INADAPTATION THEOREM IN THE DYNAMICS OF ELASTIC-WORKHARDENING STRUCTURES

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SUMMARY

This study concerns elastic perfectly plastic and elastic strain hardening structures subjected to a given history of external agencies (loads and imposed strains or displacements) whose variation is not slow so that inertia and damping effects cannot be neglected. Displacements are supposed to be small enough for the usual first order theory to be applied.

The structures considered are made discrete and the material constitutive laws are piecewise linearized. Some broad classes of discrete structural models, piecewise linear yield loci and hardening rules are allowed for. In this way the dynamic elasto-plastic response of the structure to the assigned external agencies may be studied on the basis of a convenient matrix description. A number of results concerning the dynamic behavior of plastic structures have been previously reached making use of these assumptions: Ceradini's dynamic shakedown theorem was extended to strain hardening structures, allowing for a fairly large class of hardening rules; the problem of finding the maximum load multiplier for which shakedown occurs was formulated as a linear programming problem.

In this paper a necessary and sufficient condition for inadaptation is stated and proved, inadaptation meaning that the plastic work, and, hence, some plastic deformations, increase unlimitedly in time (i.e. the structure does not shakedown). This condition may be considered as the extension to the dynamic range of the second (Koiter's) shakedown theorem of classical plasticity, to which it reduces when the variation of external agencies is slow. In the same sense Ceradini's theorem can be regarded as an extension of classical Bleich-Melan's theorem.

The proof of the theorem makes use of the aforementioned linear programming formulation of the dynamic shakedown analysis and flows from considerations pertaining to the mechanical nature of the problem and from concepts belonging to mathematical optimization theory. It is reasonable to expect that the use of the generalized inadaptation theorem in dynamics is as effective as to the use of Koiter's theorem in statics.

1. Introduction

The classical quasi-static shakedown theory of perfect plasticity is centered on two fundamental results: the static theorem which supplies sufficient and necessary conditions for adaptation or shakedown (Melan, [1]); the kinematic theorem which states sufficient and necessary conditions for inadaptation (Koiter, [2]).

The theoretical and practical importance of extending the shakedown theory to the dynamic range is clear. The breakthrough in this direction is due to Ceradini who generalized Melan's theorem, allowing for inertia and viscous forces (Ceradini, [3]). Further generalizations to hardening and softening behavior were established (Maier, [4], [5]).

Scope of this paper is to extend the second, Koiter's shakedown theorem to the dynamic range.

Imposed strains and displacements are taken into consideration among the external actions as in (Maier, [6]) and (De Donato, [7]) for the quasi-static range. A broad class of hardening rules are allowed for, particularly in view of the bounding techniques to be expounded elsewhere (Maier and Vitiello, paper L 7/4 of this Conference). Discrete structural models and piecewise linear yield loci and hardening are assumed, mainly because of the simplicity and operative value of the consequent matrix-vector description.

Notation: underlined symbols denote matrices and column-vectors: $\underline{O}$ is a matrix or vector whose entries are all zero: a dot means time derivative, a tielde means transpose.

2. Basic Relationships

Trusslike structures will be referred to first in setting up the relations which govern the dynamic behavior of a broad class of elastoplastic work-hardening systems subject to loads and imposed strains. Let $\underline{u}$ and $\underline{F}$ the n-vectors of the free nodal displacement components (degrees of freedom) and corresponding external force components. Vectors $\underline{q}$ and $\underline{Q}$ will collect all, say m, bar elongations and bar forces (generalized strains and stresses, respectively).

Compatibility and equilibrium are expressed by the following linear equations (first order theory):

$$\underline{q} = \underline{C}\,\underline{u} \qquad (1)$$

$$\tilde{\underline{C}}\,\underline{Q} + \underline{I}\,\ddot{\underline{u}} + \underline{V}\,\dot{\underline{u}} = \underline{F} \qquad (2)$$

The compatibility matrix $\underline{C}$ has rank n: $\underline{I}$ and $\underline{V}$ denote the inertia and the viscous damping matrix, respectively, masses and damping effects being assumed as concentrated in the nodes. In Eqs. (1) (2) all matrices are constant, all vectors variable in time.

