CONVERGENCE OF LUMPED FINITE ELEMENT SCHEMES FOR SELECTED INITIAL VALUE PROBLEMS

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SUMMARY

In the finite element analysis of dynamic systems, the evaluation of inertia force is usually made by the employed shape function itself, while that of stiffness is done by the spatial derivatives of the shape function. This fact considerably increases the complexity in calculation of mass matrices (or conductivity ones in the case of heat equation). Therefore, lumping is often made to overcome such shortcomings of the consistent formulation and to reduce the bandwidth of the mass matrices.

Its theoretical justification is, however, not yet sufficient in general forms except in some very simple finite element schemes. Therefore, the aim of this paper is to propose general criteria for convergence of lumped finite element schemes for selected initial value (or mixed initial-boundary value) problems. For this purpose, the result obtained by Tong *et al.* in eigenvalue problems is very helpful. It is also to be noted that the lumping in the present discussion means that of the employed finite element basis, and not that of the consistent matrices calculated.

The examples of equations to which the present criteria can be applied are:

(1) heat equation: $\mathrm{d}u/\mathrm{d}t + Au = f$, $u(0) = e$,
(2) wave equation: $\mathrm{d}^2u/\mathrm{d}t^2 + Au = f$, $u(0) = e$, $\mathrm{d}u/\mathrm{d}t(0) = g$,
(3) Schroedinger's equation: $\mathrm{d}u/\mathrm{d}t + iAu = f$, $u(0) = e$,

in which i is imaginary unit, and A is a spatial operator which is independent of time t, self-adjoint and positive-definite in a suitable real or complex Hilbert space H with its domain $D(A)$ being dense there.

The finite element schemes discussed here are the modified Faedo-Galerkin schemes in which the dynamic forces are evaluated by lumped basis. In order to assure the convergence of the schemes, we must impose as criteria some relations between the consistent and lumped expressions of the same deflection fields. These conditions consist of equivalence of norms of both expressions, which is necessary to assure the stability of the schemes, and some consistency or completeness conditions for shape functions. Then, the convergence of the schemes can be easily derived by utilizing fundamental techniques in functional analysis. Some cares to be taken in choosing lumped expressions are also given so as not to reduce the convergence rates of original consistent schemes. It is especially to be noted that the proposed criteria are usually satisfied in popular conforming shape functions of polynomial type.

1. Introduction

In the finite element analysis of dynamic phenomena, the evaluation of inertia force is usually made by the employed shape function itself, while that of stiffness by its spatial derivatives. This fact considerably increases the complexity involved in the calculation of mass matrices (or Gram's matrices in general evolution equations). Consequently, lumping is often employed to remove such a shortcoming of consistent formulation and to reduce the bandwidths of mass matrices (Clough [1]).*)

The thoretical justification of the above-mentioned procedure does not appear to be fully developed except in several types of fairly simple finite element schemes (Krieg and Key [2], Fujii [3]). Therefore, the aim of this paper is to propose general criteria for convergence of lumped finite element schemes for typical initial value (or mixed initial-boundary value) problems in mathematical physics. For this purpose, the results derived by Tong et al. ([4]) are very useful. Some cares to be taken in the choice of lumped bases are also presented so as not to reduce the convergence rate of the original consistent schemes.

2. Boundary value problems

We will first discuss typical boundary value problems with some mathematical preliminaries and convergence criteria provided.

2.1 Formulation

Let Q be a domain in R^n and H be a real or complex $L_2(Q)$-space. The scalar product and norm of H are, respectively,

$$(u,v) = \int_Q u\bar{v}\, dQ \quad , \quad \|u\| = \sqrt{(u,u)} \quad , \tag{1}$$

where both u and v are square-summable functions defined on Q and $\bar{v}$ denotes complex conjugate of v (Heider and Simpson [5]).

We are to be mainly concerned with a linear operator A which is self-adjoint and positive definite in H in the sense of Mikhlin ([6]):

$$(Au,v) = (u,Av) \quad \text{for } \forall u,\, v \in D(A) \; ; \quad (Au,u) \geqq C_1^2 \|u\|^2 \quad \text{for } \forall u \in D(A), \tag{2}$$

in which D(A) is the domain of definition of A, which is dense in H, C_1 a positive constant independent of u, and the boundary conditions for A are assumed to be homogeneous. The energy space associated with A is designated by H_A with its scalar product and norm respectively denoted as $\langle\, ,\, \rangle$ and $|||\ \ |||$.

