b}{6} (3 n^2 - 5 b^2 + 6 b n) + \frac{b}{2} r (2 n - b) \qquad \text{eq. (23)}$$

where: n is the number of degrees of freedom,

2b-1 is the total band-width,

r is the number of eigenvectors required.

For procedure II the number of operations is of the form:

$$N_{II} = N_1 + \sum_{j=1}^{L} N_2^{(j)} + T N_3 \qquad \text{eq. (24)}$$

where: L is the total number of iterations,

T is the total number of acceptance tests for eigenvectors and one has(*):

$$N_1 = \frac{b^2}{6} (3n - 2b) + \frac{b}{2} r (2n - b) \qquad \text{eq. (25)}$$

$$N_3 = b (2n - b) + \sum_{i=1}^{E} g_i^2 \qquad \text{eq. (26)}$$

$$N_2^{(j)} = (p - t_j) (m_j N_3 + 2 n p) \qquad \text{eq. (27)}$$

where: m_j is the number of power steps at the iteration j,

E is the number of elements in the structure,

g_i is the number of degrees of freedom of the i^{th} element,

t_j is the number of eigenvectors which have fulfilled the acceptance test up to iteration j,

p is the number of eigenvectors used in the iterative process (p must be always greater than r).

The number of operations for procedure III includes the arithmetic involved in the computation of the parameter μ (see eq. (10)). In the program this parameter is computed by assuming for $\bar{\omega}_p$ the estimate of ω_p obtained with procedure II stoppen when:

(*) The number of operations involved in the Jacobi step (e) (see sect. 2.1) was neglected.

$$\left| \frac{(\omega_p^{(j)})^2 - (\omega_p^{(j-1)})^2}{(\omega_p^{(j)})^2} \right| \leq \varepsilon_p \qquad \text{eq. (28)}$$

where the accuracy ε_p is not very high ($\simeq 0.1$).

The iterations required to estimate μ are not wasted; indeed the eigenvalues and eigenvectors obtained at this stage are used as starting guesses in the accelerated procedure. In this way the number of operations for procedure III is of the form:

$$N_{III} = N_1 + \sum_{j=1}^{L_1} N_2^{(j)} + \sum_{j=L_1+1}^{L} N_4^{(j)} + T\, N_3 \qquad \text{eq. (29)}$$

where: N_1, $N_2^{(j)}$ and N_3 are defined as in eq. (25), (26), (27);

L_1 is the number of iterations performed with procedure II until eq. (28) is verified

L is the total number of iterations ;

$N_4^{(j)}$ is defined as follows:

$$N_4^{(L_1+1)} = (p - t_{L_1+1})\, N_3 \qquad \text{eq. (30)}$$

$$N_4^{(j)} = (p - t_j)\,(N_3 + n(p - t_j)) \quad \text{for } j > L_1+1 \qquad \text{eq. (31)}$$

The acceptance tests used in the program are:

$$\left| \frac{(\omega_i^{(j)})^2 - (\omega_i^{(j-1)})^2}{(\omega_i^{(j)})^2} \right| \leq \varepsilon_1 \qquad (i=1,p) \qquad \text{eq. (32)}$$

$$\left\| (\omega_i^{(j)})^2\, R^{-T} M R^{-1} Y_i^{(j+1)} - Y_i^{(j+1)} \right\|_2 \leq \varepsilon_2 \left[1 - \left(\frac{\omega_i^{(j)}}{\omega_p^{(j)}} \right)^2 \right]$$

The numerical experimentation was performed with $\varepsilon_1 = 10^{-5}$ and $\varepsilon_2 = 10^{-4}$.

Table I shows the number of operations for procedures I, II and III computed according to eqs. (23), (24) and (29). The value of the accuracy used for the computation of the acceleration parameter μ is also given. The results quoted in Table I indicate that:

(i) The Householder-Givens method is the most efficient for small size problems (Case A) even if the running time of procedure III with $\varepsilon_p = 0.1$ is nearly the same.

(ii) For large size problems procedure I becomes prohibitive while with procedures II and III a time savings of a factor of about 4 and 9 is achieved in cases B and C respectively. In particular procedure III obtains a gain of about 20% with respect to procedure II. Ist must be noticed that, for the cases considered, the convergence of procedure II was sufficiently fast, the eigenvalues being spread enough $\left(\frac{\omega_5^2}{\omega_8^2} \simeq 0.45\right)$. One might expect that more significant gains can be achieved by means of procedure III when eigenvalues are closer to each other than in the cases considered.

(iii) Convergence of procedure III seems to be sensitive to parameter μ . Indeed a poor choice of $\bar{\omega}_p$ (Case A with $\varepsilon_p = 0.5$) deletes the effect of the accelerated procedure.

