DEFLECTION BOUNDING AT SHAKEDOWN UNDER THERMAL AND MECHANICAL LOADINGS

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SUMMARY

Some elements of nuclear reactors are subjected to cycles of both temperature variation as well as to cycles of repeated variable pressure of liquid and/or gas. The theory of shakedown is appropriate in such cases to estimate the structural safety of a structure under consideration.

However fundamental theorems of the theory are based on the assumption of geometric linearity. Moreover they do not provide any direct bounds to maximum strains and displacements which may occur during an arbitrary long period of service of the structure, even if shakedown is ensured.

Only recently some results have been obtained in this field. The main idea of those estimates is the following. The total displacement u in an elastic-plastic body is a sum of "elastic" displacement u^E calculated for unlimited elastic response and an additional "plastic" term u^P. The term u^E is a linear form of load parameters and the term u^P can be presented as a linear functional of plastic strain field ε^P. The constraints of the problem are as follows:

(1) Limits of variations of loads (limits for load parameters);
(2) Yield condition;
(3) Maximum value of the total energy dissipated in a shakedown process.

The constraints (2) and (3) can be presented as some functional inequalities for the load parameters and plastic strains.

In this way the problem of maximum (minimum) of the displacement at a given point of the body becomes a problem of optimal control. If the structure is described in a discrete manner one obtains then a problem of mathematical programming. This problem can be linearized, e.g. by means of linearization of the yield condition.

The present paper extends some of those methods to the case of thermal effects by including an additional term of thermal strains.

A numerical example of a circular thick-walled tube is presented. An upper bound is established for maximum radial displacement occurring when internal pressure and temperature vary arbitrarily within some prescribed limits.

Some conclusions concerning limits of validity of the theory of shakedown are given.

1. Introduction

Some elements of nuclear reactors, as well as many other systems, are subjected to cycles of both temperature variation and/or to cycles of repeated variable pressure of liquid and/or gas. When creep effects may be neglected then shakedown is considered to be an appropriate criterion of structural safety [1, 2, 3, 13].

General shakedown theorems have been extended to various thermal effects [7] and several solutions of particular problems have been obtained [2, 4, 8, 11, 12]. The solutions give safe limits of variation of the loadings and are in agreement with experimental data [2, 20]. The theorems, however, are not able to estimate an order of deformations /strains, deflections/ which can appear in a shakedown process.

During the last two years several authors [14 - 18] have obtained some results on bounding shakedown strains and displacements. However in these papers thermal effects have not been systematically taken into account.

The present paper extends the two existing methods of bounding, namely that by PONTER [16, 17] concerning the continuum description of structures and Maier's method of natural finite element [5] to the case of thermal loadings.

Also the temperature dependence of elastic and plastic moduli of materials is included into the analysis. An example of a thick-walled cylinder subject to variable internal pressure and to quasi-static variable temperature field is presented. The results seem to be of practical importance.

2. Basic Relations

The strain tensor in an elastic-plastic medium subjected to thermal actions may be presented in the form

$$\underline{\varepsilon} = \underline{\varepsilon}^E + \underline{\varepsilon}^P + \underline{\varepsilon}^T \qquad /2.1/$$

where $\varepsilon^E_{ij} = A_{ijkl}\, \sigma_{kl}$ and A_{ijkl} is the elastic moduli tensor, $\underline{\varepsilon}^T$ stands for the thermal strain and $\underline{\varepsilon}^P$ denotes the plastic strain.

By the elastic stress $\underline{\sigma}^E$ we understand the solution of the actual boundary value problem obtained under the condition $\underline{\varepsilon}^P = 0$. The tensor $\underline{\sigma}^E$, in general, differs from the actual stress tensor $\underline{\sigma}$ which concerns the solution for an elastic perfectly plastic solid, where the constitutive relations are given by a yield condition and by a flow rule.

