THE INCREMENTAL THERMOMECHANICAL THEORY OF VISCOELASTOPLASTIC SOLIDS AND SOLUTION BY FINITE ELEMENTS

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SUMMARY

A new approach to a solution of boundary value problems of thermoviscoelastoplastic structures is presented. A concept of piecewise linearization which permits a superposition of nonlinear quantities within the short discretized time increment leads to a coupling of effects of various material characteristics over the entire history domain. Thus, the task of arriving at a single constitutive equation reflecting all of these effects is replaced by one of superimposing independent constitutive equations governing elasticity, viscosity, and plasticity with one on top of the other within a small discretized time increment. Combined nature of material response is reconstructed and maintained throughout the history of load-deformation process by successively recalculating the material kernels based on the results of the previous time increment.

In order to achieve this goal we first write the constitutive relations for the free energy φ, the stress tensor σ^{ij}, the entropy η and the heat flux q as follows:

$$\varphi = \Phi[\gamma_{ij}^{(e)}, \gamma_{ij}^{(p)}, \theta, \nabla\theta, \overset{(r)}{\alpha}{}_{ij}^{(e)}, \overset{(r)}{\alpha}{}_{ij}^{(p)}] \tag{1}$$

$$\sigma^{ij} = \hat{\Sigma}[-], \eta = \hat{H}[-], q^i = \hat{Q}[-] \tag{2}$$

where γ_{ij}, θ, and $\overset{(r)}{\alpha}{}_{ij}$ in (1) are the strain tensor, absolute temperature and internal variables for viscous behavior, respectively; $^{(e)}$ and $^{(p)}$ denote "elastic" and "plastic", respectively. The symbols [-] in (2) refer to the identical functional representation as in (1). Elastic and plastic components of strain and internal variables are separated. This is because of the assumption that viscous effect could prevail in the ranges of either elasticity or plasticity and that the free energy may be expressed in the form,

$$\rho\varphi = \tfrac{1}{2}E^{ijkl}\gamma_{ij}^{(e)}\gamma_{kl}^{(e)} + \tfrac{1}{2}\overset{*}{E}{}^{ijkl}\gamma_{ij}^{(p)}\gamma_{kl}^{(p)} - B^{ij}T\gamma_{ij}^{(e)} - \overset{*}{B}{}^{ij}T\gamma_{ij}^{(p)} - \tfrac{1}{2}cT^2 - \sum_{r=1}^{n} B_{(r)}^{ij}T\overset{(r)}{\alpha}{}_{ij}^{(e)} - \sum_{r=1}^{n} B_{(r)}^{ij}T\overset{(r)}{\alpha}{}_{ij}^{(p)} + \tfrac{1}{2}\sum_{r=1}^{n}\xi_{(r)}^{ijkl}(\alpha_{ij}^{(e)} + \alpha_{ij}^{(p)})(\alpha_{kl}^{(e)} + \alpha_{kl}^{(p)}) + \sum_{r=1}^{n}\xi_{(r)}^{ijkl}(\overset{(r)}{\alpha}{}_{ij}^{(e)} + \overset{(r)}{\alpha}{}_{ij}^{(p)})(\gamma_{kl}^{(e)} + \gamma_{kl}^{(p)})$$

where c is the specific heat; T and T_o are the temperature change and reference temperture; ρ is the density; E^{ijkl}, B^{ij}, and $\xi_{(r)}^{ijkl}$ are material constants; the symbol * refers to the plastic counterpart which may be derived from any one of the acceptable plasticity theories; and $\overset{(r)}{\alpha}{}_{ij}$ is given by

$$\overset{(r)}{\alpha}{}_{ij} = \int_0^t \exp\left\{\frac{-(t-\tau)}{T_{(r)}}\right\}\dot{\gamma}_{ij}\,d\tau .$$

Here $T_{(r)}$ is the relaxation time.

With these preliminaries together with the first and second laws of thermodynamics one derives the equations of equilibrium and heat conduction. In the present study, a three-dimensional structure and numerical integration is carried out in time domain. As a result of this formulation complete transient deformation and stress fields can be calculated from the input heat source. The computed results so far obtained have confirmed the soundness of the present theory and computational schemes.

1. Introduction

Despite extensive studies in recent years on thermodynamics of non-linear dissipative media, practical applications of existing theories to solve complex engineering problems do not appear to be readily available.

Since the fundamental studies by Coleman and Gurtin [1] many investigators have attempted to incorporate the nonlinear behavior to materials with memory from the viewpoint of irreversible thermodynamic processes. An equation of evolution as proposed by Coleman and Gurtin [1] which includes the rate of change of hidden or internal state variables was used by Kratochvil and Dillon [2,3], Tseng [4], and Hahn [5]. Their work was then related to the theory of dislocations in dealing with the internal state variables. These studies may appear to be in contrast to constitutive equations of differential type proposed by Coleman and Mizel [6], Schapery [7] and Kestin [8]. However, under certain suitable conditions most of the qualitative results are similar to those obtained by Coleman and Gurtin [1]. As alternate approaches to [1-5], Perzyna and Wojno [9] used a symmetric second order tensor for a hidden variable, and Valanis [10,11] introduced a functional "endochronic theory". Liu and Lee [12] and Green and Naghdi [13] discuss the kinematical interpretation of finite strains associated with plastic strains. The viscoplastic theory by Malvern [14] and the limiting maximum stress-strain relationship by Lubliner [15] and Cristescu [16] may also be noted. More exhaustive review on the thermoviscoplasticity may be found in Oden and Bhandari [18] who used the Coleman approach and variational principle in arriving at a thermodynamic theory of viscoplasticity.

Some of the previous studies based on the free energy functional in terms of histories of strains and temperature involve considerable difficulties in handling the plastic behavior coupled with thermoviscoelasticity. The objective of the present paper, therefore, is to introduce an incremental theory which facilitates both analytical formulation and computational procedure using the finite element method. An earlier work by the authors [19] without the viscous effects substantiated the effectiveness of the present approach. It was also demonstrated that either classical theory of plasticity or dislocation theory using the work of Lindholm [20] may easily be incorporated into the final form of governing equations of equilibrium and heat conduction [19,21].

In the present study a smoothness assumption of free energy to be "continuous" functional of the past histories of "inelastic" strains is not considered. Instead, we assert that such smoothness can be assumed to be valid only for a small time interval or a fraction of loading increments. Thus, a concept of piecewise linearization which permits a superposition of nonlinear quantities within the infinitesimal or discretized history is effectively utilized such that, in essence, the difficult task of coupling the effects of thermoviscoelasticity and thermoplasticity is carried out on the basis of material kernels initially decoupled. Then, coupling or direct superposition is established by determining the plastic kernel from independent viscoelastic responses within the small time interval or loading increment. This result then is carried over to the next time increment by suitable difference operator.

