COMPUTATIONAL PROCEDURES FOR INELASTIC STRESS ANALYSIS

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SUMMARY

The principles and procedures used in inelastic finite element analysis are reviewed. The procedures considered include step by step analysis, constant stiffness iteration and Newton-Raphson iteration. It is emphasized that all procedures involve the sequential solution of only two subproblems, namely linearization (computation of the structure tangent stiffness) and state determination (computation of the new state of stress and strain for given displacement increments). The linearization problem is well known and is reviewed only briefly. However, the state determination problem is at least as important yet has been examined much less thoroughly in the literature. The state determination computations are described, and the importance of path dependence is emphasized.

A general solution strategy is proposed, in which the analyst can select essentially any combination of step by step and iterative solution procedures by specifying the values of three convergence tolerances. Added sophistication, with relatively little increase in complexity, can be obtained by allowing the analyst to specify when unloading is expected to occur and when new linearization is to be suppressed. Over-relaxation procedures to speed the convergence of constant stiffness iteration are briefly reviewed.

Within the general solution strategy, thirteen different procedures can be obtained by varying the convergence tolerance values. A simple example (tension strip with central hole) is used to compare the computational effort required for each of the procedures. The computed peak strain values and the numbers of linearization and state determination computations required for each procedure are tabulated, and the computational effort is shown to vary widely. The results indicate that iteration procedures of Newton-Raphson type are likely to be more efficient than those of constant stiffness type without over-relaxation.

1. Introduction

Solution procedures for inelastic structures will generally be of step-by-step or iterative type, with the aim being to obtain a solution satisfying (1) kinematic compatibility, (2) equilibrium, and (3) the material stress-strain relationship, including any effects of path dependence. Typically, finite element formulations will automatically be compatible kinematically, so that the problem is to find a solution in which the element stresses are (1) in equilibrium with the nodal loads and (2) consistent with the stress-strain relationship and strain history.

In this paper, the essential features of the solution process are reviewed, and it is emphasized that only two basic computational steps are involved. A solution strategy is then described with which it is possible to construct a computer program which is logically simple yet which gives the analyst considerable freedom to select the most appropriate solution procedure. An example is presented to provide information on the relative efficiency of alternative procedures. Large displacement effects are ignored herein, although the procedures are applicable, in principle, to problems in which the displacements are large.

2. Computational Steps

Concepts and procedures for the analysis of nonlinear structures are conveniently developed with the aid of one-dimensional load-displacement relationships of the type shown in Figs. 1, 2 and 3. This type of diagram will be used herein, but it must never be forgotten that the actual relationship is a complex one in multi-dimensional space. Conclusions derived from the one-dimensional representation must therefore be applied with caution.

The simplest type of step-by-step procedure is illustrated in Fig. 1. The algorithm is:

$$\Delta r_n = K_n^{-1} \Delta R_n \tag{1}$$

$$r_n = r_{n-1} + \Delta r_n; \; r_o = 0 \tag{2}$$

in which ΔR_n and Δr_n are the load and displacement increments, respectively, for the nth increment, K_n is the tangent stiffness at the beginning of the nth step, and in which K_n^{-1} implies, for a multi-dimensional problem, solution of the equilibrium equations. This procedure has been widely used in elasto-plastic analysis [1, 2, 3, 4]. It is essentially an Euler-Cauchy method, and errors are likely to be accumulated after several steps so that the solution diverges from the true load-displacement relationship.