The difference

$$\underline{\sigma}^R = \underline{\sigma} - \underline{\sigma}^E \qquad /2.2/$$

is a residual stress state satisfying the zero boundary conditions. Thus the formula /2.1/ may be presented as follows :

$$\underline{\varepsilon} = \underline{\varepsilon}^{EE} + \underline{\varepsilon}^{R} + \underline{\varepsilon}^{T} + \underline{\varepsilon}^{P} \qquad /2.3/$$

where $\underline{\varepsilon}^{EE} = \underline{A}\ \underline{\sigma}^{E}$, $\underline{\varepsilon}^{R} = \underline{A}\ \underline{\sigma}^{R}$ and the fields $\underline{\varepsilon}^{EE} + \underline{\varepsilon}^{T}$ and $\underline{\varepsilon}^{R} + \underline{\varepsilon}^{P}$ are self-compatible.

It is important to note that whenever the tensor of elastic moduli varies with temperature, the plastic strain field $\underline{\varepsilon}^{P}$ does not specify uniquely the residual stress field $\underline{\sigma}^{R}$. In this case the field $\underline{\sigma}^{R}$ depends also on the actual temperature and is susceptible to vary with temperature even for a steady plastic strain field $\underline{\varepsilon}^{P}$.

In an arbitrary process of elastic-plastic deformation the following relations must hold :

1° equilibrium equations, together with static boundary conditions;
2° geometrical compatibility conditions, together with kinematic boundary conditions;
3° actual stress tensor must not violate yield condition;
4° active process condition i.e. plastic strain rates occur only if the stress tensor is on the yield condition.

Any direct method of looking for an upper bound for extremum strains and/or dispacements will lead to an extremely complex problem of optimal control. In such a problem the conditions 1° until 4° , amended by limits of variations of loadings and temperature, play the role of constraints. However, in such formulation the problem seems rather hopeless to be handled.

3. Static Shakedown Theorem

The wellknown Bleich-Melan static shakedown theorem has been extended to thermal effects in the most general way in the paper [7] . Applications of that theorem to practical boundary value problems have been presented in the paper [8] .

The theorem states the following :

if there exist a safety factor $s > 1$ and a time-independent plastic strain field $\bar{\underline{\varepsilon}}$ such that the associated residual stress field $\bar{\underline{\sigma}}$ superimposed on variable elastic stress does not exceed the yield condition

$$F\left[\underline{\sigma}^{E}(x,t,T) + \bar{\underline{\sigma}}(x,T)\right] \leq \frac{1}{s} K(x,T) \qquad /3.1/$$

then the structure will shake down. The relation $F(\underline{\sigma}) = K(x,T)$ stands for the yield condition.

The proof of the theorem Cf. [7] gives also the following upper bound for the total energy dissipated in an arbitrary process :

$$W_p = \int_o \int_V \sigma\ \dot{\varepsilon}^{P}\ dV\ dt \leq \frac{s}{2(s-1)} \int_V \left[\bar{A}\ r^2 + A_T\ R^2\ \bar{T}\right] dV \qquad /3.2/$$

where the constants occurring are defined in the following way :

$$R = \sup_T \sup_F |\underline{\sigma}| + \sup_T |\underline{\sigma}| + \sup_t |\underline{\sigma}^E| , \quad r = \sup_T |\underline{\sigma}| ,$$

$$A_T = \sup_T \sup \underline{A}_\theta \underline{\sigma}\,\underline{\sigma} , \quad \bar{T} = \sup_t T , \quad \bar{A} = \sup \sup \underline{A}\,\underline{\sigma}\,\underline{\sigma} \qquad /3.3/$$

The symbol $\sup_T [\ldots]$ denotes the upper bound of the quantity in brackets when temperature T varies arbitrarily within the limits prescribed by the loading program. Appropriately the symbols $\sup_F [\ldots]$ and $\sup [\ldots]$ denote the respective upper bounds for cases when stress tensor may become any tensor satisfying the yield condition $F(\underline{\sigma}) = K$ or $|\underline{\sigma}| = 1$ respectively. The symbol $\sup_t [\ldots]$ denotes the upper bound of the expression in brackets over all positive times.

In other words, the constants A, A_T, R, r, $\bar{T}$ can be calculated from mechanical properties of the structure and from the limits of variation of loads and temperature.