Basic thermodynamic preliminaries and the proposed thermoviscoelastoplastic model are presented in Section 2. Applications of this model, then, to obtain equations of heat conduction and equilibrium are given in Sections 3 and 4, respectively. Finally, some example problems and concluding remarks are presented in Sections 6 and 7, respectively.

2. Thermodynamics of Inelastic Solids with Memory

We begin with the results of the first and second laws of thermodynamics. The summary is given below:

Conservation of mass $$\int_{V_0} \rho_0 dV_0 = \int_V \rho dV \tag{1a}$$

Conservation of linear and angular momentum $$\sigma^{ij}\big|_i + \rho F^j - \rho\ddot{u}^j = 0,\quad \sigma^{ij} = \sigma^{ji} \tag{1b}$$

Balance of energy $$\rho\dot{\varepsilon} = \sigma^{ij}\dot{\gamma}_{ij} + q^i\big|_i + \rho h \tag{1c}$$

Clausius-Duhem inequality $$D + \frac{1}{\theta} q^i \theta_{,i} \geq 0 \tag{1d}$$

$$D = \rho\theta\dot{\eta} - q^j\big|_j - \rho h \tag{1e}$$

Here ρ is the mass density with the subscript $_0$ indicating undeformed configuration. σ^{ij} is the second Piola-Kirchhoff stress tensor; superposed dots represent time rates; strokes and commas are covariant and ordinary differentiations; F^j and u^j are the body forces and displacements; γ_{ij} is the strain tensor; ε, h, and η are the internal energy, heat supply and entropy per unit mass; q^i is the heat flux per unit area; θ is the absolute temperature; D is the internal dissipation. It is understood for small strains $\rho_0 = \rho$ and for rectangular cartesian coordinates covariant and ordinary differentiations are the same.

The Helmholtz free energy is defined as

$$\varphi = \varepsilon - \theta\eta \tag{2}$$

which upon substitution of (1e) leads to

$$\rho\dot{\varphi} = \sigma^{ij}\dot{\gamma}_{ij} - D - \rho\eta\dot{\theta} \tag{3}$$

The basic approach to the present study, as followed in the previous work [19] by the authors, is to express the thermodynamic variables in incremental quantity for a small time interval. This permits derivation of governing equations with a minimum of mathematical manipulations and simpler computational scheme, thus avoiding complexities as occuring in the functional theory of thermodynamic processes [1-13]. It is proposed that the continuous histories of material behavior may be discretized for a small time interval Δt such that the free energy, the stress, the heat flux, the entropy, and the plastic strain rate are functions of elastic strains $\overset{(e)}{\gamma}_{ij}$, plastic strains $\overset{(p)}{\gamma}_{ij}$, absolute temperature θ and a set of internal (hidden) variables $\alpha_{ij}(r = 1,2, \ldots n) = \overset{(r)(e)}{\alpha}_{ij} + \overset{(r)(p)}{\alpha}_{ij}$ with $(_e)$ and $(_p)$ indicating "elastic" and "plastic" behavior, respectively. Separation of the internal variables into elastic and plastic components was not attempted in the previous literature. This idea, however, forms an important part of the present paper. Further elaboration will be given later. In spite of the principle of equipresence it can be shown that heat flux is a function of the temperature gradient while other quantities such as the free energy, stress and entropy are free of its dependency. The foregoing statements can be written in the form

$$\varphi(\Delta t) = \hat{\Phi}[\overset{(e)}{\gamma}_{ij}(\Delta t),\ \overset{(p)}{\gamma}_{ij}(\Delta t),\ \theta(\Delta t),\ \overset{(r)(e)}{\alpha}_{ij}(\Delta t),\ \overset{(r)(p)}{\alpha}_{ij}(\Delta t)] \tag{4a}$$

$$\sigma^{ij}(\Delta t) = \hat{\Sigma}_{ij}[\overset{(e)}{\gamma}_{ij}(\Delta t),\ \overset{(p)}{\gamma}_{ij}(\Delta t),\ \theta(\Delta t),\ \overset{(r)(e)}{\alpha}_{ij}(\Delta t),\ \overset{(r)(p)}{\alpha}_{ij}(\Delta t)] \tag{4b}$$

$$\eta(\Delta t) = \hat{H}[\overset{(e)}{\gamma}_{ij}(\Delta t),\ \overset{(p)}{\gamma}_{ij}(\Delta t),\ \theta(\Delta t),\ \overset{(r)(e)}{\alpha}_{ij}(\Delta t),\ \overset{(r)(p)}{\alpha}_{ij}(\Delta t)] \tag{4c}$$

$$q^i(\Delta t) = \hat{Q}[\overset{(e)}{\gamma}_{ij}(\Delta t),\ \overset{(p)}{\gamma}_{ij}(\Delta t),\ \theta(\Delta t),\ \nabla\theta(\Delta t),\ \overset{(r)(e)}{\alpha}_{ij}(\Delta t),\ \overset{(r)(p)}{\alpha}_{ij}(\Delta t)] \tag{4d}$$

$$\dot{\overset{(p)}{\gamma}}_{ij}(\Delta t) = \hat{\overset{(p)}{\gamma}}_{ij}[\overset{(e)}{\gamma}_{ij}(\Delta t),\ \overset{(p)}{\gamma}_{ij}(\Delta t),\ \theta(\Delta t),\ \overset{(r)(e)}{\alpha}_{ij}(\Delta t),\ \overset{(r)(p)}{\alpha}_{ij}(\Delta t)] \tag{4e}$$

$$\dot{\alpha}^{(r)}_{ij}(\Delta t) = \hat{\alpha}^{(r)}_{ij}[\gamma^{(e)}_{ij}(\Delta t),\ \gamma^{(p)}_{ij}(\Delta t),\ \theta(\Delta t),\ \alpha^{(r)(e)}_{ij}(\Delta t),\ \alpha^{(r)(p)}_{ij}(\Delta t)] \tag{4f}$$