Consider a single member, say that of index i. Its behaviour, assumed as elastic-linearly hardening, is depicted in fig.1 and can be analytically described as follows:

$$E^i\, e^i = Q^i \qquad (3)$$

$$p^i = \lambda_1^i - \lambda_2^i \qquad (4)$$

$$\dot{\lambda}_1^i \geq 0, \quad \dot{\lambda}_2^i \geq 0 \qquad (5)$$

$$\left.\begin{aligned} \varphi_1^i &\equiv Q^i - (K_1^i + H_{11}^i \lambda_1^i + H_{12}^i \lambda_2^i) \leq 0 \\ \varphi_2^i &\equiv - Q^i - (K_2^i + H_{21}^i \lambda_1^i + H_{22}^i \lambda_2^i) \leq 0 \end{aligned}\right\} \qquad (6)$$

$$\varphi_1^i \dot{\lambda}_1^i = 0, \quad \varphi_2^i \dot{\lambda}_2^i = 0 \qquad (7)$$

$$\dot{\varphi}_1^i \dot{\lambda}_1^i = 0, \quad \dot{\varphi}_2^i \dot{\lambda}_2^i = 0 \qquad (8)$$

Eq.(3) concerns the elastic part e^i of strain q^i (fig.1-a). The plastic part p^i is expressed by Eq.(4) as the result of the accumulation of contributions from both yielding modes, in tension (positive, index 1) and compression (negative, index 2); these contributions are defined by the "plastic multipliers" λ , non negative in view of Eq.(5). Eqs.(6) define the "yield functions" φ and, through the constants H, the hardening rules, $K_1^i (>0)$ and $-K_2^i$ (<0) being the yield limits at t=0 (when the λ are assumed zero). Eqs.(7) require that a mode is activated, (i.e. $\dot{\lambda} > 0$) only if the relevant current yield limit is attained ($\varphi = 0$). Eqs.(8) rule out that plastic flow ($\dot{\lambda} > 0$) and unloading ($\dot{\varphi} < 0$) occur simultaneously.

It is convenient for subsequent developments to condense all relation sets Eqs.(3) - (8) for i=1...m, into a single matrix relation set. This is easely achieved by defining the diagonal matrices $\underline{E} \equiv \text{diag}\left[E^i\right]$, $\underline{H} \equiv \text{diag}\left[H^i\right]$, $\underline{N} \equiv \text{diag}\left[1, -1\right]$, and the vectors $\underline{e}, \underline{p}, \underline{\lambda}$, $\underline{K}, \underline{\varphi}$ which collect all the scalar quantities denoted by the same letters. The "constitutive" relations for all constituents now read:

$$\underline{E}\,\underline{e} = \underline{Q} \qquad (9)$$

$$\underline{p} = \underline{N}\,\underline{\lambda} \qquad (10)$$

$$\underline{\lambda} \geq \underline{0} \qquad (11)$$

$$\underline{\varphi} \equiv \underline{\tilde{N}Q} - \underline{H}\,\underline{\lambda} - \underline{K} \leq \underline{0} \qquad (12)$$

$$\underline{\tilde{\varphi}}\,\dot{\underline{\lambda}} = 0 \qquad (13)$$

$$\dot{\underline{\tilde{\varphi}}}\,\dot{\underline{\lambda}} = 0 \qquad (14)$$

If the imposed strains ("dislocations") are defined by the m-vector $\underline{D}$, one can write:

$$\underline{q} = \underline{e} + \underline{p} + \underline{D} \tag{15}$$

The dynamic response of the truss to a given time history of external actions $\underline{F}(t)$, $\underline{D}(t)$ is governed by Eqs.(1)(2) and Eqs.(9)-(14) and by the initial conditions:

$$\underline{u}\ (t=0) = \underline{u}_o \quad , \quad \dot{\underline{u}}\ (t=0) = \dot{\underline{u}}_o \tag{16}$$

$\underline{u}_o$ and $\dot{\underline{u}}_o$ being given distributions of displacements and velocities.