Let us consider a boundary value problem

$$Au = f \qquad \text{for} \quad f \in H \quad , \tag{3}$$

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*) It is to be noted that the lumping in the present discussions means that of the employed finite element basis and not that of the calculated consistent matrix. The latter in general yields diagonal mass matrices, while the former does not necessarily give such ones except in very simple finite element schemes (Kikuchi and Ando [7]).

in which u is the exact solution and f a given function defined on Q. We will require that both f and the boundary of Q are sufficiently smooth so that u may be sufficiently smooth on $\bar{Q}$ = closure of Q.

In order to solve eq. (3) approximately by the finite element method, Q is decomposed into finite elements $\{Q_i^h\}_{i=1}^{m_h}$ on which a suitable finite element basis $\{\phi_{hj}\}_{j=1}^{n_h}$ is to be chosen. Here, h is a representative mesh size such as the maximum diameter of finite elements, and m_h and n_h are number of elements and dimension of the space spanned by this basis, respectively. We assume that each $\phi_{hj} \in H_A$, and the finite element space spanned by $\{\phi_{hj}\}$ will be refered as S^h ($\subset H_A \subset H$).

Next we consider another basis $\{\phi_{hj}^L\}_{j=1}^{n_h}$ corresponding to $\{\phi_{hj}\}$. It will be called a lumped basis and the space spanned by it will be denoted by S_L^h. It is of course required that $S_L^h \subset H$, but not necessarily that $S_L^h \subset H_A$. An element of S^h shall be designated as U_h, V_h etc., while the corresponding one in S_L^h as U_h^L, V_h^L etc. That is, we have the following relation to hold between U_h and U_h^L :

$$U_h = \sum_{j=1}^{n_h} a_{hj}\,\phi_{hj} \quad , \qquad U_h^L = \sum_{j=1}^{n_h} a_{hj}\,\phi_{hj}^L \quad , \tag{4}$$

where the coefficients a_{hj}'s are common to both consistent and lumped expressions.

Let us define consistent and lumped finite element schemes for eq. (3):

(1) Consistent scheme : The finite element solution U_h is determined by

$$\langle U_h, V_h \rangle = (f, V_h) \qquad \text{for} \quad \forall V_h \in S^h . \tag{5}$$

(2) Lumped scheme : The finite element solution $\bar{U}_h$ is defined by

$$\langle \bar{U}_h, V_h \rangle = (f, V_h^L) \qquad \text{for} \quad \forall V_h \in S^h \quad (\forall V_h^L \in S_L^h) \tag{6}$$

The existence and uniqueness of the above-defined finite element solutions can be easily proved thanks to the relation in eq. (2), while the error estimation of the consistent solution U_h can be obtained according to the standard procedure (Mikhlin [6]). Some examples of lumped bases may be found in Refs.[3], [4] and [7] .

2.2 Convergence criteria and error estimations

In order to assure the convergence of the lumped finite element schemes, some approximation properties and mutual relations should be required for S^h and S_L^h. In this sub-section, a set of convergence criteria is to be presented for sufficiently smooth solution of eq. (3).

The conditions employed throughout this paper are :

(I) For $\forall U_h \in S^h$ and the corresponding $U_h^L \in S_L^h$ holds the relation

$$C_2 \, \| U_h^L \| \leqq \| U_h \| \leqq C_3 \, \| U_h^L \| \quad , \tag{7}$$

in which both $C_2 > 0$ and $C_3 > 0$ are constants independent of U_h and h.

(II) For a sufficiently smooth exact solution of eq. (3), there exist $\hat{U}_h \in S^h$

and $\hat{U}_h^L \in S_L^h$ such that $$\|\hat{U}_h - u\| \leqq C_4(u)\, h^p \quad , \tag{8}$$
$$\|\hat{U}_h^L - u\| \leqq C_5(u)\, h^q \quad , \tag{9}$$

where $C_4(u) > 0$ and $C_5(u) > 0$ depend only on u, and $p > 0$ and $q > 0$ are independent of u and h.

