FINITE ELEMENT ELASTOPLASTIC ANALYSIS BY QUADRATIC PROGRAMMING: THE MULTISTAGE METHOD

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SUMMARY

Recent studies have shown that the quadratic programming approach to the elastoplastic analysis of structures may be competitive with respect to the well-known iterative asymptotical solution procedures (such as "tangent-modulus", "initial strain", "initial stress") for discretized elastoplastic systems. When loads increase proportionally, local unloading phenomena can be conjectured to be either absent (regular progressive yielding) or immaterial. Then holonomic (reversible) law can be assumed and, through a suitable piecewise linearization of the hardening or perfectly plastic constitutive laws, the problem can be solved in finite terms as a single convex quadratic programming problem by some standard technique.

This paper develops the quadratic programming approach for finite element models of elastoplastic continua subjected to non-proportional loading path, i.e. to significant local unloading phenomena. The nonproportional loading path is subdivided into a set of individually proportional stages. The structural response to each loading stage is determined by solving a convex quadratic program formulated on the basis of a suitable piecewise linearization of the constitutive laws. As in all other approaches, the path-dependency of the plastic behavior is not allowed for within each loading step. However, in passing from one stage to the subsequent one, the irreversible nonholonomic nature of plastic deformations is fully allowed for, simply by a suitable adjustment of the piecewise linear constitutive laws to be assumed for the subsequent stage.

The determination of the structural response to each loading step is formulated as a quadratic programming problem in two alternative ways: in the former the unknowns are the nonnegative plastic multipliers which govern the additional plastic strains developed in the step; in the latter the unknowns are the above plastic multipliers and the additional displacements. The two formulations are compared both from the theoretical and from the computational standpoint. The extension of both formulations to cases where geometry changes affect significantly the equilibrium relations is pointed out.

The method is applied to some plane stress and plane strain problems whose solutions by other approaches are available, so that comparisons become possible and are made. The quadratic programming solutions are achieved by an algorithm with finite termination (Beale's) and by a modified conjugate gradient method, and the two techniques are compared to each other.

1. Introduction

In elastoplastic analysis of discrete or discretized structures, the determination of the infinitesimal incremental response (in terms of rates) at a generic situation has been reduced in various ways to a problem in quadratic programming (see e.g. [1] [2] [3] [4]). The rate solution is used to obtain the response in terms of finite (though small) increments, in order to follow by step integration the evolution of the system under a given loading history.

Accumulative violations of the yield conditions or other difficulties arise in passing from rates to finite increments. With piecewise linear yield surfaces a finite step response can be obtained exactly by multiplying by a time interval Δt the rate solution, provided that Δt be kept small enough so that the stress point nowhere crosses a yield plane. Clearly, this criterion often leads to an exceedingly high step number for structures with many elements and many yielding modes. It is generally advantageous to subdivide "a priori" the loading history into steps whose number be independent of the discretization pattern. This is achieved in the present method, which involves piecewise linearization of the constitutive laws and rests on quadratic programming formulations, but flows from some concepts in holonomic,"deformation" plasticity theory (see e.g. [1] [3][6]) rather than from rate solutions. The irreversible, pathdependent (nonholonomic) nature of the plastic strains, is fully allowed for from stage to stage, even if not so within each step. In this respect the procedure proposed herein is similar to the methods extensively used so far in elastoplastic analysis, i.e. "tangential stiffness","initial strain" and "initial stress methods" [7] [8] [9] . However the present approach leads to numerical techniques with finite termination and known convergence characteristics.

Notation: underlined symbols denote matrices and column vectors, $\underline{0}$ is a matrix or vector whose entries are zero. A tilde $\sim$ means transpose, a dot time derivative. A vector inequality applies to each pair of corresponding components.

2. Piecewise linear constitutive laws.

2.1. Let $\underline{q}$ and $\underline{Q}$ denote c-vectors of true or generalized strains and corresponding stresses. They correspond to each other in the sense that $\tilde{\underline{Q}}\,\dot{\underline{q}}\,dt$ is proportional to the first order work performed on the material or structural element concerned. $\underline{Q}$ and $\underline{q}$ are assumed as independent and intrinsic quantities: the c components q_r are unaffected by rigid body motions, the Q_r are self-equilibrated ("natural" variables, [7]).

Consider the following set of relations:

$$\underline{q} = \underline{e} + \underline{p} \qquad (1)$$

$$\underline{e} = \underline{S}^{-1}\underline{Q} \qquad (2)$$

$$\underline{\varphi} = \tilde{\underline{N}}\,\underline{Q} - \underline{H}\,\underline{\lambda} - \underline{k} \qquad (3)$$

$$\underline{\varphi} \leq \underline{0} \qquad (4)$$

$$\underline{p} = \underline{N}\,\underline{\lambda} \qquad (5)$$

$$\dot{\underline{\lambda}} \geq \underline{0} \qquad (6)$$

$$\tilde{\underline{\varphi}}\,\dot{\underline{\lambda}} = \underline{0} \qquad (7)$$

$$\tilde{\dot{\underline{\varphi}}}\,\dot{\underline{\lambda}} = \underline{0} \qquad (8)$$

