A LINEAR PROGRAMMING APPROACH TO SHAKEDOWN ANALYSIS OF STRUCTURES

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SUMMARY

The effects of variable repeated loads on structures have been extensively investigated by several authors, and it is now widely accepted that shakedown analysis leads to meaningful information in studying the serviceability of ductile systems.

In general terms, an elastic perfectly plastic structure is said to shake down under loads varying within a given domain (whose amplitude is defined by a multiplier s) if plastic flow eventually ceases, i.e. if after a certain time the response to further loading (within the domain) is purely elastic. According to Bleich and Melan's classical theorem a structure will shake down if selfstresses can be found which lead, when superposed to the elastic response to any load in the domain, to a stress state that nowhere exceeds the yield limits.

The analytical evaluation of the stress field is difficult and can be performed only for discrete or simple continuous structures. However the above definition spontaneously suggests numerical methods based on mathematical optimization techniques.

A theory, based on finite element discretization, that in principle permits the evaluation of the shakedown load for a general continuum was proposed by Maier: if the material yield condition is piecewise linearized, a solution can be obtained corresponding to the optimal value of a linear programming problem. At the present stage, to authors knowledge, the only numerical solution so far obtained for a problem whose analytical solution is not available was given by Belytschko, who found a lower bound to s for a plane stress problem, using an equilibrated finite element model and nonlinear mathematical programming techniques.

In this paper a procedure is proposed which reduces drastically the number of constraints in Maier's approach while retaining the linear nature of the problem. The method is conceptually analogous to a procedure that already proved successful in limit analysis. Depending on the finite element model adopted, values that reasonably can be conceived as lower bounds to s can be obtained, as well as upper bounds. Some numerical results are also shown, from which the effectiveness of the method can be deduced.

1. Generalities

In the present paper two and three dimensional structures are dealt with, made discrete by the finite element method. Each of these elements is assumed subject to continuous fields of tension and deformation defined by stress and displacement functions respectively. The adopted elements can correspond either to the well known "compatibility" or "displacement" model (Zienkiewicz [1]) or to the "equilibrium" model (Fraeijs de Venbeke [2]).

If the internal tensions in the element are represented by functions in a chosen form (e.g. polynomials) there will be DP parameters needed for locating that distribution of stresses. It is now possible to define the internal stresses in the element depending on the D stress components evaluated at P suitably chosen points.

When dealing with limit shake-down analysis depending on the application of Bleich and Melan's theorem, an elastic analysis of the structure has to be performed for any individual load condition. In this paper the results of such analysis are supposed to be achieved, provided that the same kind of element models is assumed both for the elastic phase and for the subsequent shakedown analysis.

For materials that satisfy Drucker's postulate, the plasticity domain is represented by a convex region in the space of the stress components. The boundary of this region is generally defined by one or more functions of the stress variables. However, for the application of linear programming only linear relations between the variables are allowed for. It follows that these functions must either be linear or piecewise linearised.

The yield domain is thus defined as a polyhedron in the D-space of the stress components. The respect of the yield condition (called also "conformity" in the sequel) for a given stress state is ensured if the corresponding stress vector is "internal" to any yield plane. An alternative formulation, given by Zavelani [3] , suggests to express the given stress state as a linear combination, with nonnegative coefficients, of the stress states corresponding to the vertices of the polyhedron: the respect of conformity is then ensured if the sum of these coefficients does not exceed unity.

The adoption of Bleich and Melan's theorem involves the respect of conformity throughout the structure. This can be achieved by evaluating a stress vector at the nodes of a suitably chosen check grid. In general the confiance of the results strictly depends on the density of this mesh,(except when uniform or linear stress field are adopted). In this paper conformity will be ensured just in the stress points of the elements, that therefore will be also called for brevity, "check points". The above assumption does not lead to a loss of generality: it must be noted that additional check points can always be introduced, as well as a finer finite element mesh adopted, in order to obtain more precise results.

2. Linear Programming formulations of shakedown problems

When a parametric stress field is constructed and the yield condition is piecewise linearized at every check point m, as previously described, Bleich Melan's theorem leads to the following linear programming formulation for s :

$$s = \max k \qquad (1\ a)$$

$$\text{subject to} \quad \underline{E}\,\underline{\sigma} = \underline{0} \qquad (1\ b)$$

$$k\underline{M}^m + \underline{\tilde{N}}^m\,\underline{\sigma}^m \leqslant \underline{K}^m, \; m=1,\dots,M \qquad (1\ c)$$

$$k \geqslant 0 \qquad (1\ d)$$

where k is a load domain multiplier, the G eqs.(1b) impose that the DM-vector $\underline{\sigma}$ of stresses at the check point represent a self-equilibrating stress state, and the YM eqs.(1c) impose, at every m, conformity, with respect to a piecewise linear yield condition, of the elastic response (contained in $\underline{M}^m$) added to a self-stress state.