It would be easy to show that discrete models with lumped compliances (such as those illustrated in fig.2a) and finite element models with piecewise linear displacements (fig.2b) are governed by the same set of relations with a simple re-interpretation of symbols, provided that a suitable piecewise linearization of the yield loci be adopted. Thus a fairly general basis has been set up for the theoretical developments which follow.

3. Dynamic Inadaptation Theorems

For the class of discrete structures specified, the shakedown phenomenon can be mathematically characterized as follows:

$$\underline{\lambda}_\infty \equiv \lim_{t\to\infty} \underline{\lambda}\ (t) < \underline{\infty} \quad \text{(bounded)} \tag{17}$$

Clearly Eq.(17) implies that the total dissipated work $\tilde{\underline{K}}\,\underline{\lambda}$ is bounded in time.

Let us define some quantities and notions of use in subsequent developments.

(1) The linear elastic dynamic response of the structure to the given $\underline{F}(t)$ and $\underline{D}(t)$ and to some convenient initial conditions $\underline{u}_o^*, \dot{\underline{u}}_o^*$ will be referred to as a fictitious process. An asterisk will mark all the quantities pertaining to such a process: in particular $\underline{Q}^{e*}(t)$, $\underline{F}_I^*(t)$, $\underline{F}_v^*(t)$ will denote, respectively, stresses, inertia forces and damping forces generated.

(2) Consider the matrix:

$$\underline{A} \equiv \underline{H} - \tilde{\underline{N}}\underline{Z}\underline{N} \tag{18}$$

where $\underline{Z} \equiv \underline{E}\ \underline{C}\ (\tilde{\underline{C}}\ \underline{E}\ \underline{C})^{-1}\tilde{\underline{C}}\ \underline{E} - \underline{E}$ is the operator such that $\underline{Z}\ \underline{D}$ represent the elastic stress response in the structure due to imposed inelastic strains $\underline{D}$. A quasi-static plastic flow process $\dot{\underline{\lambda}}'(t) \geqslant \underline{0}$ over some time interval t_1, t_2, characterized by the properties:

$$\int_{t_1}^{t_2} \dot{\underline{\lambda}}'(t)\,dt \equiv \underline{\lambda}' \qquad \text{such that } \underline{A}\,\underline{\lambda}' = 0 \tag{19}$$

will be called an admissible yielding cycle. A prime will mark all the quantities (displacements, plastic strains, residual stresses, etc.) that would be generated by the plastic strains $\underline{N}\,\underline{\lambda}'$, should they be imposed as initial strain rates on the structure supposed elastic.

The following sufficient and necessary conditions for inadaptation can be proved.

(A) The structure does not shakedown (i.e. inadaptation will occur) if, for all t^*, $\underline{u}_o^*$, $\dot{\underline{u}}_o^*$ an admissible cycle starting at $t_1 \geqslant t^*$ can be found such that:

$$\int_{t_1}^{t_2} (\underline{\tilde{F}} + \underline{\tilde{F}}_I^* + \underline{\tilde{F}}_v^*)\, \dot{\underline{u}}^{p'} dt + \int_{t_1}^{t_2} \underline{\tilde{D}}\, \dot{\underline{Q}}^{p'} dt > \underline{\tilde{k}}\, \underline{\lambda}' \qquad (20)$$

(B) When $\underline{A}$ is symmetric and positive semidefinite if the structure does not shakedown, then for all t^*, $\underline{u}_o^*$, $\dot{\underline{u}}_o^*$ the inequality:

$$\int_{t_1}^{t_2} (\underline{\tilde{F}} + \underline{\tilde{F}}_I^* + \underline{\tilde{F}}_v^*)\, \dot{\underline{u}}^{p'} dt + \int_{t_1}^{t_2} \underline{\tilde{D}}\, \dot{\underline{Q}}^{p'} dt \leqslant \underline{\tilde{k}}\, \underline{\lambda}' \qquad (21)$$

will be violated by some admissible cycle starting at $t_1 \geqslant t^*$.