Table II compares the running time obtained for cases B and C by applying procedures II (standard band algorithm) and IV (frontal technique). The results obtained show that running

times are comparable, with a possible trend for the frontal technique to become advantageous for large size problems. Such result matches the conclusions that can be drawn from the solution of elastostatic structural problems.

Finally it must be noticed that the use of the frontal technique in a modular computing system can be very attractive even if, in some cases, a banded algorithm is more convenient. Indeed the frontal technique easily permits extending the number of modules, as new requirements arise, with minimum programming effort.

4.2. Seismic analysis results

As an example of the type of results obtained by the program, the earthquake analysis of the concrete gravity dam section, shown in fig. 1, was performed. The response spectrum used is shown in fig. 2 and refers to a maximum ground acceleration of 0.24 g and to a maximum velocity of 12 inch/sec.

The earthquake force was assumed to act only along the x direction and the dynamic response was computed by superposition of the first five modes. The computations were performed for the three finite element idealizations A, B, C.

Table III quotes the first five frequencies obtained for the three cases. It can be seen that the frequencies obtained in cases B and C are nearly the same, while they are lower than those relevant to case A by at most 10%. Also the earthquake responses calculated for the idealizations B and C are very similar, while the values obtained in case A are lower than those relevant to cases B and C by at most 4% for nodal displacements and 15% for nodal stresses. As an illustrative example, some results obtained for case B are shown. Fig. 3 shows the edge displacements, fig. 4 shows the contour lines for τ_{max}.

5. Conclusions

The article has presented the preliminary results of a study aimed at evaluating different numerical methods for the computation of the vibration frequencies and modes of an undamped structure. The research has been considered of primary importance for setting up a general computing system designed to perform the seismic analysis of complex structures using the finite element method.

Among the algorithms considered, the Householder-Givens method and the simultaneous iteration method have been chosen to perform numerical experimentations.

Moreover, a new version of the latter procedure has been proposed which improves the convergence rate of the iterative process by using the properties of the Chebyshev polynomials. The numerical experimentation has been carried out by using a first version of the computing system which permits, at present, seismic analysis of plane stress and plane strain structures. From the results available at this time the following conclusions can be drawn:

(i) As far as the number of degrees of freedom is small (≤ 100) comparable running times were obtained by using all the numerical methods considered, although the Householder-Givens method probably is the most advisable.

(ii) For large size problems the Householder-Givens method requires such a high computing time that it soon becomes impracticable. On the contrary the simultaneous iteration

scheme and its accelerated version work well; in particular the accelerated procedure seems to be more efficient. In the cases considered a time savings of about 20% was achieved but one may expect that more significant gains can be obtained if the method is used to compute eigenvalues close to each other.

(iii) The particular structure of the simultaneous iteration scheme or of its accelerated version allows the use of the frontal technique so that it is not necessary to consider the stiffness and mass matrices as banded. Moreover this feature is very interesting to set up a modular computing system because the flexibility of this technique permits extending the number of modules with minimum programming effort.

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TABLE I

NUMBER OF OPERATIONS FOR PROCEDURES I, II, III

Case	n	b	r	p	N_I	N_{II}	N_{III}
A	84	22	5	7	$0.512 \cdot 10^6$	$0.652 \cdot 10^6$	$0.64 \cdot 10^6$ (ε_p=0.5) $0.534 \cdot 10^6$ (ε_p=0.1)
B	360	94	5	7	$39.83 \cdot 10^6$	$9.27 \cdot 10^6$	$7.58 \cdot 10^6$ (ε_p=0.1)
C	624	124	5	7	$195.29 \cdot 10^6$	$22.05 \cdot 10^6$	$18.52 \cdot 10^6$ (ε_p=0.1)

TABLE II

COMPUTING TIME (CP) FOR PROCEDURES II and IV (IBM 360/65)

Case	Procedure II (sec)	Procedure IV (sec)
B	640	722
C	1458	1342

TABLE III

FREQUENCIES FOR CASES A, B, AND C (rad/sec)

ω_i \ Case	A	B	C
1	20.31	19.71	19.64
2	50.77	47.56	47.48
3	60.65	60.21	60.19
4	94.73	86.17	86.05
5	136.00	129.97	129.26