4. Discrete Case

In many practical cases a given structure can be considered, with a sufficient accuracy, as a finite assembly of n elements connected and interacting at special points called nodes. The elements are sufficiently small to assume their stress and strain states to be homogeneous within each element. The states are thus described by generalized stresses Q_i^k and generalized strains q_i^k, $k = 1,\ldots,n$; $i = 1,\ldots,v$. The external loads F_s act exclusively at nodes, $s = 1,\ldots,m$, and respective displacements of the nodes in the directions of the loads are u_s. Every element remains completely in one of the two admissible states : elastic or plastic. We assume also discrete distribution of temperature. This is admissible for sufficiently slow temperature variations.

Maier [5] has introduced such a model and successfully applied in elastic-plastic analysis of structures. The general symbolism holds naturally in case of structures which, by their very nature, should be described in a discrete manner e.g. trusses and frames.

Following the symbolism by Maier, all the relations are presented in matrix form. To make the formulas as short as possible let us introduce the following matrices :

- $\underline{F}$ – vector of nodal loads, $[m,1]$; $\underline{u}$ – vector of nodal displacements ;
- $\underline{Q}$ – vector of generalized stresses $[vn,1]$;
- $\underline{q}$ – vector of generalized strains, $[vn,1]$;
- $\underline{C}$ – rectangular matrix of static and kinematic compatibility, $[vn,m]$;
- $\underline{A}$ – nonsingular, positively defined, square matrix of elastic moduli, $[vn,vn]$;
- $\underline{K}$ – vector of plastic constants, $[pn,1]$;
- $\underline{L}$ – rectangular matrix, $[pn,vn]$;
- $\underline{\lambda}$ – vector of modes of plastic flow, $[pn,1]$;
- $\underline{\tau}$ – vector of thermal expansion coefficients, $[vn,n]$;

$\underline{\theta}$ - vector of temperature distribution, $[n,1]$;
Now the basic relations may be presented as follows. The superscript "T" denotes transposition of a matrix.

Equilibrium equations

$$\underline{F} = \underline{C}^T \underline{Q} \quad . \qquad /4.1/$$

Compatibility conditions

$$\underline{q} = \underline{C}\,\underline{u} \quad . \qquad /4.2/$$

Strain decomposition

$$\underline{q} = \underline{q}^E + \underline{q}^T + \underline{q}^P, \qquad /4.3/$$

where the successive terms are defined by :

Hooke's law

$$\underline{q}^E = \underline{A}\,\underline{Q} \; , \qquad /4.4/$$

thermal expansion law

$$\underline{q}^T = \underline{\tau}\,\underline{\theta} \; , \qquad /4.5/$$

associated flow rule, integrated with respect to time

$$\underline{q}^P = \underline{L}^T \underline{\lambda} \quad . \qquad /4.6/$$

Linearized yield condition

$$\underline{L}\,\underline{Q} - \underline{K} \leq \underline{0} \qquad /4.7/$$

here the $\underline{K}$ is a function of temperature.

Active loading criterion

$$\underline{\lambda}^T \left[\underline{L}\,\underline{Q} - \underline{K}\right] \leq \underline{0} \quad , \quad \underline{\lambda} \geq \underline{0} \; . \qquad /4.8/$$

The static shakedown theorem [7] , recalled in the Introduction can be directly formulated in terms of the discrete model and the respective upper bound for energy dissipated is obtained :

$$W_p = \int_0^t \underline{Q}^T \underline{\dot{q}}^P \, dt = \int_0^t \underline{\dot{\lambda}}^T \underline{K} \, dt \leq \frac{s}{2(s-1)} \left[\underline{r}^T \underline{A}\,\underline{r} + \underline{R}^T \underline{B}\,\underline{R} \right] = a\,\frac{s}{s-1} \qquad /4.9/$$

where the sens of the s, $\underline{r}$, $\underline{A}$, R, is analogous to that defined by /3.3/ and the $\underline{B}$ is matrix containig extremum values of temperature derivatives of the elastic moduli $\underline{A}$ multiplied by maximum values of temperature $\underline{\theta}$ appropriate for an element considered.