It is important to note here that all variables above are incremental quantities with the range of smoothness confined within the small interval Δt. The internal variables $\alpha^{(r)}_{ij} = \alpha^{(r)(e)}_{ij} + \alpha^{(r)(p)}_{ij}$ are assumed to represent the time dependent viscous behavior. A superposition of $\alpha^{(r)(e)}_{ij}$ and $\gamma^{(e)}_{ij}$ implies viscoelastic properties whereas that of $\alpha^{(r)(p)}_{ij}$ and $\gamma^{(p)}_{ij}$ refers to viscoplastic behavior. For materials exhibiting no viscous properties, we would have $\alpha^{(r)}_{ij}$ removed from eq. (4a-e) above. The preceding arguments immediately suggest that one feasible form of the free energy is the superimposed sum of all individual contributions of linear and nonlinear behavior for the small time interval Δt, so that

$$\begin{aligned}\rho\varphi(\Delta t) = {} & \frac{1}{2}E^{ijk\ell}\gamma^{(e)}_{ij}(\Delta t)\gamma^{(e)}_{k\ell}(\Delta t) + \frac{1}{2}\overset{*}{E}{}^{ijk\ell}\gamma^{(p)}_{ij}(\Delta t)\gamma^{(p)}_{k\ell}(\Delta t) - B^{ij}T(\Delta t)\gamma^{(e)}_{ij}(\Delta t) - \overset{*}{B}{}^{ij}T(\Delta t)\gamma^{(p)}_{ij}(\Delta t) \\ & - \frac{cT^2(\Delta t)}{2T_0} - \sum_{r=1}^{n} B^{ij}_{(r)}T(\Delta t)\alpha^{(r)(e)}_{ij}(\Delta t) - \sum_{r=1}^{n}\overset{*}{B}{}^{ij}_{(r)}T(\Delta t)\alpha^{(r)(p)}_{ij}(\Delta t) + \frac{1}{2}\sum_{r=1}^{n}\xi^{ijk\ell}_{(r)}\Big\{\alpha^{(r)(e)}_{ij}(\Delta t)\alpha^{(r)(e)}_{k\ell}(\Delta t) \\ & + \alpha^{(r)(p)}_{ij}(\Delta t)\alpha^{(r)(p)}_{k\ell}(\Delta t)\Big\} + \sum_{r=1}^{n}\xi^{ijk\ell}_{(r)}\Big\{\alpha^{(r)(e)}_{ij}(\Delta t)\gamma^{(e)}_{k\ell}(\Delta t) + \alpha^{(r)(p)}_{ij}(\Delta t)\gamma^{(p)}_{k\ell}(\Delta t)\Big\}\end{aligned} \tag{5}$$

where $E^{ijk\ell}$ and B^{ij} are tensors of elastic and thermoelastic moduli; $\overset{*}{E}{}^{ijk\ell}$ and $\overset{*}{B}{}^{ij}$ are tensors of plastic and thermoplastic moduli, their explicit forms to be derived later; c is the specific heat; T and T_0 are the temperature change and reference temperature; $\xi^{ijk\ell}_{(r)}$ is the array of material stiffness constants related to the internal variables. The specific forms of $\alpha^{(r)(e)}_{ij}$ and $\alpha^{(r)(p)}_{ij}$ are taken as

$$\alpha^{(r)(e)}_{ij}(\Delta t) = \int_0^t \exp\left(\frac{-(t-\tau)}{T_{(r)}}\right)\dot{\gamma}^{(e)}_{ij}(\Delta t,\tau)d\tau,\quad \alpha^{(r)(p)}_{ij}(\Delta t) = \int_0^t \exp\left(\frac{-(t-\tau)}{T_{(r)}}\right)\dot{\gamma}^{(p)}_{ij}(\Delta t,\tau)d\tau \tag{6}$$

where τ is the time variable and $T_{(r)}$ is the relaxation time. It should be noted here that a product of $\gamma^{(e)}_{ij}\gamma^{(p)}_{k\ell}$ is not included in (5) because of lack of its explicit form of kernel, but the role of this term will be adequately accommodated by introducing the concept of equivalent yield stress in the computational process. This argument applies to $\alpha^{(r)(e)}_{ij}\alpha^{(r)(p)}_{k\ell}$ also.

Rewriting (3) likewise for a small time interval Δt and decomposing the total strain into elastic and plastic components,

$$\rho\dot{\varphi}(\Delta t) = \sigma^{ij}(\Delta t)\Big\{\dot{\gamma}^{(e)}_{ij}(\Delta t) + \dot{\gamma}^{(p)}_{ij}(\Delta t)\Big\} - D(\Delta t) - \rho\eta(\Delta t)\dot{T}(\Delta t) \tag{7}$$

Dropping (Δt) for simplicity and in view of eq. (4), eq. (7) becomes

$$\rho\left\{\frac{\partial\varphi}{\partial\gamma^{(e)}_{ij}}\dot{\gamma}^{(e)}_{ij} + \frac{\partial\varphi}{\partial\gamma^{(p)}_{ij}}\dot{\gamma}^{(p)}_{ij} + \frac{\partial\varphi}{\partial T}\dot{T} + \frac{\partial\varphi}{\partial\alpha^{(r)(e)}_{ij}}\dot{\alpha}^{(r)(e)}_{ij} + \frac{\partial\varphi}{\partial\alpha^{(r)(p)}_{ij}}\dot{\alpha}^{(r)(p)}_{ij}\right\} - \sigma^{ij}\dot{\gamma}^{(e)}_{ij} - \sigma^{ij}\dot{\gamma}^{(p)}_{ij} + D + \rho\eta\dot{T} = 0 \tag{8}$$

Performing the partial differentiation in (8) and requiring that (8) must be valid for all arbitrary values of $\dot{\gamma}^{(e)}_{ij}$ and $\dot{T}$, we obtain

$$\sigma^{ij} = E^{ijk\ell}\gamma^{(e)}_{k\ell} - B^{ij}T + \sum_{r=1}^{n}\xi^{ijk\ell}_{(r)}\alpha^{(r)(e)}_{k\ell} \tag{9}$$

$$\rho\eta = B^{ij}\gamma^{(e)}_{ij} + \overset{*}{B}{}^{ij}\gamma^{(p)}_{ij} + \frac{cT}{T_0} + \sum_{r=1}^{n}B^{ij}_{(r)}\alpha^{(r)(e)}_{ij} + \sum_{r=1}^{n}\overset{*}{B}{}^{ij}_{(r)}\alpha^{(r)(p)}_{ij} \tag{10}$$