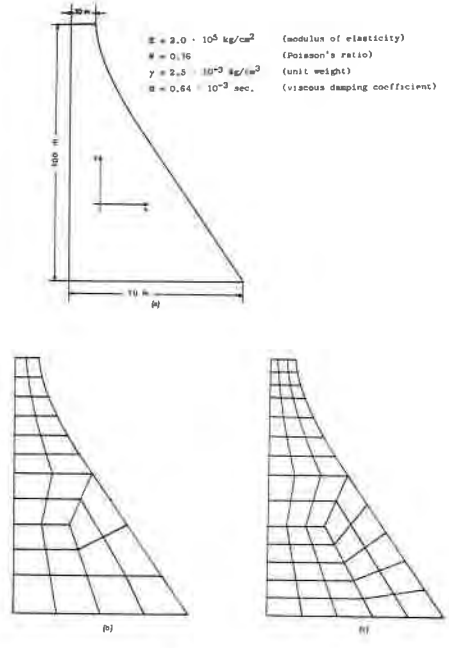

Fig. 1 Concrete gravity dam cross section.
(a) Material properties and geometry.
(b) Finite element idealization: linear elements (Case A), cubic elements (Case B).
(c) Finite element idealization: cubic elements (Case C).

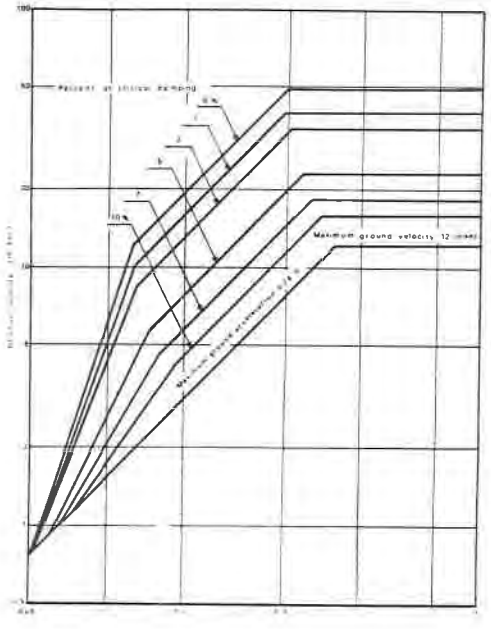

Fig. 2 Design response spectrum.

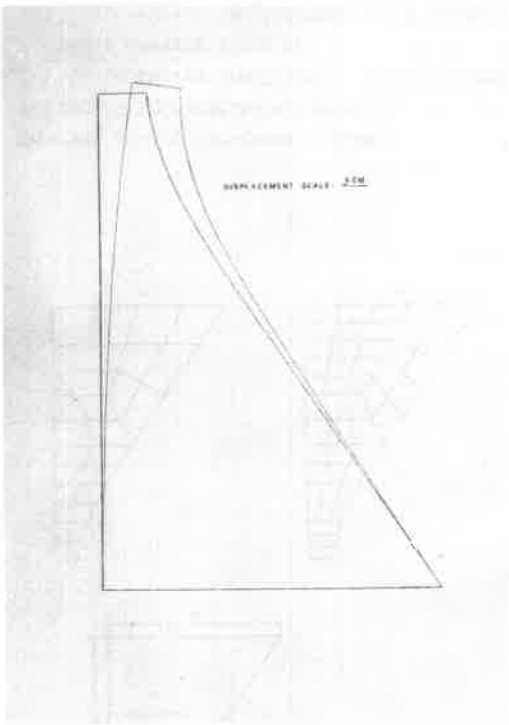

Fig. 3 Edge displacements.

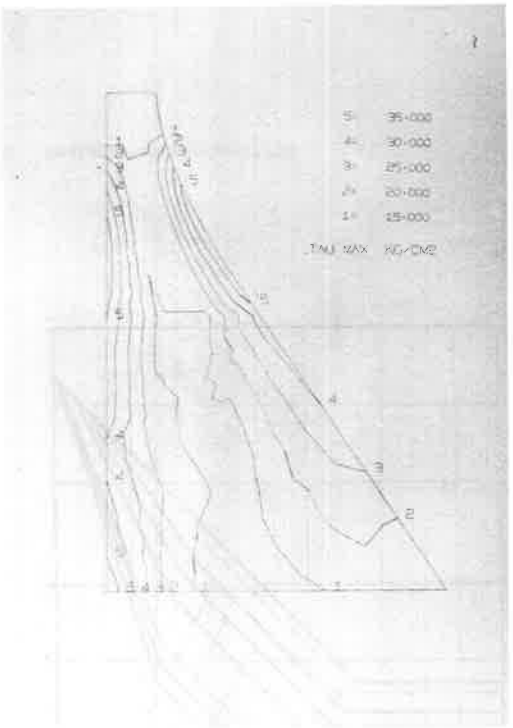

Fig. 4 Contour lines of τ_{max}.