By following the procedure used in papers [15] and [19] we arrive at the expressions decomposing the displacement as well as the generalized stress vectors :

$$\underline{u} = \underline{u}^E + \underline{u}^P = \left[\underline{C}^T \underline{A}^{-1} \underline{C}\right]^{-1} \left[\underline{F} + \underline{C}^T \underline{A}^{-1} \underline{\tau}\,\underline{\theta}\right] + \left[\underline{C}^T \underline{A}^{-1} \underline{C}\right]^{-1} \underline{C}^T \underline{A}^{-1} \underline{L}^T \underline{\lambda} =$$
$$= \left[\underline{C}^T \underline{A}^{-1} \underline{C}\right]^{-1} \left[\underline{F} + \underline{C}^T \underline{A}^{-1} \left(\underline{\tau}\,\underline{\theta} + \underline{L}^T \underline{\lambda}\right)\right],$$
$$\underline{Q} = \underline{Q}^E + \underline{Q}^R = \underline{A}^{-1} \; \underline{C} \left[\underline{C}^T \underline{A}^{-1} \underline{C}\right]^{-1} \left[\underline{F} + \underline{C}^T \underline{A}^{-1} \underline{\tau}\,\underline{\theta} - \underline{\tau}\,\underline{\theta}\right] +$$
$$+ \underline{A}^{-1} \left\{ \underline{C} \left[\underline{C}^T \underline{A}^{-1} \underline{C}\right]^{-1} \underline{C}^T \underline{A}^{-1} \underline{L}^T \underline{\lambda} - \underline{L}^T \underline{\lambda} \right\} . \qquad /4.10/$$

This way the displacement and stress are expressed as linear forms of nodal loads, temperature and plastic deformations.

The nodal load vector $\underline{F}$ and the temperature vector $\underline{\Theta}$ are linear forms of k external factors μ_i, consisting load parameter vector $\underline{\mu}$, [19] . Thus

$$\underline{F} = \underline{f}\,\underline{\mu} \qquad \underline{f}\ [m,k] \qquad /4.11/$$

$$\underline{\Theta} = \underline{\vartheta}\,\underline{\mu} \qquad \underline{\vartheta}\ [n,k] \qquad /4.12/$$

Limits of variation of the load parameters are prescribed by

$$\underline{\mu}^- \leq \underline{\mu} \leq \underline{\mu}^+ \qquad \underline{\mu}\ [k,1] \qquad /4.13/$$

In an arbitrary process of elastic-plastic deformation the equilibrium equations /4.1/, compatibility conditions /4.2/, yield condition /4.7/ and the active process criterion /4.8/ must hold. Moreover, if the given limits of variation of loads and temperature allow for shakedown then the total energy dissipated /4.9/ is bounded. Therefore the problem of an upper /lower/ bound for the s-th component of the vector $\underline{u}$, equal to u_s is a problem of upper /lower/ bound of the selected s-th component of the expression /4.10/ under the constraints /4.1/, /4.2/, /4.7/, /4.8/, /4.13/. According to /4.8/ the inequality /4.9/ may be written down in the following way

$$\underline{\lambda}^T\underline{k} \leq \int_0^\infty \dot{\underline{\lambda}}^T\,\underline{K}\,dt \leq a\,\frac{s}{s-1} \qquad /4.14/$$

where the $\underline{k}$ is any constant matrix such that $\underline{k} \leq \underline{K}$ for all temperatures $\underline{\Theta}$ resulting from /4.13/ and /4.12/.

If, in an optimisation process, one drops down a constraint then the resulting, modified feasible domain is not smaller than the original feasible domain. Therefore,if we neglect the active process constraint /4.8/ in an optimization of u_s , we obtain an upper /though usually not an actual/ bound of the displacement u_s. In the case considered we obtain a linear programming problem. Namely :

$$\text{find} \quad \max \left[\underline{C}^T\underline{A}^{-1}\underline{C}\right]^{-1}\left[\underline{f}\,\underline{\mu} + \underline{C}^T\underline{A}^{-1}\underline{\tau}\,\underline{\vartheta}\,\underline{\mu} + \underline{C}^T\underline{A}^{-1}\underline{L}^T\underline{\lambda}\right]\Big|_s \qquad /4.15/$$