The problem of determining the tangent stiffness for any given state may be termed the linearization problem. Symbolically, K_n is given as

$$K_n = \Sigma \int B^T D_n B dV \tag{3}$$

in which the integration is over each element volume; the summation implies assembly of the element stiffnesses to give the structure stiffness; B = strain-displacement transformation matrix; and D_n defines the tangent stress-strain relationship

$$d\sigma = D_n \, d\varepsilon \tag{4}$$

for the state at the beginning of the nth step. For an elastic material D_n simply defines the elastic stress-strain relationship. For a material which is yielding, D_n is given by the well known expression [4, 5, 6]

$$D_n = D_e - \frac{D_e(\partial f/\partial\sigma)(\partial f/\partial\sigma)^T D_e}{(\partial f/\partial\sigma)^T D_e(\partial f/\partial\sigma) - (\partial f/\partial\varepsilon_p)^T(\partial f/\partial\sigma)} \tag{5}$$

in which D_e = elastic stress-strain matrix; $(\partial f/\partial\sigma)$ = vector of derivatives of the yield function with respect to stress; and $(\partial f/\partial\varepsilon_p)$ = vector of derivatives with respect to plastic strain, which accounts for strain hardening.

Because D_n varies with the state of stress and strain, it will generally vary over the element volume. It may be assumed to be constant (using, say, the state at the element center), or may be interpolated using the values at the element nodes, or, for isoparometric elements with Gauss integration, may be evaluated at each integration point and incorporated directly into the numerical integration. This last procedure has the advantage that the precision with which the spread of yielding through an element is monitored can be changed simply by varying the integration order. It is not difficult to develop a computer program in which the integration order is specified by the analyst at execution time.

After eq. (1) has been solved for Δr, it is necessary to compute the corresponding changes in stress and strain, and hence obtain the new state. This will be termed the state determination problem. This problem is at least as important as linearization, but has received less attention in the literature. The solution is obtained in two stages, as follows, with an optional third stage

$$\Delta r \xrightarrow[B]{\text{Stage 1}} \Delta\varepsilon \xrightarrow[D]{\text{Stage 2}} \Delta\sigma \xrightarrow[B^T]{\text{Stage 3}} R^I \tag{6}$$

Stage 1 involves only kinematics, specifically the strain-displacement transformation. Stage 2 involves only the stress-strain relationship, and is considered further in the following section. When these two stages are complete, a new state of stress and strain has been computed, and the linearization for the next step can be carried out. As indicated in Fig. 1, however, there will generally be an error at the end of any linear step, resulting from the fact that the true behavior within a load increment is not linear. The third stage of the state determination procedure serves to calculate magnitude of the error, and involves little additional computation. For each state of stress there is a unique value of the external load required to satisfy the equations of equilibrium. This load will be termed the internal resisting load, R^I, to distinguish it from the external applied load, R. The magnitude of R^I is calculated by considering the equilibrium relationship between stresses and loads, with the equation

$$R^I = \Sigma \int B^T \sigma dV \tag{7}$$

in which the integration is over each element and the summation over all elements. The equilibrium error is then given by

$$R^U = R - R^I \tag{8}$$

in which R^U = unbalanced load. As is well known, the accuracy of the simple step by step procedure may be improved by calculating the unbalanced load at the end of each linear step and adding this load to the following load increment, as illustrated in Fig. 1, so that Eq. (1) becomes

$$\Delta r_n = K_n^{-1}(\Delta R_n + R_n^U) \tag{9}$$

Calculation of the unbalanced load also permits solutions to be obtained by iteration. The iteration may be of constant stiffness (initial stress) type [7] as in Fig. 2 or of Newton-Raphson type [8] as in Fig. 3. The algorithm for constant stiffness iteration is based on

$$\Delta r_n = K_1^{-1}(R - R_n^I) = K_1^{-1} R_n^U \tag{10}$$

in which K_1 = stiffness of elastic structure and R_n^I = internal resisting load at beginning of step n. For Newton-Raphson iteration the algorithm is based on

$$\Delta r_n = K_n^{-1} R_n^U \tag{11}$$

in which K_n is the tangent stiffness at the beginning of step n. Combinations of constant stiffness and Newton-Raphson iteration may be used, and over-relaxation schemes may be used to speed convergence. Both of these are considered in a later section.