(III) Any $V_h \in S^h$ satisfies $$|(f, V_h^L - V_h)| \leqq C_6(f)\, h^r \, \|V_h\| \quad , \tag{10}$$

where $C_6(u) > 0$ depends only on f and $r > 0$ is independent of V_h and h.

As a consequence of the condition (II), we find
$$\|\hat{U}_h - u\| \leqq C_1^{-1} C_4(u)\, h^p \tag{11}$$
thanks to the following relation derived from ineq. (2) :
$$\|v\| \geqq C_1 \|v\| \qquad \text{for each } v \in H_A \quad . \tag{12}$$
It is also to be noted that a sufficient condition for (III) is

(III)* $$\|V_h - V_h^L\| \leqq C_7\, h^r \, \|V_h\| \qquad \text{for each } V_h \in S^h \; . \tag{13}$$

Here, $C_7 > 0$ is independent of V_h and h.

The above condition may be conveniently employed when H_A is equivalent to a certain Sobolev space ([8]). Ineq. (10) may be easily derived from ineq. (13) by utilizing Cauchy-Schwartz' inequality.

It is especially worth mentioning that the above criteria are usually satisfied in popular conforming shape functions of polynomial type. A concrete example is given by Fujii ([3]) with regard to the piecewise linear shape function and the corresponding piecewise constant one for R^n-simplex element, and the present discussions may be regarded as a generalization of his results.

Next we will consider the convergence of the lumped finite element scheme under the above-mentioned conditions.

Theorem-1 The finite element solution $\overline{U}_h \in S^h$ defined by eq. (6) satisfies the error estimations
$$\|\overline{U}_h - u\| \leqq C_4(u)\, h^p + C_6(f)\, h^r \quad , \tag{14}$$
$$\|\overline{U}_h^L - u\| \leqq C_1^{-1}(C_4(u)\, h^p + C_6(u)\, h^r) \quad , \tag{15}$$
$$\|\overline{U}_h - u\| \leqq C_1^{-1} C_2^{-1}(2C_4(u) h^p + C_6(f) h^r) + C_5(u) h^q \quad . \tag{16}$$

Proof We have from eqs. (5) and (6)
$$\|U_h - \overline{U}_h\|^2 = (f, U_h - \overline{U}_h - U_h^L + \overline{U}_h^L) \quad . \tag{17}$$
Applying ineq. (10) to the above yields
$$\|U_h - \overline{U}_h\| \leqq C_6(f) h^r \quad . \tag{18}$$
Because U_h is the best approximation of u in S^h in the metric of H_A, we have
$$\|U_h - u\| \leqq C_4(u) h^p \tag{19}$$
from ineq. (8). Applying the triangle inequality to $\|\overline{U}_h - u\|$ with ineqs. (18) and (19) leads to ineq. (14). Inequality (15) is obvious from ineq. (12). On the other hand, we find
$$\|\overline{U}_h^L - u\| \leqq \|\overline{U}_h^L - \hat{U}_h^L\| + \|\hat{U}_h^L - u\| \leqq C_2^{-1}\|\overline{U}_h - \hat{U}_h\| + \|\hat{U}_h^L - u\|$$

$$\leqq C_2^{-1}(\| \overline{U}_h - u \| + \| \hat{U}_h - u \|) + \| \hat{U}_h^L - u \| \qquad (20)$$

by utilizing ineq. (7) and the triangle inequality. Substituting the estimations (9), (11) and (15) into ineq. (20), ineq.(16) may be immediately obtained. This completes the proof.

As seen from Theorem-1, lumping does not essentially bring loss of accuracy if $r \geqq p$ at least in the metric of H_A. The above results for boundary value problems will play an important role in the discussions of mixed initial-boundary value problems given in the following section. They might be also useful for evaluating errors induced by lumping of equivalent nodal forces as given in Ref.[9] .