In Eqs.(20) and (21) $\dot{\underline{u}}^{p'}$ denotes a vector of "plastic" displacement rates generated by the admissible cycle; $\dot{\underline{Q}}^{p'}$ denotes the vector of corresponding stress rates.

The presence of t^* in the statements needs a comment. According to the mathematical definition (17) of the shakedown phenomenon, the structure will certainly shakedown if an external action history that would lead to inadaptation of indefinitely repeated, smooths down and ceases after a certain time; this shows that fulfillment of Eq.(20) for some t_1 would not necessarily imply inadaptation. The proofs of statements (A) and (B), omitted here for brevity, rest on a linear programming formulation of the search for the safety factor on the basis of the first dynamic shakedown theorem.

By assuming $\underline{F}_I^* = \underline{F}_v^* = \underline{0}$, both Statements (A) and (B) coincide with the corresponding theorems for the quasi-static range (Maier,[10]). The inequality to which (20) reduces has to be satisfied for at least one of the external actions sequences that are expected to occur during the life time of the structure. When also $\underline{H} = \underline{0}$ (perfect plasticity) and $\underline{D} = \underline{0}$ (no dislocations) the two statements further specialize to Koiter's classical shakedown theorem (Koiter,[2]). When constant external actions are considered, the kinematic theorem of limit analysis is obtained. It can be easily verified that in this case the term involving $\underline{D}$ disappears from Eqs.(20),(21), as well expected. It is worth noting that the restriction that matrix $\underline{A}$ Eq.(19) be symmetric positive semidefinite, intervenes only in statement B (necessary condition for inadaptation, i.e. sufficient for shakedown), not in statement A.

This restriction rules out hardening rules with nonsymmetric $\underline{H}$ and implies nonnegative second order work in any incremental process whatever set of yielding mode may become active (overall stability see Maier, [10]. The above requirement is always fulfilled in perfect plasticity $\underline{H} = \underline{0}$ in the absence of geometric effects on the equilibrium equations. It is readily seen that, when $\underline{A}$ is symmetric positive semidefinite, the characterization Eq.(19) of admissible cycle implies the following two circumstances: (i) $\underline{H}\,\underline{\lambda}' = 0$, i.e. all yield loci must return at t_2 to the shapes and positions they had at t_1; (ii) $\underline{Z}\,\underline{N}\,\underline{\lambda}' = 0$, i.e. the total plastic strains generated by the cycle must be stressless (compatible).

4. Procedures of Dynamic Shakedown Analysis Resting on the Inadaptation Theorems.

Making use of the virtual work principle, the following equation can be proved:

$$\int_{t_1}^{t_2} (\tilde{F} + \tilde{F}_2^{*} + \tilde{F}_v^{*})\,\dot{u}^{p\,\prime}\,dt + \int_{t_1}^{t_2} D\dot{Q}^{p\,\prime}\,dt = \int_{t_1}^{t_2} \underline{\tilde{Q}}^{e*}\,\underline{N}\,\underline{\dot{\lambda}}'\,dt \qquad (22)$$

where $\underline{Q}^{e*}(t)$ represents the stress history in the fictitious process with the initial conditions $\underline{u}_o^{*}, \underline{\dot{u}}_o^{*}$. The "vector of maximal projections" will be defined with reference to all yielding modes (fig.3) as:

$$\underline{\tilde{M}}^{*} \equiv [\ldots M_j^{i} \ldots\ldots],\text{with } M_j^{i} = \max_{t \geq t^{*}} \tilde{N}_j^{i}\,\underline{Q}^{e*}(t) \qquad (23)$$