$$\begin{aligned}D = {} & -\overset{*}{E}{}^{ijk\ell}\dot{\gamma}^{(p)}_{ij}\gamma^{(p)}_{k\ell} + \overset{*}{B}{}^{ij}T\dot{\gamma}^{(p)}_{ij} + \sigma^{ij}\dot{\gamma}^{(p)}_{ij} - \sum_{r=1}^{n}\xi^{ijk\ell}_{(r)}\Big\{\dot{\alpha}^{(r)(e)}_{ij}\alpha^{(r)(e)}_{k\ell} + \dot{\alpha}^{(r)(p)}_{ij}\alpha^{(r)(p)}_{k\ell} + \dot{\alpha}^{(r)(e)}_{ij}\gamma^{(e)}_{k\ell} + \dot{\alpha}^{(r)(p)}_{ij}\gamma^{(p)}_{k\ell} \\ & + \dot{\gamma}^{(p)}_{ij}\alpha^{(r)(p)}_{k\ell}\Big\} + \sum_{r=1}^{n}B^{ij}_{(r)}T\dot{\alpha}^{(r)(e)}_{ij} + \sum_{r=1}^{n}\overset{*}{B}{}^{ij}_{(r)}T\dot{\alpha}^{(r)(p)}_{ij}\end{aligned} \tag{11}$$

It is also seen that the internal dissipation depends on the energy associated with plastic strains and elastic and plastic components of internal variables.

To obtain a numerically more manageable form of the internal variable $\alpha^{(r)}_{k\ell}$ of (6), we introduce a discretized form of $\dot{\gamma}_{k\ell}$ with a linear variation in a small time domain Δt. This operation leads to

$$\alpha^{(r)}_{k\ell}(s) = \overset{(r)}{A}\alpha^{(r)}_{k\ell}(s-1) + \overset{(r)}{B}\dot{\gamma}_{k\ell}(s-1) + \overset{(r)}{C}\dot{\gamma}_{k\ell}(s) \tag{12a}$$

$$\overset{(r)}{\dot\alpha}_{k\ell}(s) = \overset{(r)}{A}\,\overset{(r)}{\dot\alpha}_{k\ell}(s-1) + \overset{(r)}{B}\,\ddot\gamma_{k\ell}(s-1) + \overset{(r)}{C}\,\ddot\gamma_{k\ell}(s)$$

where s represents any discretized time step and

$$\overset{(r)}{A} = \exp\left(\frac{-\Delta t}{T_{(r)}}\right), \quad \overset{(r)}{B} = T_{(r)}\left(\overset{(r)}{\Phi} - \overset{(r)}{A}\right), \quad \overset{(r)}{C} = T_{(r)}\left(1 - \overset{(r)}{\Phi}\right), \quad \overset{(r)}{\Phi} = \frac{T_{(r)}}{\Delta t}\left(1 - \overset{(r)}{A}\right)$$

Substituting (12) into (11) gives

$$\begin{aligned} D(s) = & -\overset{*}{E}{}^{ijk\ell}\dot\gamma^{(p)}_{ij}(s)\gamma^{(p)}_{k\ell}(s) + \overset{*}{B}{}^{ij}T(s)\dot\gamma^{(p)}_{ij}(s) + \sigma^{ij}(s)\dot\gamma^{(p)}_{ij}(s) - \sum_{r=1}^{n}\xi^{ijk\ell}_{(r)}\Bigg\{\left(\overset{(r)}{A}\,\overset{(r)(e)}{\dot\alpha}_{ij}(s-1) + \overset{(r)}{B}\,\ddot\gamma^{(e)}_{ij}(s-1)\right. \\ & \left.+ \overset{(r)}{C}\,\ddot\gamma^{(e)}_{k\ell}(s)\right)\left(\overset{(r)}{A}\,\overset{(r)(e)}{\alpha}_{k\ell}(s-1) + \overset{(r)}{B}\,\dot\gamma^{(e)}_{k\ell}(s-1) + \overset{(r)}{C}\,\dot\gamma^{(e)}_{k\ell}(s) + \gamma^{(e)}_{k\ell}\right) + \left(\overset{(r)}{A}\,\overset{(r)(p)}{\dot\alpha}_{ij}(s-1) + \overset{(r)}{B}\,\ddot\gamma^{(p)}_{ij}(s-1)\right. \\ & \left.+ \overset{(r)}{C}\,\ddot\gamma^{(p)}_{ij}(s)\right)\left(\overset{(r)}{A}\,\overset{(r)(p)}{\alpha}_{k\ell}(s-1) + \overset{(r)}{B}\,\dot\gamma^{(p)}_{k\ell}(s-1) + \overset{(r)}{C}\,\dot\gamma^{(p)}_{k\ell}(s) + \gamma^{(p)}_{k\ell}\right) + \dot\gamma^{(p)}_{ij}\left(\overset{(r)}{A}\,\overset{(r)(p)}{\alpha}_{k\ell}(s-1) + \overset{(r)}{B}\,\dot\gamma^{(p)}_{k\ell}(s-1)\right. \\ & \left.+ \overset{(r)}{C}\,\dot\gamma^{(p)}_{k\ell}(s)\right)\Bigg\} + \sum_{r=1}^{n} B^{ij}_{(r)}T\left(\overset{(r)}{A}\,\overset{(r)(e)}{\dot\alpha}_{ij}(s-1) + \overset{(r)}{B}\,\ddot\gamma^{(e)}_{ij}(s-1) + \overset{(r)}{C}\,\ddot\gamma^{(e)}_{ij}(s)\right) + \sum_{r=1}^{n}\overset{*}{B}{}^{ij}_{(r)}T\left(\overset{(r)}{A}\,\overset{(r)(p)}{\dot\alpha}_{ij}(s-1) + \overset{(r)}{B}\,\ddot\gamma^{(p)}_{ij}(s-1) + \overset{(r)}{C}\,\ddot\gamma^{(p)}_{ij}(s)\right) \end{aligned}$$

We now consider for a moment a linear thermoelastic behavior alone, and then let the load increase incrementally and permit the material to undergo plastic strains in increments. In this process we avoid a need for considering a yield function combining viscous and plastic behavior. Rather it is the purpose of the present study to determine status of yielding in increments based on incremental viscoelastic stresses. This treatment makes it possible to use any plasticity theory completely independent of viscoelastic behavior. A superposition of these distinctively different properties then requires the incremental stresses to assume the form

$$d\sigma^{ij} = E^{ijk\ell}d\gamma_{k\ell} - B^{ij}dT + \sum_{r=1}^{n}\xi^{ijk\ell}_{(r)}\,d\overset{(r)}{\alpha}_{k\ell} + \overset{*}{E}{}^{ijk\ell}\,d\gamma_{k\ell} - \overset{*}{B}{}^{ij}dT \tag{13}$$

Here we may use von Mises yield criteria and after some algebra [19,23-26], we obtain

$$\overset{*}{E}{}^{ijk\ell} = -H^{-1}E^{ijmn}Z_{pq}Z_{mn}E^{k\ell pq}, \quad \overset{*}{B}{}^{ij} = -H^{-1}B^{mn}Z_{mn}Z_{k\ell}E^{ijk\ell}, \quad H = E_{(p)} + Z_{rs}Z_{tu}E^{rstu}$$

in which $E_{(p)}$ is the plastic modulus, Z_{pq} is defined as $Z_{ij} = (3/2\bar\sigma)(\partial J/\partial\sigma^{ij})$, with $\bar\sigma$ = equivalent yield stress and J = second deviatoric stress invariant. Details of these derivations are given in [19].