subject to the constraints

$$\underline{L}\,\underline{A}^{-1}\underline{C}\left[\underline{C}^T\underline{A}^{-1}\underline{C}\right]^{-1}\left[\underline{C}^T\underline{A}^{-1}(\underline{L}^T\underline{\lambda} + \underline{\tau}\,\underline{\vartheta}\,\underline{\mu}) + \underline{f}\,\underline{\mu} - \underline{\tau}\,\underline{\vartheta}\,\underline{\mu} - \underline{L}^T\underline{\lambda}\right] \leq \underline{k} \qquad /4.16/$$

$$\underline{\lambda}^T\underline{k} = a\,\frac{s}{s-1} \qquad /4.17/$$

$$\underline{\mu}^- \leq \underline{\mu} \leq \underline{\mu}^+ \quad , \quad \underline{\lambda} \geq \underline{0} \qquad /4.18/$$

Standard procedures applied in solving linear programming problems usually require for all variables occurring to be non-negative. To satisfy this requirement it suffices to introduce the following new variables

$$\bar{\underline{\mu}} = \underline{\mu} - \underline{\mu}^- \qquad /4.19/$$

In many problems the constraint /4.16/ can turn out not to be very stringent and may be omitted. In any case neglecting this constraint gives a safe /upper/ bound for maximum deflection Cf. [18] . In such case the problem of optimisation can be disseparated :

$$\max u_s = \max u_s^E + \max u_s^P \qquad /4.20/$$

5. Continuum Case

As it was said in the Introduction, direct handling of the problem of extremum deformations is very difficult in cases which cannot be discretized. Neverthless Ponter [16,17] succeded in obtaining some bounds by appropriate transformations of Virtual Work Principle, Drucker's postulate and some other relationships.

Following his way we can obtain a result appropriate for the case when yield condition as well as elastic moduli tensor vary with temperature. To make the derivation simpler it is assumed that there are no initial stress in the structure /Ponter has assumed an initial residual stress distribution/. There is no difficulty to extend the present analysis to that case.

In the analysis it is assumed that stress fields

$$t_{ij} + \sigma^E_{ij} + \bar{\sigma}_{ij} \qquad \text{and} \qquad s\left[\sigma^E_{ij} + \bar{\sigma}_{ij}\right] \qquad /5.1/$$

do not violate the given yield condition. The stress state t_{ij} results from a fictive steady loading T_i in perfectly elastic way and $\eta_{ij} = A_{ijkl}\, t_{kl}$ are respective /elastic/ strains and v_i - respective displacements. The reason of introducing the fictive loading T_i will become clear at the end of our considerations. The $\bar{\sigma}_{ij}$ denotes the same steady residual stress state as in the static shakedown theorem /3.1/ and the $s>1$ is a safety factor.

Let us consider now the non-negative functional which was used to prove the static shakedown theorem, Cf. [7] .

$$Y = \frac{1}{2}\int_V A_{ijkl}\,(\sigma^R_{ij} - \bar{\sigma}_{ij})\,(\sigma^R_{kl} - \bar{\sigma}_{kl})\,dV +$$

$$+ \frac{1}{2}\int_V \int_0^t \dot{A}_{ijkl}\,(\sigma^R_{ij} - \bar{\sigma}_{ij})\,(\sigma^R_{kl} - \bar{\sigma}_{kl})\,dt\,dV +$$

$$+ \frac{1}{2}\int_V A_T\, R^2\, \bar{T}\, dV \geqq 0 \qquad /5.2/$$

Its time derivative is :

$$\dot{Y} = \int_V A_{ijkl}(\sigma^R_{ij} - \bar{\sigma}_{ij})(\dot{\sigma}^R_{kl} - \dot{\bar{\sigma}}_{kl})dV + \int_V \dot{A}_{ijkl}(\sigma^R_{ij} - \bar{\sigma}_{ij})\cdot$$

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$$\cdot(\sigma^R_{kl} - \bar{\sigma}_{kl})\,dV = \int_V (\sigma^R_{ij} - \bar{\sigma}_{ij})\left[A_{ijkl}(\sigma^R_{kl} - \bar{\sigma}_{kl})\right]^{\bullet} dV =$$

$$= - \int_V (\sigma^R_{ij} - \bar{\sigma}_{ij})(\dot{\varepsilon}^P_{ij} - \dot{\varepsilon}_{ij})\,dV = - \int_V (\sigma^R_{ij} - \bar{\sigma}_{ij})(\dot{\varepsilon}^E_{ij} + \dot{\varepsilon}^T_{ij})\,dV ==$$

$$\leq 0 \qquad /5.3/$$

The last two steps follow from Virtual Work Principle and from assumptions of the static shakedown theorem.