3. Mixed initial-boundary value problems

In this section, we will treat three types of mixed initial-boundary value problems in mathematical physics :

(1) heat equation : $du/dt + Au = f \quad (0 \leqq t \leqq T) \ , \ u(0) = g \ .$ (21)

(2) wave equation : $d^2u/dt^2 + Au = f \quad (0 \leqq t \leqq T) \ ,$ $u(0) = g \ , \ du/dt(0) = e \ .$ (22)

(3) Schrödinger's equation : $du/dt + i^* Au = f \quad (0 \leqq t \leqq T) \ , \ u(0) = g$.(23)*)

In the above, t is time, T upper bound of t, i^* imaginary unit, A is the same as introduced in section 2 and independent of t, and $f(t) \in H$, $g \in H_A$ and $e \in H$. We will again be interested in sufficiently smooth solutions of the above equations.

The convergence criteria in section 2-2 are to be employed here again, and the problems are to be considered in the spaces $L_\infty(0,T;H)$ and $L_\infty(0,T;H_A)$ with their norms respectively defined by

$$[\![u]\!] = \sup_{0 \leqq t \leqq T} \| u(t) \| \quad (< +\infty) \qquad (24)$$

and

$$|\!|\!| u |\!|\!| = \sup_{0 \leqq t \leqq T} ||| u(t) ||| \quad (< +\infty) \ . \qquad (25)$$

For simplicity, only Schrödinger's equation will be treated in this section, while the other two may be done similarly.

3.1 Formulation

The finite element solution $U_h(t) \in S^h$ is to be determined by solving

$$(dU_h^L/dt, V_h^L) + i^* \langle U_h, V_h \rangle = (f, V_h^L) \quad \text{for } \forall V_h \in S^h \text{ and } \forall t \in [0,T] \qquad (26\text{-}1)$$

with approximate initial condition $U_h(0) = G_h \in S^h$. (26-2)

We choose G_h in such a way that it satisfies ineqs.(8) and (9), that is,

$$||| G_h - g ||| \leqq C_4(g)\, h^p \ , \ \| G_h - g \| \leqq C_5(g)\, h^q \ . \qquad (27)$$

The above-introduced scheme is a kind of modified Faedo-Galerkin's scheme (Lions [10]) in which the external source and the time derivative are evaluated by the lumped basis and no discretization is made with respect to t. The existence and uniquness of the solution of the approximate equation may be easily proved by the general theory for ordinary differential equations of normal form (Pontrjagin [11]).

*) The source term f is usually zero in quantum mechanics.

3.2 Stability

This sub-section deals with the stability of the finite element solution by means of energy method.

Theorem-2 (stability of finite element solution)

The finite element solution U_h defined by eq. (26) is stable in the following sense :

$$[\![U_h^L]\!] \leqq C_8(\| G_h^L \| + [\![f]\!]) \quad , \tag{28}$$

$$[\![U_h]\!] \leqq C_9(\| G_h \| + [\![f]\!]) \quad , \tag{29}$$

in which C_8 and C_9 are positive constant dependent only on T.

Proof Equating V_h to U_h in eq.(26-1), we get

$$(dU_h^L/dt, U_h^L) + i^* \| U_h \|^2 = (f, U_h^L) \quad . \tag{30}$$

Taking the real part of the above equation and using some familiar inequalities yield

$$d/dt \| U_h^L \|^2 \leqq [\![f]\!]^2 + \| U_h^L \|^2 \quad . \tag{31}$$

Solving this inequality, we easily have ineq.(28), from which follows ineq. (29) due to the relations in ineq. (7). This concludes the proof.

The above theorem shows the stability in $L_\infty(0,T;H)$, and its results are to be made full use of in the error estimations of the finite element solution. It is evident that the present discussion requires only the condition (I) in section 2.2.

Furthermore, we have the following result in the case when f = 0.

Theorem-3 The finite element solution U_h of eq. (26) with f = 0 satisfies

$$\| U_h(t) \| = \| G_h \| \quad \text{for } {}^\forall t \in [0,T] \text{ , hence } [\![U_h]\!] = \| G_h \| . \tag{32}$$

Proof Equating V_h to dU_h/dt in eq.(26-1) gives

$$\| dU_h^L/dt \|^2 + i^* \langle U_h, dU_h/dt \rangle = 0 \quad . \tag{33}$$

Taking the imaginary part of the above, we find that

$$\mathrm{Re} \langle U_h, dU_h/dt \rangle = \tfrac{1}{2}\, d/dt \| U_h \|^2 = 0 \quad , \tag{34}$$

from which we obtain the desired results. This completes the proof.