3. Heat Conduction Equations

It should be noted that if in elastic range $\overset{(r)}{\alpha}_{k\ell}$ is $\overset{(r)(e)}{\alpha}_{k\ell}$ contributing to viscoelastic stress which will then be used to calculate $\overset{*}{E}{}^{ijk\ell}$, $\overset{*}{B}{}^{ij}$, etc. If, however, the state of stress is in the plastic range, then $\overset{(r)}{\alpha}_{k\ell}$ is $\overset{(r)(p)}{\alpha}_{k\ell}$ contributing to dissipation during the plastic deformation. This reasoning is the basis for distinguishing the elastic and plastic components of the internal variables.

A local form of the conservation of energy in terms of the entropy, internal dissipation and heat variables as defined in (1e) and (10) will be used to derive governing heat conduction equations and subsequently the finite element model. Rewriting (1e) for a small time interval Δt in rectangular cartesian coordinate, we have

$$\rho\theta(\Delta t)\dot\eta(\Delta t) - q^{j}(\Delta t)_{,j} - \rho h(\Delta t) - D(\Delta t) = 0 \tag{14}$$

In view of (10) and dropping (Δt) for simplicity, (14) assumes the following form:

$$\begin{aligned}(T_0+T(s))\,\Big\{&(B^{ij} + \overset{*}{B}{}^{ij} + \overset{*}{\beta}{}^{ij})\,\dot\gamma_{ij}(s) + (\tilde B + \tilde\beta)\dot T(s) + \frac{c\dot T(s)}{T_0} + \sum_{r=1}^{n}B^{ij}_{(r)}\overset{(r)(e)}{\dot\alpha}_{ij}(s) \\ &+ \sum_{r=1}^{n}\overset{*}{B}{}^{ij}_{(r)}\overset{(r)(p)}{\dot\alpha}_{ij}(s)\Big\} - q^{i}_{,i}(s) - \rho h(s) - D(s) = 0\end{aligned} \tag{15}$$

where

$$\overset{*}{\beta}{}^{ij} = H^{-1}\overset{*}{B}{}^{ij}Z_{pq}Z_{k\ell}E^{pqk\ell}, \quad \tilde B = H^{-1}B^{ij}Z_{ij}Z_{pq}B^{pq}, \quad \tilde\beta = H^{-1}\overset{*}{B}{}^{ij}Z_{ij}Z_{pq}B^{pq}$$

It can easily be shown that $\overset{*}{B}{}^{ij}$, $\overset{*}{\beta}{}^{ij}$, $\tilde B$, and $\tilde\beta$ need be evaluated only for anisotropic solids, their values being zero for isotropic solids. Since elastic strains can always be determined from the elastic constitutive law and the total strains from the displacements by solving the equilibrium equations, it is always possible to calculate the plastic strains as a difference between the total and elastic strains.

In order to apply the finite element method to (15) it is first necessary to introduce interpolation functions associated with nodal values of temperature and displacements in the form, $T = \Omega_R T^R$, $u_i = \psi_{iN} u^N$. For the 8 node isoparametric element, we have $^R = 1,\ldots 8$, $^N = 1,\ldots 24$, $_i = 1,2,3$. Ω_R and ψ_{iN} are the normalized interpolation functions for temperature and displacements, respectively.

The strain-displacement relationship is given by

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i} u_{m,j}), \text{ or } \gamma_{ij} = A_{Nij} u^N + C_{NMij} u^N u^M$$

where A_{Nij} and C_{NMij} are the strain transformation operators

$$A_{Nij} = \frac{1}{2}(\psi_{iN,j} + \psi_{jN,i}) \quad , \quad C_{NMij} = \frac{1}{2}\psi_{KN,i}\psi_{KM,j}$$

Of all the possible avenues for solving the differential equation (15), Galerkin's method appears to be most convenient when combined with finite elements. Let the local residual of (15) be R and we require this residual to be orthogonal to the subspace spanned by the function Ω_R for each finite element. This yields

$$\langle \hat{R}, \Omega_R \rangle = \int_V \hat{R}\Omega_R \, dv = 0$$

or

$$\int_V \Big[\Big(T_0 + \Omega_U T^U(s)\Big)\Big\{(B^{ij} + \overset{*}{B}_{ij} + \overset{*}{\beta}^{ij})(A_{Mij} + 2C_{MPij}u^P)\dot{u}^M(s) + (\tilde{B}+\tilde{\beta})\Omega_s \dot{T}^s(s) + \frac{c}{T_0}\Omega_S \dot{T}^S(s) + \sum_{r=1}^{n} B^{ij}_{(r)} \overset{(r)(e)}{\alpha}_{ij}(s) + \sum_{r=1}^{n} \overset{*}{B}^{ij}_{(r)} \overset{(r)(p)}{\alpha}_{ij}(s)\Big\}\Omega_R - \Big\{q^j_{,j}(s) + \rho h\Big\}\Omega_R + D(s)\,\Omega_R\Big] dv = 0 \tag{31}$$