An iterative procedure can be assumed to have converged when the unbalanced load becomes acceptably small. The authors prefer to place a limit on the largest term in the vector R^U, although the Euclidean norm or some other property of the vector might be limited. Appropriate convergence limits can be selected by determining a permissible error in the computed stress and multiplying it by the area tributary to a typical nodal point. Tests for convergence can also be based on the increments of node displacement or element strain. However, such tests are inherently more risky, as convergence in some cases may be slow and the iteration may be terminated too early.

3. State Determination

Stages 1 and 3 of the state determination calculation are based on kinematics and statics, respectively, and are generally straightforward. However, care must be taken in completing stage 2, which involves the stress-strain relationship. Fig. 4 shows a von Mises yield surface in the stress deviator plane, and illustrates a typical state determination calculation for an elastic perfectly plastic material. At the beginning of the load step the state of stress is at A, and the material is elastic. For the computed strain increment let the new state of stress be at B, which is beyond the yield surface and hence is incorrect. Let the actual stress be at C, which is reached by following some indeterminate strain path. To find an approximation to point C, it is essential to make assumptions about the strain path. Two possible alternative assumptions are: (1) that all components of strain increase in proportion during the step, in which case the stresses would follow the path A-D-C; and (2) that the stress follows the path A-C until the yield surface is reached and then remains at C. For any given strain increment, these two alternative procedures will generally lead to different locations of point C, although from a practical point of view the differences are likely to be small. Of course, neither location is necessarily correct. Computational schemes for either alternative can be formulated, but the first alternative is conceptually simpler and more reasonable physically. Only this procedure is considered herein.

Point D is easily found from geometrical considerations, and that proportion of $\Delta\varepsilon$ used to reach the yield surface follows directly. The remaining stress change, $\Delta\sigma'$, is given by

$$\Delta\sigma' = \int D_T d\varepsilon \tag{12}$$

in which the components of $d\epsilon$ are assumed to vary proportionately, but in which D_T, the tangent stress-strain relationship, depends on the location of the stress point on the yield surface, and hence on $\Delta\sigma$. The integration must therefore be carried out numerically. One procedure is to subdivide $\Delta\epsilon$ into a number of equal parts and use Euler integration. Alternatively, the authors have found that fourth-order Runge-Kutte integration works well. No matter which method is used, the computational effort is small, and in all cases some modification of the final $\Delta\sigma$ value will usually be necessary to place the new stress point exactly on the yield surface.

If the stress point at the beginning of any step is on the yield surface, a test for unloading must first be made. This is easily done by computing the stress increase assuming that the material is elastic, and computing the corresponding change in the yield function. If the stress point moves inside the yield surface the material is unloading, otherwise it is continuing to yield. The subdivision of the strain increment into elastic and elasto-plastic parts proceeds as before.

The calculation of $\Delta\sigma$ from $\Delta\epsilon$ may be of a "path dependent" or "path independent" type. In the path independent type, the strain increment computed at each iteration is added to the strain increment from all previous iterations for the current load case, and the stress change is computed from this accumulated strain increment. In the path dependent procedure, a stress change corresponding to the strain increment in each iteration is computed, and the stress increments are combined to give the increment for the load case. In a study using Newton-Raphson iteration, the authors [8] found that much faster convergence was obtained with the path dependent procedure. However, the study was limited in scope, and further investigations are needed.

4. Over-Relaxation

Constant stiffness iteration has the advantage that the formulation and triangulation of the structure stiffness matrix (i.e. linearization) needs to be carried out only once. The method is therefore attractive, especially for large problems. However, iteration with constant stiffness converges more slowly than Newton-Raphson iteration, and hence requires more state determination calculations. Unless steps can be taken to improve the convergence rate with constant stiffness iteration, it is doubtful whether this procedure can be as efficient as Newton-Raphson iteration (see example in this paper).

An over-relaxation scheme has been proposed by Nayak and Zienkiewicz [9]. In effect, this scheme modifies the computed displacement increment, Δr, in each step so that it approximates the increment which would be computed by Newton-Raphson iteration. The "constant stress" [10,11] iteration procedure can also be interpreted as an over-relaxation process.