Eq.(1) subdivides $\underline{q}$ into the elastic strains $\underline{e}$ end plastic strains $\underline{p}$. The former part $\underline{e}$ is characterized by Eq.(2), where $\underline{S}$ represents the elastic stiffness matrix, assumed as symmetric, positive definite. Eq.(3) defines the y-vector $\underline{\varphi}$ of the "yield functions" or "plastic potentials" as linear function of $\underline{Q}$ and of the y-vector $\underline{\lambda}$ of "plastic multipliers", which are parameters depending on the previous history of plastic deformations: $\underline{k}$ is a vector of positive constants, $\underline{N}$ and $\underline{H}$ are constant matrices. Eqs.(3) and (4) combined define the polyhedrical (piecewise linear) instantaneous yield surface, for the instant t at which the plastic multipliers are $\underline{\lambda}$. Eq.(5) specifies that the plastic strain vector $\underline{p}$ is a linear combination, through $\underline{\lambda}$, of the outward normal vectors to the yield planes (which form the faces of the above defined yield locus).

Eqs.(6)-(8) represent flow rules: Eq.(6) requires that the contribution to $\dot{\underline{p}}$ of each yield plane is directed as the outward vector to that plane.

Eq.(7), because of (4) and (6), implies $\varphi_j \dot{\lambda}_j = 0$ for any j and, hence, selects as potentially active at any t only the y'yield planes which contain the current stress point $\underline{Q}$ ($\varphi_j = 0$ for j=1...y'). Eq. (8) rules out that plastic flow ($\dot{\lambda}_j > 0$) and "local unloading" ($\dot{\varphi}_j < 0$, i.e. the fact that the stress point looses contact with the j-th plane) occur simultaneously for the same yield plane. In fact, as a consequence of (4), $\dot{\underline{\varphi}}' \leq \underline{0}$ for the y' activable yield planes, $\dot{\underline{\lambda}}'' = \underline{0}$ for the y"= y - y' non activable yield planes (such that $\varphi_j < 0$, j=1...y''): it follows that Eq.(8) implies $\dot{\varphi}_j \dot{\lambda}_j = 0$ for any j.

The workhardening matrix $\underline{H}$ governs the translations of the yield planes due to plastic flow. Many hardening rules may be implemented by suitable choice of $\underline{H}$. We will refer here to the class corresponding to symmetric, positive semidefinite matrices $\underline{H}$, among which the most important in practice are:

perfect plasticity, $\underline{H} = \underline{0}$; Koiter's rule of non-interacting yield modes, $\underline{H} = \text{diag}\left[H_j\right]$: Prager's kinematic hardening rule $\underline{H} = H\,\underline{\tilde{N}}\,\underline{N}$, with $H > 0$, [6] .

As a conclusion, the relation set (1)-(8) describes a broad class of path-dependent (nonholonomic) stress-strain relationships, characterized by translating-interacting yield planes [6] . Most elastic-plastic laws are amenable to an approximate description of this type, through suitable piecewiselinearizations.

2.2. Assume a stress path $\underline{0} \rightarrow \underline{Q}$ along which the stress point does not loose contact with any yield plane after this has been activated: in other terms let any yielding-unloading sequence be ruled out ("regular progression" path).

This means that $\lambda_j > 0$ implies $\dot{\varphi}_j = 0$ for any j and t; hence $\dot{\tilde{\varphi}}\underline{\lambda} = 0$ which, combined with Eq.(7), requires that:

$$\tilde{\underline{\varphi}}\,\underline{\lambda} = 0 \qquad (9)$$

It follows from Eq.(6) that

$$\underline{\lambda} \geqslant \underline{0} \qquad (10)$$

The strains $\underline{q}$ generated by a regular progression path $\underline{0} \rightarrow \underline{Q}$,turn out to depend on $\underline{Q}$ through the relations (1)-(4), (9), and (10). These relations are all in finite terms and path-independent and will be called piecewise linear holonomic laws "associated" to the nonholonomic ones defined by Eqs. (1)-(8).

Consider now a stress path $\underline{Q}_a \longrightarrow \underline{Q} = \underline{Q}_a + \Delta\underline{Q}$ and again rule out yielding unloading sequences within this path (but not unloading with respect to yield planes possibly activated before it starts in $\underline{Q}_a$). It is readily seen that the stress and strain increments $\Delta\underline{Q}$ and $\Delta\underline{q}$ are related to each other by the following laws:

$$\Delta\underline{q} = \underline{S}^{-1}\Delta\underline{Q} + \underline{N}\,\Delta\underline{\lambda} \qquad (11)$$

$$\Delta\underline{\varphi} = \underline{\tilde{N}}\,\Delta\underline{Q} - \underline{H}\,\Delta\underline{\lambda} \qquad (12)$$

$$\Delta\underline{\varphi} \leqslant -\underline{\varphi}_a \qquad (13)$$

$$\Delta\underline{\lambda} \geqslant \underline{0} \qquad (14)$$

$$\Delta\tilde{\underline{\varphi}}\,\Delta\underline{\lambda} = -\tilde{\underline{\varphi}}_a\,\Delta\underline{\lambda} \qquad (15)$$