It is not difficult to check, that, according to the formula /2.3/ :

$$\int_o^t \int_V t_{ij}(\dot{\varepsilon}^E_{ij} + \dot{\varepsilon}^T_{ij})\,dV\,dt = \int_V t_{ij}(\varepsilon^E_{ij} + \varepsilon^T_{ij})\,dV = \int_V t_{ij}(\varepsilon^{EE}_{ij} + \varepsilon^T_{ij})\,dV +$$

$$+ \int_V t_{ij}\,\varepsilon^R_{ij}\,dV = \int_S T_i\,u^E_i\,dS + \int_V t_{ij}A_{ijkl}\,\sigma^R_{kl}\,dV = \int_S T_i\,u^E_i\,dS +$$

$$+ \int_V \sigma^R_{ij}\,\eta_{ij}\,dV = \int_S T_i\,u^E_i\,dS \qquad /5.4/$$

According to /5.1/ and to Drucker's postulate

$$\left[(\sigma^E_{ij} + \bar{\sigma}_{ij}) - \sigma_{ij} + t_{ij}\right]\dot{\varepsilon}^P_{ij} = (\bar{\sigma}_{ij} + t_{ij} - \sigma^R_{ij})\,\dot{\varepsilon}^P_{ij} \leq 0 \qquad /5.5/$$

By integrating this over the structure and with respect to time we arrive at

$$\int_o^t \int_V (\bar{\sigma}_{ij} + t_{ij} - \sigma^R_{ij})\,\dot{\varepsilon}^P_{ij}\,dV\,dt \leq 0 \qquad /5.6/$$

According to /5.2/ and /5.3/ we obtain also :

$$\int_o^t \int_V (\bar{\sigma}_{ij} - {}^R_{ij} + t_{ij})(\dot{\varepsilon}^E_{ij} + \dot{\varepsilon}^T_{ij})\,dV\,dt = \int_o^t \int_V (\bar{\sigma}_{ij} - \sigma^R_{ij})(\dot{\varepsilon}^E_{ij} +$$

$$+ \dot{\varepsilon}^T_{ij})\,dV\,dt + \int_o^t \int_V t_{ij}(\dot{\varepsilon}^E_{ij} + \dot{\varepsilon}^T_{ij})\,dV\,dt = Y(0) - Y(t) +$$

$$+ \int_S T_i\,u^E_i\,dS \qquad /5.7/$$

Adding together the relations /5.6/ and /5.7/ results in

$$\int_o^t \int_V (\bar{\sigma}_{ij} - \sigma^R_{ij} + t_{ij})\,\dot{\varepsilon}_{ij}\,dV\,dt \leq Y(0) - Y(t) + \int_S T_i u^E_i\,dS \qquad /5.8/$$

However :

$$\int_0^t \int_V (\bar{\sigma}_{ij} - \sigma^R_{ij} + t_{ij})\, \dot{\varepsilon}_{ij}\, dV\, dt = \int_0^t \int_S T_i \dot{u}_i\, dS\, dt = \int_S T_i u_i\, dS \qquad /5.9/$$

Therefore, at last, we obtain

$$\int_S T_i u_i\, dS \leq Y(0) - Y(t) + \int_S T_i\, u_i^E\, dS \qquad /5.10/$$

According to the result /3.2/ the inequality /5.10/ gives :

$$\int_S T_i u_i\, dS \leq Y(0) + \int_S T_i u_i^E\, dS \leq \int_S T_i u_i^E\, dS + \frac{1}{2} \int_V (\bar{A}\, r^2 + A_T\, R^2\, \bar{T})\, dV \qquad /5.11/$$

The inequalities /5.10/ and/or /5.11/ provide a local displacement bound when T_i becomes a concentrated force or a bound for average displacement over a surface portion when T_i becomes a uniformly distributed load.