A further procedure is illustrated in Fig. 5. After each displacement increment is computed, it is scaled so that the unbalanced load, R^U, is minimized. In the one-dimensional diagram of Fig. 5, R^U is reduced to zero at each step, and convergence is rapid. In a multi-dimensional problem it is not possible to reduce all terms in R^U to zero, but only to minimize the Euclidean norm of R^U, with the result that the rate of convergence is not improved as dramatically as indicated in Fig. 5. Nevertheless, studies of small problems have shown substantial improvements in the rate of convergence. It can be shown that if R^U_n is the original unbalanced load vector for step n, and if ΔR^I_{n1} is the increase in internal

resisting load computed at the end of the normal iteration step, then the scale factor, k_n, to be applied to Δr_n is

$$k_n = \frac{(R_n^U)^T(\Delta R_{nl}^I)}{(\Delta R_{nl}^I)^T(\Delta R_{nl})} \qquad (13)$$

It is important to note that this procedure requires two state determination calculations in each step.

A further important point with over-relaxation procedures is that the displacements and strains may not converge monotonically towards the final values. Hence, if a path dependent state determination solution is used, there is a risk of obtaining grossly incorrect results. The use of a path independent procedure is therefore necessary.

5. A Solution Strategy

Within a single computer program it is possible to permit the user to select any one of a wide variety of solution procedures by specifying the magnitudes of three convergence tolerances, as follows.

(1) Final convergence tolerance, t_f. If the largest unbalanced load is less than this value, the solution for the current loading has converged with sufficient accuracy, and the next loading is considered.

(2) Constant stiffness tolerance, t_c. If the largest unbalanced load is less than this tolerance, the stiffness matrix is not reformed for the next iteration step, but the solution iterates with the same stiffness as in the current step. If t_c is set equal to or less than t_f, a new stiffness will be formed at each step, corresponding to Newton-Raphson iteration. On the other hand, if t_c is set very large the iteration will be of constant stiffness type throughout. If a value of intermediate magnitude is chosen, the iteration will be of a mixed type, with Newton-Raphson iteration until the maximum unbalance is less than t_c, and then constant stiffness iteration to final convergence. That is, the stiffness will be reformed during the early steps of the iteration, when the stiffness changes are large, but the stiffness will remain constant during later steps, when the stiffness changes are small.

(3) Step by step convergence tolerance, t_s. This tolerance is used when a loading is applied in a number of increments. At the end of any increment, if (a) the largest unbalanced load is less than t_s, and (b) this is not the last increment, then the next increment is added. If t_s is equal to or less than t_f, the solution will iterate to final convergence within each increment. On the other hand, if t_s is very large the solution will proceed immediately to the next increment (step by step solution with equilibrium correction) and will iterate to final convergence only within the last increment. If a value of intermediate magnitude is chosen, the procedure will iterate within any step until the maximum unbalance is less than t_s and will then apply the next load increment. The iteration may be of Newton-Raphson or constant stiffness type, depending on the value of t_c.

Further sophistication, with little additional complexity, can be introduced by means of two stiffness reformulation indicators which are defined for each new load case or load increment as follows.

(1) Stiffness Re-Use Indicator. At the beginning of a new load increment, the solution algorithm would normally be arranged to recalculate the tangent structure stiffness.

However, through a re-use indicator the algorithm can be made to use the stiffness at the end of the preceding step. Thus, if Newton-Raphson iteration has been used for the previous loading, it is probable that the change in stiffness will be small and the recalculation can therefore be suppressed. Alternatively, if constant stiffness iteration using the initial undeformed stiffness is required over several loadings, the re-use indicator can be used to ensure retention of this original stiffness.

(2) Unloading Indicator. If cyclic loading is being applied, it is known that removal or reversal of the load will cause all elements to unload and become elastic. This knowledge can be conveyed to the solution algorithm by means of an unloading code, which ensures that a new tangent stiffness assuming elastic material is used for the first step of the new loading. The use of an accurate initial tangent stiffness is therefore ensured.