If we introduce the linear Fourier law, $q^i = \varkappa^{ij} T_{,j}$ where $\varkappa^{ij}$ is the thermal conductivity matrix, it is possible to write

$$\int_V q^j_{,j}\Omega_R \, dv = \int_A q^j n_j \Omega_R \, dA - \int_V q^j \Omega_{R,j} \, dv = \int_A q^j n_j \Omega_R \, dA - \int_V \varkappa^{ij}\Omega_{R,i}\Omega_{S,j} \, dv T^S$$

subject to the boundary conditions $q^j n_j = -q - \bar{\alpha}(T-T')$ where n_j is the unit normal to the surface, $\bar{\alpha}$ is the film coefficient, T and T′ are the element temperature and the ambient temperature respectively and q is the value of heat flux on an element surface. The final form of the local finite element heat conduction equation becomes

$$N_{RS}\dot{T}^S(s) + R_{RS}T^S(s) = P_R^{(Q)}(s) + P_R^{(q)}(s) + P_R^{(c)}(s) + P_R^{(TEP)}(s) + P_R^{(TVP)}(s) + P_R^{(D)}(s) \tag{17}$$

in which heat capacity matrix, $N_{RS} = \int_V c\,\Omega_R \Omega_S \, dv$,

conductivity matrix, $R_{RS} = \int_V \varkappa^{ij}\Omega_{R,i}\Omega_{S,j} \, dv + \int_A \bar{\alpha}\Omega_R \Omega_S \, dA$

volume heat supply vector, $P_R^{(Q)}(s) = \int_V \rho h(s)\,\Omega_R \, dv$

Surface heat flux vector, $P_R^{(q)}(s) = -\int_A \Omega_R (q(s) - \bar{\alpha}T'(s)) \, dA$

pseudo heat capacity vector, $P_R^{(c)}(s) = -\int_V \frac{c}{T_0}\Omega_R \, dv T(s)\,\dot{T}$

pseudo thermoelastoplastic coupling vector,

$$P_R^{(TEP)}(s) = \int_V \Big(T_0 + T(s)\Big)\Omega_R (B^{ij} + \overset{*}{B}{}^{ij} + \overset{*}{\beta}{}^{ij})(A_{Mij} + 2C_{MPij}u^P)\, dv\,\dot{u}^M(s) + \int_V (T_0 + T(s)\Omega_R(\tilde{B} + \tilde{\beta})\Omega_S \, dv\,\dot{T}^S(s)$$

pseudo thermoviscoplastic coupling vector,

$$P_R^{(TVP)}(s) = \int_V \Big(\sum_{r=1}^{n} B^{ij}_{(r)} \overset{(r)(e)}{\alpha}_{ij}(s) + \sum_{r=1}^{n} \overset{*}{B}{}^{ij}_{(r)} \overset{(r)(p)}{\alpha}_{ij}(s)\Big)\Omega_R \, dv$$

pseudo thermoviscoplastic dissipation vector, $P_R^{(D)}(s) = \int_V D(s)\Omega_R \, dv$

To the knowledge of the authors, the above heat conduction equation for thermoviscoelastoplasticity with a complete list of explicit terms ready for computation is presented here for the first time.

4. Equilibrium Equations

It is understood that the heat conduction equation derived earlier results from the second law of thermodynamics, whereas equilibrium equations can be obtained from the first law of thermodynamics or the minimum potential energy principle. The finite element equilibrium equation may be written in the form

$$\int_V \sigma^{ij} \frac{\partial \gamma_{ij}}{\partial u^N} dv = F_N^{(a)} \tag{18}$$

where $F_N^{(a)}$ is the nodal load vector. Now taking a variation of (18) to obtain incremental form, we have

$$\int_V d\sigma^{ij} \frac{\partial \gamma_{ij}}{\partial u^N} dv + \int_V \sigma^{ij} d \left\{\frac{\partial \gamma_{ij}}{\partial u^N}\right\} dv = dF_N^{(a)} \tag{19}$$

It is seen that the first term and second term in the left-hand side of (19) represent incremental changes in stresses for material nonlinearity and incremental changes in strains for geometric nonlinearity, respectively. In view of (13), we rewrite (19) in the form

$$\int_V \Big[(E^{ijk\ell} + \overset{*}{E}{}^{ijk\ell})\big(A_{Nk\ell} d\overset{N}{u}(s) + 2C_{NMk\ell}\overset{N}{u}(s) d\overset{M}{u}(s)\big) - (B^{ij} + \overset{*}{B}{}^{ij})\Omega_R d\overset{R}{T}(s) + \sum_{r=1}^{n} \xi^{ijk\ell}_{(r)} \Big\{ \overset{(r)}{A} d\overset{(r)}{\alpha}_{k\ell}(s-1)$$
$$+ \overset{(r)}{B} A_{Nk\ell} d\dot{u}^N(s-1) + 2\overset{(r)}{B} C_{NMk\ell}\big(u^M(s-1)d\dot{u}^N(s-1) + \dot{\overset{M}{u}}(s-1)d\overset{N}{u}(s-1)\big) + \overset{(r)}{C} A_{Nk\ell} d\dot{u}^N(s) + 2\overset{(r)}{C} C_{NMk\ell}\big(\overset{M}{u}(s)d\dot{\overset{N}{u}}(s)$$
$$+ \dot{\overset{M}{u}}(s) d\overset{N}{u}(s)\big)\Big\} \big(A_{Mij} + 2C_{Mpij}u^P\big)\Big] dv + \int_V \big[2\sigma^{ij}(s)\big] C_{NMij}\, du^M(s) dv = dF_N^{(a)}(s) \tag{25}$$

Introducing interpolation functions into the strains and subsequently to the incremental stress (13) the equilibrium equation (19) may be written in a compact form,

$$J_{MN} d\dot{\overset{M}{u}}(s) + (\overset{(e)}{K}_{NM} + \overset{(G)}{K}_{NM} + \overset{(p)}{K}_{MN}) d\overset{M}{u}(s) = d\overset{(a)}{F}_N(s) + d\overset{(TEP)}{F}_N(s) + d\overset{(V)}{F}_N(s) + d\overset{(N)}{F}_N(s) \tag{20}$$

where J_{MN}, $\overset{(e)}{K}_{MN}$, $\overset{(G)}{K}_{MN}$, and $\overset{(p)}{K}_{MN}$ are the viscosity matrix, linear stiffness matrix, geometric stiffness matrix and plastic stiffness matrix, defined as

$$J_{MN} = \int_V \sum_{r=1}^{n} \xi^{ijk\ell}_{(r)} \overset{(r)}{C} A_{Nk\ell} A_{Mij} dv, \quad \overset{(e)}{K}_{MN} = \int_V E^{ijk\ell} A_{Mij} A_{Nk\ell} dv,$$

$$\overset{(G)}{K}_{MN} = \int_V 2\sigma^{ij} C_{NMij} dv, \quad \overset{(p)}{K}_{MN} = \int_V \overset{*}{E}{}^{ijk\ell} A_{Mij} A_{Nk\ell} dv$$

and $d\overset{(TEP)}{F}_N$ and $d\overset{(V)}{F}_N$ are the pseudo thermoelastoplastic load vector and pseudo viscous load vector given by

$$d\overset{(TEP)}{F}_N(s) = \int_V \big(B^{ij} + \overset{*}{B}{}^{ij}\big)\, \Omega_R \big(A_{Nij} + 2C_{NMij}\overset{M}{u}(s) dv\big) d\overset{R}{T}(s)$$

$$d\overset{(V)}{F}_N(s) = \int_V \sum_{r=1}^{n} \xi^{ijk\ell}_{(r)} \{\overset{(r)}{A} d\overset{(r)}{\alpha}_{k\ell}(s-1) + \overset{(r)}{B} A_{Mk\ell}(d\dot{\overset{M}{u}}(s-1)\} A_{Nij} dv$$

All terms not represented so far may be grouped in $d\overset{(N)}{F}_N(s)$ which may be called the pseudo higher order nonlinear load vector. This vector, however, may be dropped because of its negligible contribution.