The method presented is applicable only to a limited class of problems as in the case of concentrated force the fictive stresses t_{ij} usually contain some singularities.

6. Example

A thick-walled circular cylinder in plane strain state is subjected to cycles of internal pressure and temperature varying within prescribed limits. The problem seems to be of practical importance not only in reactor structural analysis. Similar problems, with temperature dependence of material characteristics taken into account, were analyzed in various papers Cf. [8, 11, 12] .

Let the internal, external and running radii be a, b and r respectively ; internal temperature be θ, external being nought ; and internal pressure being p . Then the temperature and elastic stress fields for separated actions of the temperature and of the pressure are as follows:

$$T(r) = \theta\, \frac{\ln r/b}{\ln a/b} \qquad /6.1/$$

$$\sigma^T_r(r) = -\frac{E\,\theta\,\alpha}{2(1-\nu)}\, \frac{a^2}{b^2-a^2} \left[1 - \frac{b^2}{r^2} + \frac{b^2-a^2}{a^2 \ln a/b} \ln r/b \right] \qquad /6.2/$$

$$\sigma^T_\varphi(r) = -\frac{E\,\alpha\,\theta}{2(1-\nu)}\, \frac{a^2}{b^2-a^2} \left[1 + \frac{b^2}{r^2} + \frac{b^2-a^2}{a^2 \ln a/b} (1 + \ln r/b) \right]$$

$$\sigma^T_z(r) = -\frac{E\,\alpha\,\theta}{2(1-\nu)}\, \frac{a^2}{b^2-a^2} \left[1 + \frac{b^2-a^2}{a^2 \ln a/b} \left(\frac{1}{2} + \ln r/b \right) \right]$$

$$\sigma_r^P(r) = p\,\frac{a^2}{b^2 - a^2}\left[1 - \frac{b^2}{a^2}\right]$$

$$\sigma_\varphi^P(r) = p\,\frac{a^2}{b^2 - a^2}\left[1 + \frac{b^2}{a^2}\right]$$

$$\sigma_z^P(r) = p \qquad /6.3/$$

The Tresca yield condition for plane strain has the form

$$-2\,k \leq \sigma_\varphi - \sigma_r \leq 2\,k \qquad /6.4/$$

and the yield stress k dependence on temperature is assumed to be linear

$$k(T) = k_o\,(1 - n\,T) \qquad /6.5/$$

where n – material constant ; elastic moduli are assumed to remain constants.

Let the temperature θ and the pressure p vary independently within the limits :

$$0 \leq p \leq p_o$$
$$0 \leq \theta \leq \theta_o \qquad /6.6/$$

By assuming steady residual stresses in the form

$$\bar{\sigma}_r(r) = \frac{2[k_o - p_o\, b^2/(b^2 - a^2)]}{2\,b^2/a^2 + (b^2 - a^2)/a^2 \ln a/b}\left[1 - \frac{b^2}{r^2} + \frac{b^2 - a^2}{a^2 \ln a/b}\ln r/b\right]$$

$$\bar{\sigma}_\varphi(r) = \frac{2[k_o - p_o\, b^2/(b^2 - a^2)]}{2\,b^2/a^2 + (b^2 - a^2)/a^2 \ln a/b}\left[1 + \frac{b^2}{r^2} + \frac{b^2 - a^2}{a^2 \ln a/b}(1 + \ln r/b)\right]$$

$$\bar{\sigma}_z(r) = \frac{4[k_o - p_o b^2/(b^2 - a^2)]}{2\,b^2/a^2 + (b^2 - a^2)/a^2 \ln a/b}\left[1 + \frac{b^2 - a^2}{a^2 \ln a/b}\left(\frac{1}{2} + \ln r/b\right)\right] \qquad /6.7/$$

we are able to **demonstrate** that the structure considered shakes down to the limits /6.6/ if the quatities θ_o and p_o satisfy the following

$$\theta_o\left[n + \frac{E\alpha a^2}{2(1-\nu)(b^2 - a^2)}\left(2\,b^2/a^2 + (b^2 - a^2)/a^2 \ln a/b\right)\right] + p_o\frac{b^2}{b^2 - a^2} \leq 2\,k_o \qquad /6.8/$$

To bound the maximum radial displacement which may occur in an arbitrary shakedown process let us use another steady pressure q as the fictive loading.