Over-relaxation schemes of the type which modify the computed displacement increment can also be incorporated readily into the computer logic. In all cases the solution process always consists of some sequence of linearization and state determination calculations. Selection of the computational logic to ensure that all possibilities are accounted for is nevertheless an interesting exercise.

6. Comparative Example

If only the three tolerances, t_f, t_c and t_s are used, and all other refinements are ignored, a total of thirteen different solution procedures can be identified, depending on whether the tolerances have "small", "moderate or "large" magnitudes. These thirteen procedures are summarized in Table 1. To obtain an indication of the relative efficiency of different solution procedures, a plate with a central circular hole, as shown in Fig. 5a, has been analyzed using each of these thirteen procedures. This plate has been previously studied both experimentally and theoretically [7]. The finite element mesh for one quarter of the plate is shown in Fig. 5b. Plane stress isoparametric elements were used, with 2 x 2 Gauss integration and four added incompatible modes to improve element flexibility. The purpose of the example is to compare methods rather than to obtain accurate results, and hence the mesh is coarse. It may be noted, however, that results in close agreement with those for a finer mesh were obtained [8], and that the number of iterations required for convergence did not increase as the mesh was refined.

Linear kinematic strain hardening, with a hardening modulus equal to 3.2 percent of the initial elastic modulus was assumed. Analyses were carried out up to an average stress, σ_{av}, on the gross section of 0.5 times the uniaxial yield stress. For methods 1, 2 and 3 the load was applied in a single step. For the remaining methods, a load of one half of the maximum was applied, followed by five equal increments up to the maximum. The computed longitudinal strains at the most highly strained integration point (point X in Fig. 5b) are compared in Table 2. The total numbers of iterations and the numbers of tangent stiffness reformulations are shown in Table 3.

The convergence tolerances were as follows: (a) small: corresponding to a stress error of approximately 0.05 percent; (b) moderate: corresponding to a stress error of approximately 25 percent; (c) large: essentially infinite. These are not necessarily the most appropriate tolerances for practical use. In particular, the small tolerance is very stringent.

The most accurate solution procedure obviously is Method 8, and the results for this method can be used as a control. It can be seen from Table 2 that Method 2 failed to converge adequately even after 35 iteration cycles. It should be emphasized, however, that no over-relaxation was used in this study. Over-relaxation schemes can be expected to produce major improvements in the convergence of constant stiffness iteration. Method 4 was significantly less accurate than the other methods, but nevertheless gave acceptable results. All other methods gave results in close agreement with each other for the final loading. Methods 4, 5, 6 and 7 are identical except for the last load increment, and predict a strain path which is significantly in error. Methods 11, 12 and 13 are substantially in error at $\sigma_{av} = 0.35\sigma_y$, but improve in subsequent steps. Method 9 is significantly less accurate than Methods 8 and 10.

Table 3 gives an indication of relative computational effort. It is not possible to recommend a "best" procedure, because the total solution time in any problem will depend on the relative times taken to solve the linearization and state determination problems. Nevertheless, it is believed that an experienced analyst, with the help of the type of information in Tables 2 and 3, should be able to choose an appropriate procedure for any type of problem.

7. Conclusion

The techniques applied in inelastic analysis have been reviewed, and it has been shown that all analysis procedures reduce to a sequence of linearization and state determination calculations. A general solution strategy has been proposed. Comparisons of several procedures included in this strategy show that the solution effort can vary widely. Because of the large number of variables, it is not possible to identify a single best procedure. Nevertheless, it is believed that an experienced analyst should have little difficulty in selecting an efficient procedure for any given problem.