Within the class of all admissible yielding cycles with $t_1 \geq t^{*}$, it is clearly possible to single out at least one for which each plastic multiplier rate $\dot{\lambda}_j^{i}$ becomes $\neq 0$ only when $\tilde{N}_j^{i} Q^{e*}(t)$ attains its maximum M_j^{i} over $t \geq t^{*}$ (maximal cycles). For such cyclea the work defined by Eq.(22) can be expressed also in the form:

$$\int_{t_1}^{t_2} \underline{\tilde{Q}}^{e*}\,\underline{N}\,\underline{\dot{\lambda}}'\,dt = \underline{\tilde{M}}^{*}\,\underline{\lambda}' \qquad (24)$$

and it exceeds the corresponding work pertaining to any other nonnmaximal cycle. Consider the linear programming problem:

$$\left.\begin{array}{l} \omega(\underline{u}_o^{*}, \underline{\dot{u}}_o^{*}, t^{*}) = \min\limits_{\lambda'} \underline{\tilde{K}}\,\underline{\lambda}' \quad , \text{ subject to:} \\ \underline{M}^{*}\,\underline{\lambda}' = 1, \qquad \underline{A}\,\underline{\lambda}' = \underline{0}, \qquad \underline{\lambda}' \geq \underline{0} \end{array}\right\} \qquad (25)$$

Note that ω is a function $\underline{u}_o^{*}, \underline{\dot{u}}_o^{*}$, t^{*} because vector $\underline{M}^{*}$ depends on these quantities through $\underline{Q}^{e*}$ (t). Let $\xi \geq 0$ be a common multiplier of the external actions and initial condition. Safety factor s is the value such that for $\xi < s$ the system shakes down, for $\xi > s$ it does not.

On the basis of the inadaptation theorems of Sec.3, it can be shown that the solution of the linear program (25) provides the following information on s.

(a) Systems for which $\underline{A}$ is symmetric positive semidefinite and the external actions are periodic. Let vector $\underline{M}^*$ be calculated if $V \neq 0$ on the basis of the uniqualy defined periodic stress response $Q^{e*}(t)$ in a fictitious process under the given F(t), D(t); if $\underline{V} = \underline{0}$, on the basis of the stress response in the forced vibration of the system. Then the optimal value ω of problem (25) turns.out to coincide with s.

(b) Structures with positive semidefinite $\underline{A}$, under general (nonperiodic) external actions. Then the optimal value ω calculated according to Eq.(25) on the basis of an arbitrary choice of $\underline{u}_o^*, \dot{\underline{u}}_o^*, t^*$ represents a lower bound for s ($\omega \leq s$).

(c) Systems with general $\underline{A}$, subjected by periodic external actions. The optimum ω obtained as in (a) for $\underline{V}=\underline{0}$ or $\underline{V} \neq \underline{0}$ is an upper bound on s ($\omega \geq s$).

No information can be derived from (25) for s in cases for which neither $\underline{A}$ is symmetric positive semidefinite not the external actions are periodic.

It is worth noting that ω (and, hence, s when the restriction on $\underline{A}$ is fulfilled) becomes ∞ if $\underline{A}$ is positive definite. The linear programming formulation (25) could be seen to be dual (in the sense of the optimization theory) to the formulation of the shakedown analysis on the basis of the generalized Melan-Ceradini theorem. Finally, note that in perfect plasticity ($\underline{H} = \underline{0}$) problem (25) can be conveniently cast in the form:

$$\omega = \min_{\lambda'} \tilde{\underline{k}} \underline{\lambda}' \quad , \text{ subject to:} \qquad \underline{M}^* \lambda' = 1 \; , \; \underline{C}\,\underline{u} = \underline{N}\,\underline{\lambda}' \quad , \quad \underline{\lambda}' \geq \underline{0} \tag{26}$$