5. Solution Procedure

For convenience let us rewrite here the equations of heat conduction and equilibrium in a compact form

$$N_{RS}\dot{T}^S(s) + R_{RS}\overset{S}{T}(s) = P_R(s) \tag{21}$$

$$J_{MN} d\dot{u}^M(s) + K_{MN} du^M(s) = dF_N(s) \tag{22}$$

where P^R and F_N are the sum of all vectors appearing in (17) and (20), respectively; and K_{MN} is the sum of all stiffness matrices.

These vectors contain not only the input but also the histories of past stresses, strains, and temperature. It should be noted that values of strains, displacements and temperature with (s) implying the current time increment can be lagged one time step in order to facilitate a direct numerical integration of the equations (21) and (22). The direct numerical integration may be based on linear variations of the displacement rate du^M and temperature rate T^s within a time increment. Recall that we used a linear variation of strain rate γ_{ij} earlier to discretize the internal variable $\overset{(r)}{\alpha}_{ij}$ for explicit integration and this corresponds to our present assumption of a linear variation of the displacement rate. This leads to a recurrence formula for displacements,

$$\overset{M}{du}(s) = (J_{MN} + \frac{\Delta t}{2} K_{MN})^{-1} \{F_N(s) - K_{MN}(\frac{\Delta t}{2} \overset{M}{du}(s-1) + \overset{M}{du}(s-1)\} + du(s-1)\frac{\Delta t}{2} + du(s-1) \qquad (23)$$

Similarly,

$$\overset{M}{T}(s) = (N_{RS} + \frac{\Delta t}{2} R_{RS})^{-1} \{P_R(s) - R_{RS}(\frac{\Delta t}{2} T^s(s-1) + T^s(s-1)\} + T(s-1)\frac{\Delta t}{2} + T(s-1) \qquad (24)$$

In the earlier paper [19] however, the authors used a linear variation of temperature rather than the temperature rate within a time increment originally proposed by Wilson and Nickell [22]. This is in the form,

$$\overset{M}{T}(s) = 2(R_{RS} + \frac{2}{\Delta t} N_{RS})^{-1} \{P_R(s-\tfrac{1}{2}) + \frac{2}{\Delta t} N_{RS} T^s(s-1)\} - T^s(s-1) \qquad (25)$$

which is simpler and perhaps more stable than (24). Another attractive numerical integration scheme is the fourth order Runge-Kutta method, tried in the present analysis and proved to give almost identical results as those discussed in (23) and (24).

Mechanical or thermal loading or both can be treated simultaneously. First either nodal displacements or nodal temperatures are calculated depending on initial and boundary conditions and input data available. Based on either one of these quantities the other may then be calculated. These histories then will be carried over to the next time increment with the process being repeated as far as desired. The yield conditions are checked with the equilibrium equation in connection with stresses, element by element [23,24,25]. If any element has yielded the plastic matrices $\overset{*}{E}^{ijk\ell}$, $\overset{*}{B}^{ijk\ell}$, etc., must be constructed for that element and the standard iterative cycles are repeated until convergence is achieved within the time interval [19].

In the present study, we use the three-dimensional linear isoparametric function for both temperature and displacement, and the integration for spatial domain is carried out by an 8 point Gaussian quadrature [26].

6. Applications

Although external mechanical loadings can easily be applied, an example given here is intended for determining temperature distributions, deformations and stresses only due to a heat supply specified to a three dimensional body as shown in Figure 1. The computer program is capable of also handling any type of geometrical and thermal boundary conditions including the surface exposed to time dependent ambient temperature.

The three dimensional solid divided into 144 finite elements (equal size) is fixed at both ends of the x-coordinate where no temperature is permitted to change. All surfaces are insulated with a reference temperature of $T_0 = 27^\circ$ C. Other data are given in Figure 1. An integration time increment of .01 hr. was used, which appears to provide a good solution stability. The transient temperature distributions are shown in Figure 2 The curves indicate the temperature rise from the reference temperature $T_0 = 27^\circ$ C. It is noted that effects of plasticity on temperature distributions are negligible.

Figure 3 shows the transient displacements, w in the z-direction along the x-axis (symmetrical results beyond 600 mm). It appears that the thermoviscoelastoplastic deformation is larger than the thermoviscoelastic deformation in general. However, this is not the case at the center (x = 600mm) where the heat source is provided during the early stage of deformation. This is an indication that time must elapse before plastic deformation can occur.

To study effects of viscosity on displacements the results for elastic and elastoplastic analyses are compared in Figure 4. At t = .4 hours it is clear that the elastoplastic displacements are largest, followed by the elastic, viscoelastoplastic, and viscoelastic displacements as expected.

The observation noted in Figure 3 is further emphasized in Figure 5 and Figure 6. Here, for a given magnitude of stresses to develop, it takes larger strain or more time for thermoviscoelastoplastic behavior than for thermoviscoelastic behavior. In Fig. 5, the stresses in the x-direction, σ_x for the top (first) layer are in tension and increase with time. It is interesting to note that thermoviscoelastoplastic stresses are much smaller except at the center for .1 hour. For the third layer (see Figure 6) where the stresses are locked the center portion develops larger stresses as expected.

To give an idea of the stress-strain relationship which may be indicated by the stress-time relationship, Figure 7 shows the equivalent stress-time curves for the element A as shown in Figure 1. The plastic failure occurs after approximately 3 minutes at approximately the prescribed yield stress of 2.5 kg/mm^2 and the thermoviscoelastoplastic stresses gradually increase and level off whereas the thermoelastic stresses keep increasing but are expected to decay when temperature equilibrium is reached. Once again effects of viscosity are demonstrated by a comparison with elastic and elastoplastic cases, and it is clear that slightly higher stresses are associated with viscosity.