The polyhedra defined by eqs. (1c) can alternatively be represented by their V^m vertices, whose coordinates are collected in a $(D+1)\times V^m$ matrix $\begin{bmatrix}\underline{T}^m\\ \underline{L}^m\end{bmatrix}$ so that $\begin{Bmatrix}\underline{\sigma}^m\\ k\end{Bmatrix} = \begin{bmatrix}\underline{T}^m\\ \underline{L}^m\end{bmatrix}\underline{\alpha}^m$, with $\underline{\alpha}^m \geqslant \underline{0}$.

Eqs. (1) can therefore be replaced by:

$$s = \max k \qquad (2\ a)$$

$$\text{subject to} \quad \underline{E}\,\underline{T}\,\underline{\alpha} = \underline{0} \qquad (2\ b)$$

$$\left.\begin{aligned} &k = \underline{L}^m\,\underline{\alpha}^m && (2\ c)\\ &\sum_1^{V^m}{}_i\,\alpha_i^m \leqslant 1 && (2\ d)\end{aligned}\right\}\ m=1,\dots,M$$

$$k \geqslant 0 \qquad (2\ e)$$

$$\underline{\alpha} \geqslant \underline{0} \qquad (2\ f)$$

where eqs. (2 b) and 2 c) replace eqs. (1 b) and (1 c) respectively.

Also the programs obtained by eqs.(1) and eqs.(2) by means of formal dualization are of interest, both computationally and in the interpretation of the results. They read:

$$s = \min \sum_1^M{}_m \underline{\tilde{K}}^m \underline{\lambda}^m \qquad (3\ a)$$

$$\text{subject to} \quad \underline{\tilde{E}}\,\underline{u} - \underline{N}\,\underline{\lambda} = \underline{0} \qquad (3\ b)$$

$$\sum_1^M{}_m \underline{\tilde{M}}^m\,\underline{\lambda}^m \geqslant 1 \qquad (3\ c)$$

$$\underline{\lambda} \geqslant \underline{0} \qquad (3\ d)$$

and

$$s = \min \sum_{1}^{M} {}_m P^m$$

$$\text{subject to} \quad \underline{U}\, P^m \geqslant \tilde{\underline{T}}^m \underline{\epsilon}^m + \tilde{\underline{L}}^m \underline{\varphi}^m, \quad m=1,\dots,M \qquad (4\ a)$$

$$\underline{\epsilon} = \tilde{\underline{E}}\ \underline{u} \qquad (4\ b)$$

$$\sum_{1}^{M} {}_m \varphi^m \geqslant 1 \qquad (4\ c)$$

$$P^m \geqslant 0 \qquad (4\ d)$$

By mechanically interpreting the dual variables, it can be shown that both eqs. (3) and eqs. (4) represent linear programming formulations of Koiter's shakedown theorem (Koiter [4]) . Hence $\tilde{\underline{K}}^m \underline{\lambda}^m$ and P^m represent the plastic energy dissipated at point m, and check points where this quantity is greater than zero identify a (not necessarily unique) shakedown mechanism.

3. Comparison between the given formulations

It was already mentioned that the critical factor in solving Linear Programming problems is the number of constraints. It was suggested that the computer time required to reach the optimum is roughly proportional to the cube of this number while it increases only linearly with the size of the variable vector. On this ground, only formulations (2) and (3) appear to be usable computationally, while formulations (1) and (4) are mainly of theoretical interest, and their practical validity is limited to the interpretation of the solution.

We shall first compare the two computational formulations with reference to the size of the problems. Giving the symbols the meaning of Section 2, program has DM+1 constraints and YM+G+1 variables (an additional slack variable being needed to have sign constrained $\underline{u}$).

On the other hand in form (2) the number of constraints is G+2M, while the dimension of the variable vector can be roughly taken as MV+1, V being the number of vertices of polyhedron. It can be shown that V is at most V+Y-D, but is usually of this order: hence the dimension of vector $\left\{\begin{matrix}\alpha \\ k\end{matrix}\right\}$ is greater than the dimension of $\left\{\begin{matrix}u \\ \underline{\lambda}\end{matrix}\right\}$.

For what concerns the number of constraints, formulation (3) is definitly preferable when D $\leqslant$ 2. For plane stress or plane strain problems (D = 3),much depends on how the elements are connected with each other and with the ground; usually the two formulations have roughly the same number of constraints. For general 3-dimensional continuous (D=6) formulation (2) permits a substantial gain in this sense.