3.3 Convergence

Now we are going to evaluate the convergence rate of the finite element solution by utilizing the results in the preceding sub-section.

Theorem-4 (Error estimations of finite element solution)

For a sufficiently smooth exact solution u of eq. (23), the finite element solution $U_h(t) \in S^h$ constructed by the present lumped scheme satisfies the following error error estimations under the assumptions made in sections 2 and 3:

$$[\![U_h - u]\!] \leqq C_{10}(u)\ (h^p + h^q + h^r) \quad , \tag{35-1}$$

$$[\![U_h^L - u]\!] \leqq C_{11}(u)\ (h^p + h^q + h^r) \quad , \tag{35-2}$$

where the positive constants $C_{10}(u)$ and $C_{11}(u)$ depend only on T and u (and f).

Proof Let us define $\widetilde{U}_h(t) \in S^h$ by

$$\langle \widetilde{U}_h, V_h \rangle = 1/i^*(f - du/dt, V_h^L) \qquad \text{for } {}^\forall V_h \in S^h . \tag{36}$$

Applying the results of Theorem-1 to the above gives

$$\|\tilde{U}_h - u\| \leqq C_{12}(u)(h^p + h^r) \quad , \tag{37-1}$$

$$\|\tilde{U}_h^L - u\| \leqq C_{13}(u)(h^p + h^q + h^r) \quad , \tag{37-2}$$

where $C_{12}(u)$ and $C_{13}(u)$ are positive constants dependent only on u.
Differentiating eq. (36) in t yields

$$i^*\langle d\tilde{U}_h/dt, V_h\rangle = (df/dt - d^2u/dt^2, V_h^L) \quad \text{for } \forall V_h \in S^h, \tag{38}$$

from which we obtain

$$\|d\tilde{U}_h^L/dt - du/dt\| \leqq C_{13}(du/dt)(h^p + h^q + h^r) \quad . \tag{39}$$

Subtracting eq. (36) from eq. (26-1) provides with

$$(d\Delta U_h^L/dt, V_h^L) + i^*\langle \Delta U_h, V_h\rangle = (d\,\Delta u/dt, V_h^L) \quad \text{for } \forall V_h \in S^h, \tag{40}$$

in which

$$\Delta U_h = U_h - \tilde{U}_h \quad , \quad \Delta u = u - \tilde{U}_h^L \quad . \tag{41}$$

Applying the results of Theorem-2 to eq. (40) with ineqs. (27), (37) and (39) taken into account, we can easily obtain ineq. (35), thus the proof being completed.

As seen from Theorem-3, the order of errors is essentially $O(h^s)$ with $s = \min(p,q,r)$, from which we notice that the condition (III) plays a fundamental role in time-dependent problems like in stationary ones. On the other hand, that of the original consistent scheme is apparently $O(h^p)$, so lumping brings no loss of accuracy if $q \geqq p$ and $r \geqq p$. However, it appears that the condition $r \geqq p$ becomes more difficult to hold as p increases. If so, lumping will undoutedly loose one of its merits, i.e., achievement of simplification of schemes without sacrificing convergence rate, especially when finite elements of higher accuracy are employed.

3.4 Comments on heat and wave equations

Similar techniques to those employed in the preceding sub-sections are available for both wave and heat equations with essentially the same results derived.

In the case of heat equation, we can further obtain

$$\| U_h \| \leqq C_{14}(|||G_h||| + \|f\|) \tag{42}$$

and

$$\| U_h - u \| \leqq C_{15}(h^p + h^q + h^r) \tag{43}$$

for the finite element solution $U_h(t) \in S^h$ determined by an approximate equation similar to eq. (26) and the exact solution u of eq. (21).

Similarly, we have for wave equation

$$K(U_h(t)) \leqq C_{16}(K(U_h(0) + \|f\|) \tag{44}$$

and

$$K(U_h(t) - u(t)) \leqq C_{17}(h^p + h^q + h^r) \quad , \tag{45}$$

where

$$K(u(t)) = \sqrt{\|du/dt(t)\|^2 + |||u(t)|||^2} \quad . \tag{46}$$

Therefore, the error estimations in section 3.3 remain the same irrespectively of the types of evolution equations.