Lastly, yielded regions based on viscoelastoplasticity are shown in Figure 8. Yielding starts about halfway between the edges and the center and gradually spreads to both directions. A complete yielding occurs after approximately 0.28 hours.

It is important to note that the present procedure takes into account the rate-dependent plasticity by means of direct superposition of thermoviscoelastic behavior and thermoelastoplastic behavior within a small time increment, and, that this coupling and histories are carried over to the next time increments.

Unfortunately, the authors are unable to locate reliable experimental data or analytical results of other investigations if available to compare with our results. A future verification of the present results is needed.

7. Concluding Remarks

An incremental theory of thermoviscoelastoplasticity, the finite element formulation, and an example problem have been presented. An assumption of smoothness of free energy functional containing inelastic behavior with the current values of thermodynamic process taken as a "continuous functional" of past histories is considered invalid. Rather, such smoothness is assumed to be acceptable only for a small time interval.

In conclusion, this incremental theory requires a minimum of experimental data and is quite flexible in incorporating any type of yield criterion, a classical theory of plasticity or dislocation theory. The results of an example based on the classical theory of plasticity presented here appear to be reasonable although a verification by experimental data in the future is needed.

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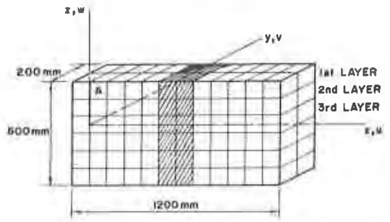

DISPLACEMENT BOUNDARY CONDITIONS:

u(t) = v(t) = w(t) = 0	at x = 0 and x = 1200 mm
u(t) = 0	at x = 600 mm
v(t) = 0	at y = 0
w(t) = 0	at z = 0

TEMPERATURE BOUNDARY CONDITION AND HEAT INPUT:

Insulated on all the surfaces and T(t = 0) = 0 everywhere. T(t) = 0 at x = 0 and 1200 mm. ρh = 300(kg/hr mm) in shaded area.

CONSTANTS:

$E = 7.1 \times 10^{3}$ (kg/mm²), $E(p) = 1.0 \times 10^{3}$ (kg/mm²), $\sigma_Y = 2.5$(kg/mm²), $\nu = 0.3$, $\alpha = 6.5 \times 10^{-6}$ (/°C), $\kappa = 2.2 \times 10^{4}$ (kg/hr °C), $c = 0.07$(kg/mm² °C), $T_0 = 27$(°C), $T_{(r)} = 7.04 \times 10^{-3}$ (hrs.), (r = 1,2,3), $\Delta t = 0.01$ (hr.).

FIGURE 1 GEOMETRY AND INPUT DATA

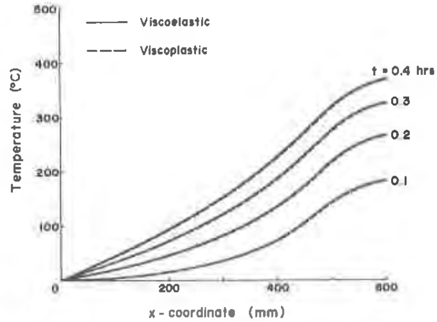

FIGURE 2 TEMPERATURE DISTRIBUTION ALONG y = z = 0

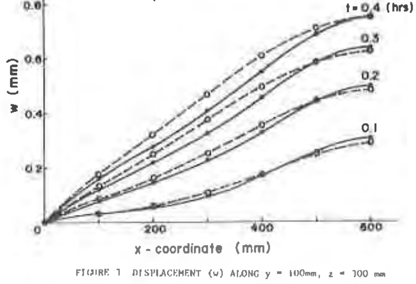

FIGURE 3 DISPLACEMENT (w) ALONG y = 100mm, z = 100 mm

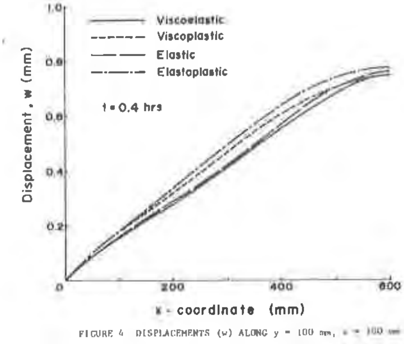

FIGURE 4 DISPLACEMENTS (w) ALONG y = 100 mm, z = 100 mm

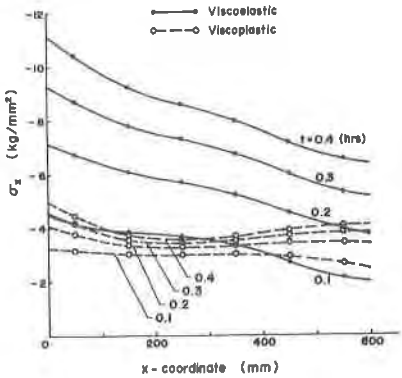

FIGURE 5 STRESS (σ_x) DISTRIBUTION ALONG THE 1st LAYER

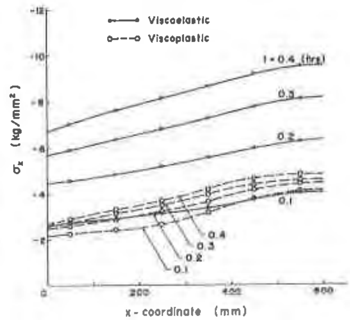

FIGURE 6 STRESS (σ_x) ALONG THE 3rd LAYER

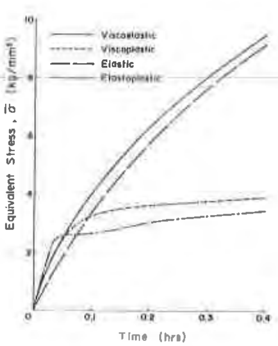

FIGURE 7 EQUIVALENT STRESS ($\bar{\sigma}$) OF ELEMENT A

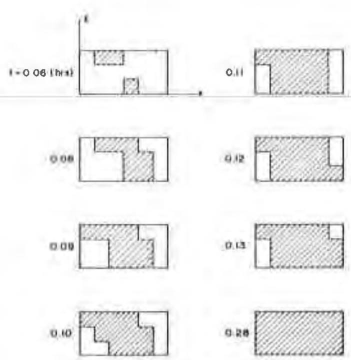

FIGURE 8 YIELDED REGIONS