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TABLE 1

SOLUTION PROCEDURES

Method Number	Convergence Tolerances			Remark
	t_1	t_2	t_3	
1	arbitrary	small	small	Newton-Raphson Iteration
2	arbitrary	large	small	Constant Stiffness Iteration
3	arbitrary	moderate	small	Mixed Iteration of Methods 1 and 2
4	large	large	large	Step by Step with equilibrium correction
5	large	small	small	Step by Step, Newton-Raphson Iteration at last increment
6	large	large	small	Step by Step, Constant Stiffness Iteration at last increment
7	large	moderate	small	Step by Step, Mixed Iteration at last increment
8	small	small	small	Step by Step, Newton-Raphson Iteration within each increment
9	small	large	small	Step by Step, Constant Stiffness Iteration within each increment
10	small	moderate	small	Step by Step, Mixed Iteration within each increment
11	moderate	small	small	Newton-Raphson Iteration to tolerance t_s in all except last increment
12	moderate	large	small	Constant Stiffness Iteration to tolerance t_s in all except last increment
13	moderate	moderate	small	Mixed Iteration to tolerance t_s in all except last increment

TABLE 2

STRAIN (ϵ/ϵ_y) AT POINT X

Method	Load σ_{av}/σ_y					
	0.25	0.3	0.35	0.4	0.45	0.5
1						4.5614
2						3.8841*
3						4.5556
4	0.2988	1.1146	1.5940	2.2420	3.2966	4.2973
5	0.9288	1.1146	1.5940	2.2420	3.2966	4.6097
6	0.2988	1.1146	1.5940	2.2420	3.2966	4.5997
7	0.2988	1.1146	1.5940	2.2420	3.2966	4.6096
8	0.2988	1.2288	1.8574	2.5559	3.4314	4.5753
9	0.2988	1.2174	1.6879	2.4964	3.3940	4.4565
10	0.2988	1.2287	1.8607	2.5564	3.4320	4.5757
11	0.2988	1.1146	1.5940	2.5535	3.4359	4.5870
12	0.2988	1.1146	1.5940	2.5575	3.4720	4.5531
13	0.2988	1.1146	1.5940	2.5536	3.4369	4.5877

*Did not converge

TABLE 3

COMPARISON OF SOLUTION EFFORT

$\frac{\sigma_{aV}}{\sigma_y}$	0.25		0.3		0.35		0.4		0.45		0.5		Total	
Method	NRS*	NI*	NRS	NI	NRS	NI	NRS	NI	NRS	NI	NRS	NI	NRS	NI
1											7	7	7	7
2											1	>35	1	>35
3											2	15	2	25
4	1	1	1	1	1	1	1	1	1	1	1	1	6	6
5	1	1	1	1	1	1	1	1	1	1	4	4	9	9
6	1	1	1	1	1	1	1	1	1	1	1	15	6	20
7	1	1	1	1	1	1	1	1	1	1	3	4	8	9
8	1	1	3	3	5	5	3	3	4	4	4	4	20	20
9	1	1	1	8	1	8	1	8	1	14	1	15	6	54
10	1	1	3	3	3	4	2	3	3	4	4	4	16	19
11	1	1	1	1	1	1	4	4	4	4	4	4	15	15
12	1	1	1	1	1	1	1	8	1	15	1	15	6	41
13	1	1	1	1	1	1	2	3	3	4	4	4	12	14

*NRS = No. of times structure stiffness reformed.
NI = No. of Iteration cycles.