In the case of internal pressure the t_r, t_φ , t_z stresses will be expressed by the formulae /6.3/ with p being replaced by q. For external pressure, needed for bounding the displacement $u(b)$, one obtains

$$t_r = q\,a^2\left[b^2/r^2 - 1\right]/(b^2 - a^2)$$
$$t_\varphi = q\,b^2\left[a^2/r^2 - 1\right]/(b^2 - a^2) \qquad t_z = q \qquad /6.9/$$

If one substitutes the residual stresses /6.7/ into the $Y(0)$ expression, given by the formula /5.2/, then the following result will be obtained :

$$Y(0) = \frac{1}{2}\int A_{ijkl}\,\bar{\sigma}_{ij}\,\bar{\sigma}_{kl}\,dV =$$

$$= \frac{\pi}{E}\int_a^b \left[\,\bar{\sigma}_r^2 + \bar{\sigma}_\varphi^2 + \bar{\sigma}_z^2 - 2\nu(\bar{\sigma}_r\bar{\sigma}_\varphi + \bar{\sigma}_\varphi\bar{\sigma}_z + \bar{\sigma}_r\bar{\sigma}_z)\right] r\,dr =$$

$$= \frac{\pi}{E}\,\frac{4[k_0 - p_0 b^2/(b^2-a^2)]^2}{(2\,b^2/a^2 + b^2-a^2\,/a^2\ln a/b\,)^2}\,b^2\Big\{(6 + 6\beta + 5\beta^2)(b^2-a^2)/2b^2 +$$

$$- 2\beta\ln a/b + (b^2-a^2)/a^2 + (5 + 4\beta)(a^2/b^2 - a^2\ln(a/b)/\,b^2 - \tfrac{1}{2}) +$$

$$+ 3\beta^2(a^2\ln(a/b)/b^2 - a\ln^2(a/b)/b) - 2\nu\left[(4 + \beta^2)(b^2-a^2)/2b^2 +\right.$$

$$\left.- 5\beta\,a^2\ln(a/b)/b^2 - \tfrac{5}{2}\beta^2 a\ln^2(a/b)\right]\Big\},\quad \beta = (b^2-a^2)/a^2\ln\tfrac{a}{b}\,. \qquad /6.10/$$

Elastic radial displacements u^E are as follows

$$u^E(a) = \frac{a}{E}\Big\{p(b^2+a^2)/(b^2-a^2) - E\alpha\theta/\,2(1-\nu)(b^2-a^2)\left[2(1-2\nu)b^2 +\right.$$

$$\left.+ (1-\nu)(b^2-a^2)/\ln(a/b)\right]\Big\}$$

$$u^E(b) = \frac{b}{E}\Big\{p(b^2+a^2)/(b^2-a^2) - \left[2 + (b^2-a^2)/a^2\ln(a/b)\right]E\alpha\theta a^2/2(b^2-a^2)\Big\}$$

/6.11/

The formulae /5.11/, /6.2/, /6.3/, /6.9/ and /6.11/ allow to bound the maximum total radial displacements for chosen numbers θ_0, p_0 satisfying , with some safety margin, the shakedown condition /6.8/ by appropriate choise of the number q in the /6.2/ or /6.9/ respectively.

For example, for material characteristics $E = 2.1\ 10^6$ at, $k_0 = 3000$at $n = 1/300^\circ C$ /these relate to a steel of eleveted strength/ one obtains that the point

$\theta_0 = 200^\circ C$, $p_0 = 246$ at

is on the shakedown curve /6.8/. By taking e.g.

$\theta_0 = 200^\circ C$, $p_0 = 236$ at, $q = 10$ at

one satisfies the conditions /5.1/ and the following bound for the maximum radial displacement at $r = a$ can be obtained

$$Y(0) = 1.35\ \pi\ a^2.1\ \text{at}$$

$$2\pi a q\, u(a) \leq Y(0) + 2\pi a q\, u^E(a)$$, and finally

$$u(a) \leq 0.0675\ a + u^E(a) \qquad /6.12/$$

An analogous bound can be obtained easily for $r = b$.

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