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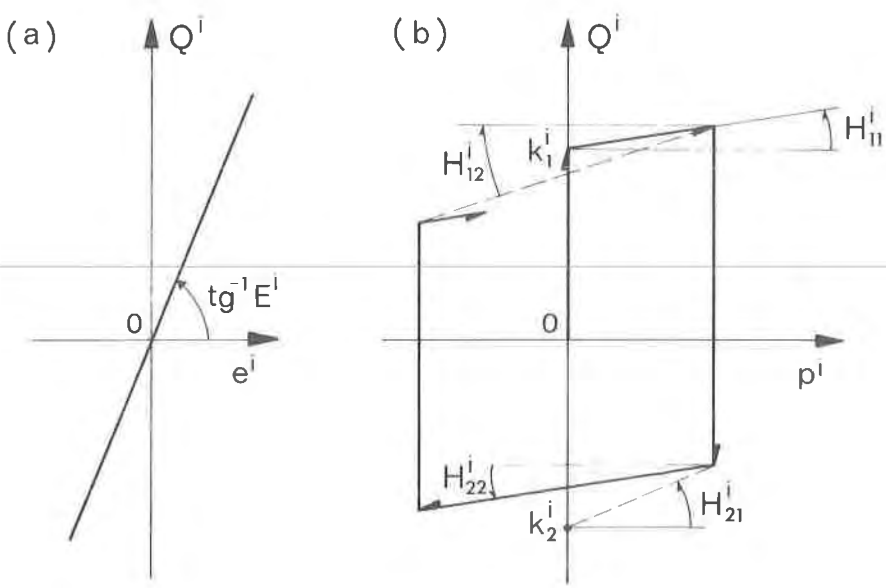

fig. 1

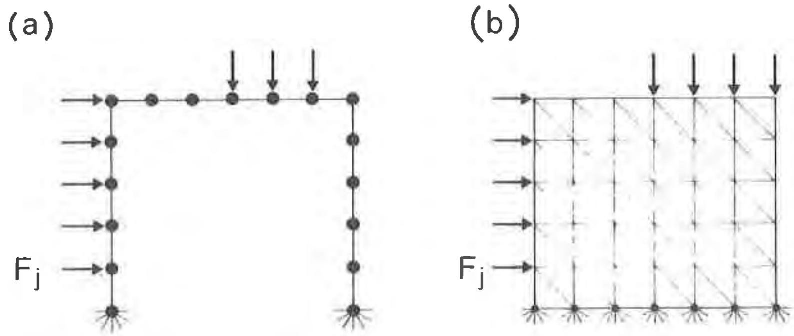
(a)
(b)
F_j
F_j

fig. 2

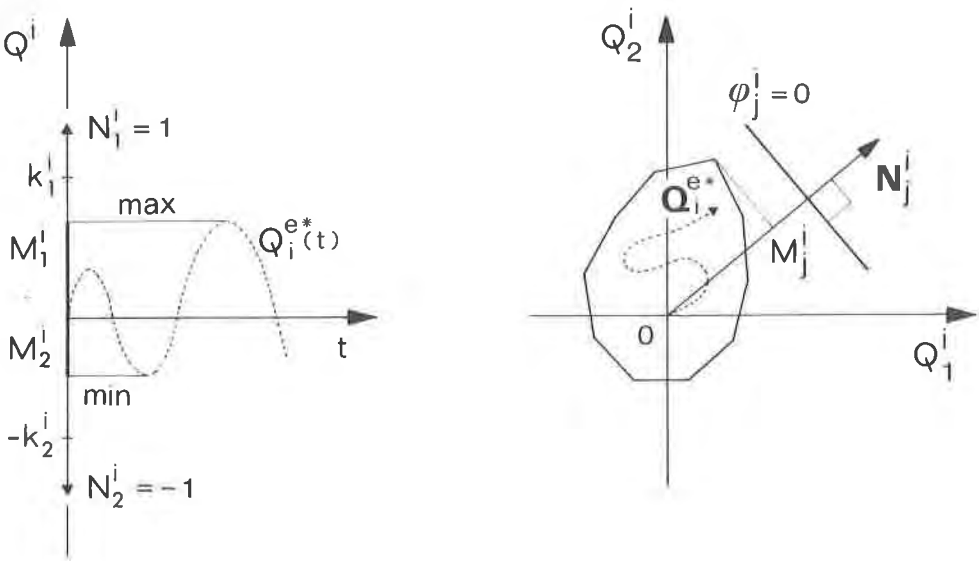
Q^i
$N_1^i = 1$
k_1^i
max
M_1^i
$Q_i^{e*}(t)$
M_2^i
t
min
$-k_2^i$
$N_2^i = -1$
Q_2^i
$\varphi_j^i = 0$
$\mathbf{N}_j^i$
$\mathbf{Q}_i^{e*}$
M_j^i
0
Q_1^i

fig. 3