Computationally speaking, formulation (2) presents an advantage in the fact that the origin of the $\left\{\begin{matrix}\alpha\\ k\end{matrix}\right\}$ space is feasible. Since most linear programming algorithms assume the origin of the space as starting point, the search for a feasible solution (the so called "first phase" of the simplex method) is avoided.

On the other hand, form.(2) requires as a preliminary step the evaluation of vertex coordinates, that are obtained as the solution of YM Linear Programs. They have only few variables (D+1) and constraints (Y) and computational experience shows that only a limited number of pivotal steps is required by each program (at most 15 for plane stress problems). Moreover the computational effort spent in determining vertex coordinates increases only linearly with M (and hence with the size of the problem). In spite of these considerations, this effort might become significant for large M, so that formulation (2) seems to be preferable only when dealing with three-dimensional continua.

However, when M is large both formulations imply the numerical solution of large Linear Programming problems, so that further reductions often are needed. Usually they are performed on the number of variables, and it turns out that reduced programs can be obtained in an easier and more reliable way from form.(2) than from (3).

4. Applications

The methods outlined in the preceding sections were given some numerical applications in order to prove their effectiveness and to check the results.

The discretization of the structures to be analysed was performed by adopting constant stress-constant strain compatible elements. This is the simplest available model, since only one check point per element is needed to define the entire stress field and to ensure the respect of conformity throughout. If the exact (continuous) elastic solution were available, any result provided by this model should be considered as an upper bound to the true value, depending on local equilibrium being violated somewhere at the interfaces. It is reasonable to expect that this conclusion holds true in spite of the elastic solution being obtained in an approximate way, using the same finite element model.

A first set of examples, that are not quoted in detail here, was concerned with a multisupported beam, subjected to multiple concentrated loads in the middle of each span. The purpose of this set of applications was, on the one hand, to compare the results with those obtained in the classical beam theory (an upper bound was reached, with a moderate gap between the two values), on the other hand to check the computational effectiveness and the numerical values of the results obtained with formulation (2), compared to the static approach (1). As well expected the results were perfectly coincident, while the computer time had a tendency to increase very quickly when adopting formulation (1).

A second set of applications was performed on a structure that had been already investigated, in its shake-down behaviour, by T. Belytschko [6] . The structure consists of the thin square slab with a central circular hole of fig.1. The slab is supposed to be loaded by the uniform tractions T_1 and T_2 applied on the edges normal to the x_1 and x_2 axis respectively.

In the loading program the two load components are considered to vary within 0 and an assigned value. A discretization mesh was adopted as shown in fig.2 with 66 elements and 47 nodes in each quadrant. A piecewise linearization of the Von Mises yield condition was used, with 14 faces and 18 vertices.

The behavior of the structure was analysed in the elastic, in the shakedown and in the limit field. The results are shown in diagram in fig.3.

As far as the elastic field is concerned each point of the curve of the "elastic solutions" has coordinates corresponding to the limit values that the loads can attain when the ratio T_1,T_2 is prescribed. The dotted line of the "elastic domain" bounds the field inside which an elastic solution exists for any loading path leading to the final values T_1,T_2.

In the curve of the "shake-down domain" each point corresponds to the maximum values that T_1,T_2 can attain in a prescribed loading program, giving rise to shake-down.

When limit analysis is accepted, the curve of the "limit solutions" provides the values of T_1,T_2 for which collapse takes place, while the dotted curve of the "limit domain" includes the points for which any loading path leading to T_1,T_2 is such to ensure safety against collapse.

Finally, the curve of the "kinematic bound" represents the values T_1,T_2 corresponding to a kinematically admissible solution of the problem: this solution can be considered trivial, as it was obtained simply by considering a mechanism for which plastic flow occurs along one axis of the structure.

A comparison between the present results and those quoted by Belytschko [6] shows up some significant differences.

With reference to the kinematic curve of fig.3 it can be noted that it provides an upper bound to the safety factor that is smaller than the lower bound of Belytschko [6] fig.3. All the other results obtained by Belytschko provide lower multipliers than those shown in this paper. It can be thought that the former represent a lower bound to the true values, because of the lack of compatibility in the model. On the contrary, the present results may be considered upper bounds, depending on the fact that compatibility and conformity are respected throughout while equilibrium is not.