3.5 Applications of the present theory

This sub-section deals with some concrete examples of finite element schemes to which the present theory is applicable.

As a first application, let us consider the simplex finite element in R^n with the piecewise linear basis and the corresponding piecewise constant lumped one. In this approximation, the value of U_h^L at a point P in Q_i^h with simplex coordinates $(z_1, z_2, \cdots, z_n, z_{n+1})$ is equal to the value of U_h at α -th node of Q_i^h if $0 \leqq z_k \leqq \frac{1}{2}$ for $k = 1,2,\cdots,n+1$ with $k \neq \alpha$. If H_A is taken to be equivalent to $W_2^1(Q)$, we can easily establish the condition (I) and obtain that $p = q = r = 1$ under the so-called uniformity condition for mesh division (Strang 12). The cases for n = 1, 2 are treated by Fujii (3) in connection with wave equations.

As a second example, we consider the HCT triangular finite element scheme for plate bending (Clough and Tocher [13]) provided with the lumped basis in Ref.[7]. In this element, the conforming shape functions are piecewise incomplete polynomials of third order, while the lumped shape functions are complete linear polynomials in each sub-element as shown in Fig. . If H_A is equivalent to $W_2^2(Q)$ with $Q \subset R^2$ and the uniformity condition holds, we can again assure the condition (I) and find that $p = 1$ and $q = r = 2$.

Therefore, it can be concluded that both of the above approximations suffer no loss of convergence rates as far as the results of section 3.3 are concerned.

4. Concluding remarks

The convergence of lumped finite element schemes for mixed initial-boundary value problems has been discussed with the method of error estimations provided. Although the proposed criteria seem to be fairly general for practical purposes, it might be of future interest whether the condition (III) can be replaced with better ones which enable to improve error estimations. The choice of suitable finite difference schemes for time derivatives is also to be studied to assure the feasibility of the finite element method for time dependent phenomena, and the present results will be expected to give basic informations for this purpose. Some examples of schmes for Schrödinger's equation are given in Ref.[14] with numerical results.
Finally it appears that the lumping approximation has close relations to the so-called partial approximation in "abstract finite element method" employed by mathematicians. (Aubin[15]).

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Appendix Lumped expression of shape functions for triangular plate bending element

The distribution of u(x,y) in the sub-region i-$\overline{k+3}$-G-$\overline{j+3}$ shown in Fig. 1 is approximated by

$$u = u_i + 2A\left\{(c_k L_j - c_j L_k)\frac{\partial u}{\partial x}\Big|_i + (b_j L_k - b_k L_j)\frac{\partial u}{\partial y}\Big|_i\right\} , \qquad \text{(A-1)}$$

in which

L_i = area coordinate (i = 1, 2, 3),

$u_i, \frac{\partial u}{\partial x}\Big|_i, \frac{\partial u}{\partial y}\Big|_i$ = values of u, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ at node i (=1, 2, 3),

A = area of triangle 1-2-3,

$b_i = \partial L_i/\partial x$,

$c_i = \partial L_i/\partial y$,

G = center of gravity of triangle 1-2-3,

4, 5, 6 = midpoints of sides of triangle,

(i,j,k) = even permutation of (1,2,3).

The above lumped expression can be used together with the HCT basis.

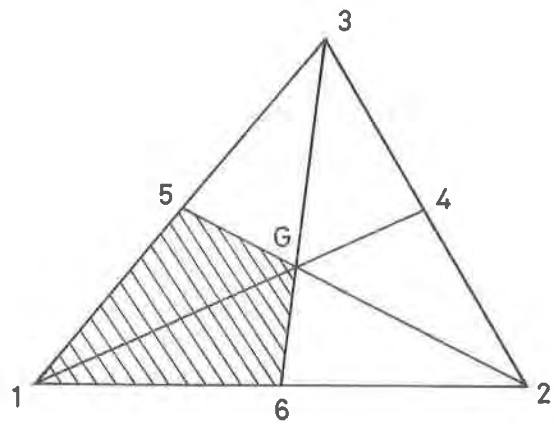

Fig. 1 Triangular finite element divided into sub-regions