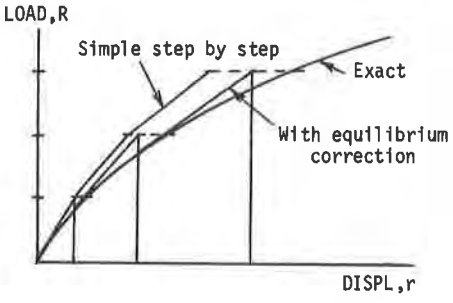

FIGURE 1 STEP BY STEP

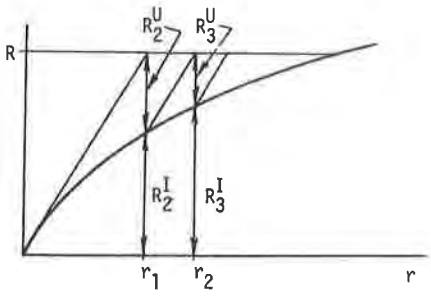

FIGURE 2 CONSTANT STIFFNESS ITERATION

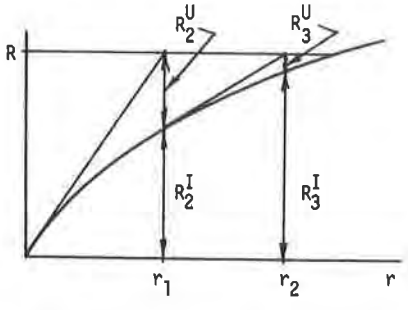

FIGURE 3 NEWTON-RAPHSON ITERATION

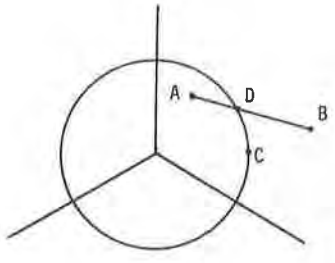

FIGURE 4 STATE DETERMINATION

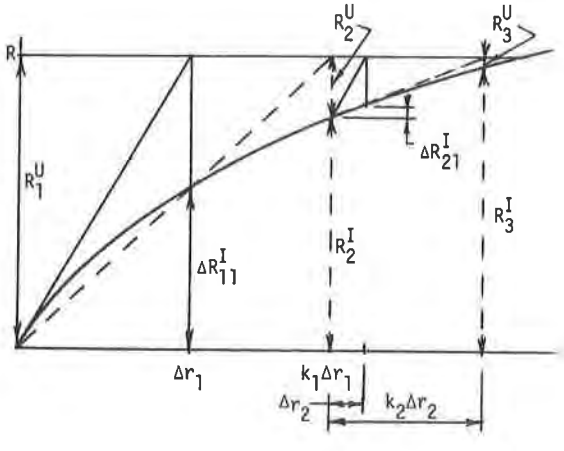

FIGURE 5 AN OVER-RELAXATION SCHEME

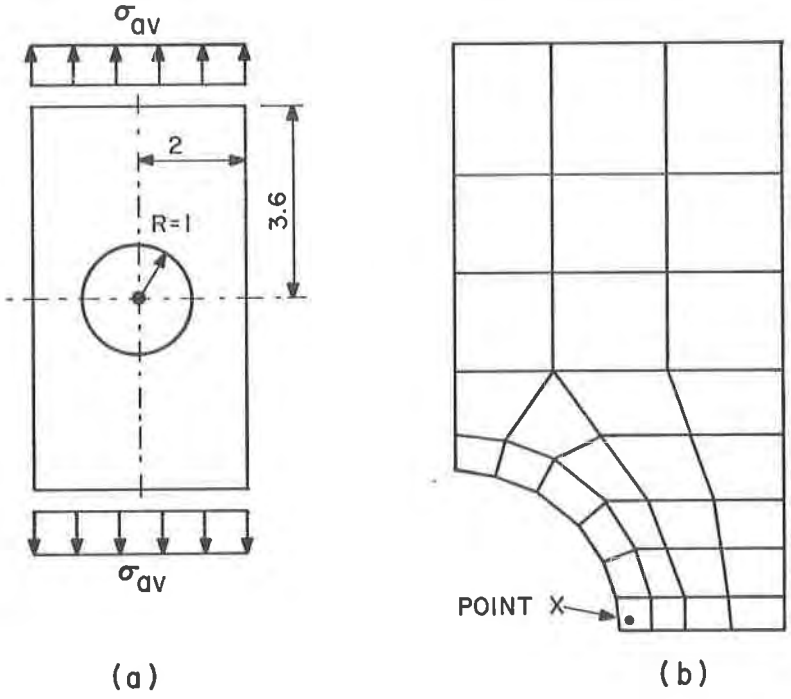

FIGURE 6 EXAMPLE