As far as shake-down is concerned, the numerical results showed that for any of the considered loading programs, a limit situation is reached when alternating plasticity takes place at point A in fig.3. It appears reasonable to assume that this failure modality represents the critical event in the continuous structure as well, so that the true shakedown multiplier can be obtained when an analytical elastic solution of the problem is available.On the basis of the elastic solution obtained by Howland (Belytschko [6]),the shake down factor would be.470 σ_0 : with his finite element tecnique, Belytschko obtained a shake-down multiplier that was about 9% lower than the analytical bound, while the present application gave a 7% higher value. Both results seem to confirm the above remarks concerning the bounding of the exact solution and show a good approximation to it.

Similar remarks can be done with reference to the elastic curves.

It can be noted that both the elastic and the limit solution curves are such as to admit higher values of the load multiplier when two load components are present, as well expected when considering the characteristics of the structure and of the yield domain.

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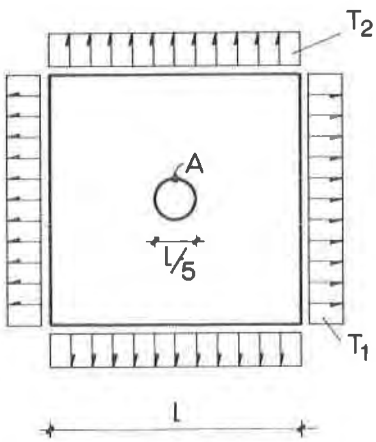

Figure 1

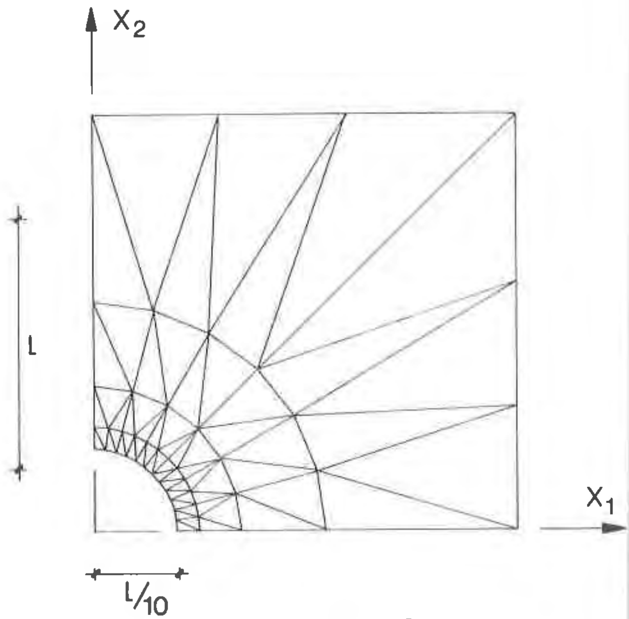

Figure 2

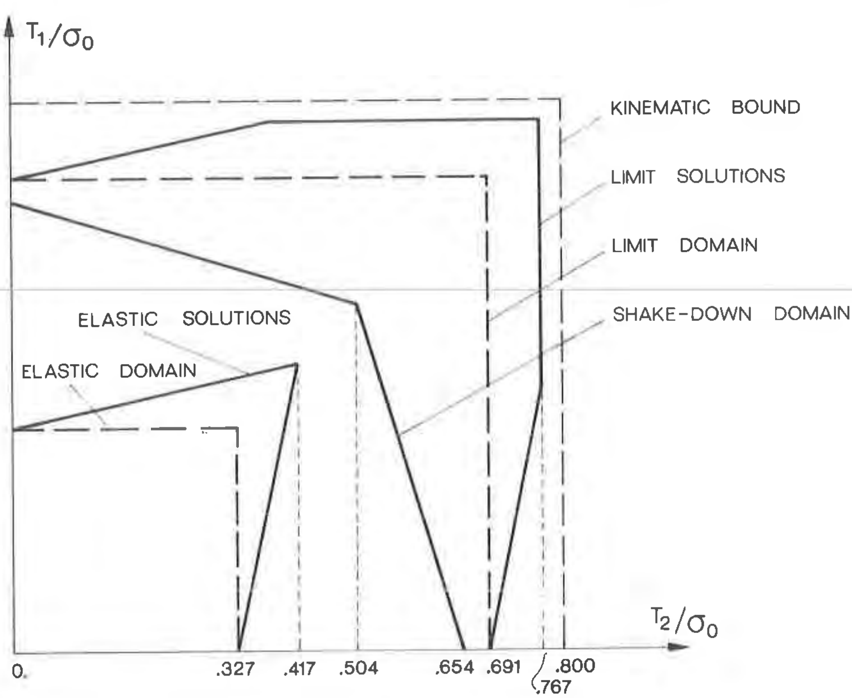

Figure 3