Eq.(11) is an obvious consequence of Eqs.(1)(2)(4). Eq.(13), where $\underline{\varphi}_a$ denotes the yield function vector at the starting instant t_a, flows from Eq.(3); Eq.(14) from (6). The regular regression hypothesis within the path means that $\dot{\varphi}_j\,\Delta\lambda_j = 0$ for any j at any $t \geqslant t_a$; hence $\dot{\tilde{\underline{\varphi}}}\,\Delta\underline{\lambda} = 0$. By associating this equation to Eq.(7) written as $(\tilde{\underline{\varphi}}_a + \Delta\tilde{\underline{\varphi}})\,\Delta\dot{\underline{\lambda}} = 0$, and by integrating with respect to time, we obtain Eq.(15). The relation set (11)-(15) governing finite increments of the variables can be called for its very origin, the "stepwise holonomic" laws associated to the nonholonomic" laws Eqs. (1)-(8).

3. Linear complementarity problem formulations.

Let $\underline{u}$ represent the vector of the n free independent displacements of a discrete structural model; $\underline{F}$ will denote the vector of corresponding external force components. Let us subdivide a given loading history $\underline{F}(t)$ in s stages, each of which is, or can be regarded as, individually proportional:

$$\underline{F}(0)\ldots\underline{F}(t_h)\ldots\underline{F}(t_n)\ ,\ \Delta\underline{F}_h \equiv \underline{F}(t_h) - \underline{F}(t_{h-1}) \tag{16}$$
$$\underline{F}(t) = \underline{F}(t_{h-1}) + (t - t_{h-1})\,\Delta\underline{F}_h\,\Delta t_h^{-1} \quad \text{for } t_{h-1} \leq t \leq t_h$$

Assume that each one of the, say, m constituents of the structure in point is governed by stress-strain laws of the type Eqs.(1)-(8). The "element behaviour" in terms of "natural" generalised stresses and strains has a straightforward meaning and flows directly from the material behavior, for structures like trusses and constant strain finite element models (triangles or tetrahedra) of two-or threedimensional continua [7] . These structures will be referred to here for brevity; however, the procedure proposed can be extended with inessential adjustements to any discrete structural model.

Let $\bar{\underline{q}}$, $\bar{\underline{Q}}$, $\bar{\underline{\varphi}}$, $\bar{\underline{\lambda}}$ be supervectors formed by collecting in the same order the vectors $\underline{q}, \underline{Q}, \underline{\varphi}, \underline{\lambda}$ pertaining to all elements; e.g. $\tilde{\bar{\underline{q}}} \equiv \left[\tilde{\underline{q}}^1\ldots\tilde{\underline{q}}^m\right]$. The structural response to a generic h-th loading stage will be governed, under the small deformation hypothesis, by the compatibility and equilibrium equations (subscript h omitted for simplicity):

$$\Delta\bar{\underline{q}} = \underline{C}\cdot\Delta\underline{u} \tag{17}$$
$$\tilde{\underline{C}}\cdot\Delta\bar{\underline{Q}} = \Delta\underline{F} \tag{18}$$

Under the hypothesis that no sequence yielding-unloading occurs within the h-th loading stage, the stepwise holonomic laws Eqs.(11)-(15) will be used. These can be condensed into a single relation set for all the m constituents, if one defines the block-diagonal matrices $\bar{\underline{N}} = \text{diag}\left[\underline{N}^i\right]$, $\bar{\underline{H}} = \text{diag}\left[\underline{H}^i\right]$, $\bar{\underline{S}} \equiv \text{diag}\left[\underline{S}^i\right]$. Through trivial changes of Eqs.(11)-(15), such as the substitution $\underline{\varphi} = \underline{\varphi}_{h-1} + \Delta\underline{\varphi}$, the condensed relation set reads:

$$\Delta\bar{\underline{Q}} = \bar{\underline{S}}\ \Delta\bar{\underline{q}} - \bar{\underline{S}}\,\bar{\underline{N}}\,\Delta\bar{\underline{\lambda}} \tag{19}$$
$$\bar{\underline{\varphi}} = \tilde{\bar{\underline{N}}}\ \Delta\bar{\underline{Q}} - \bar{\underline{H}}\ \Delta\bar{\underline{\lambda}} + \bar{\underline{\varphi}}_{h-1} \tag{20}$$
$$\bar{\underline{\varphi}} \leq \underline{0}\ ,\quad \Delta\bar{\underline{\lambda}} \geq \underline{0}\ ,\quad \tilde{\bar{\underline{\varphi}}}\Delta\bar{\underline{\lambda}} = 0 \tag{21}$$

Substituting Eq.(17) into (19) and, thereafter, Eq.(19) into (18) and (20), we obtain:

$$\underline{K}\ \Delta\underline{u} - \tilde{\underline{C}}\ \bar{\underline{S}}\ \bar{\underline{N}}\cdot\Delta\bar{\underline{\lambda}} = \Delta\underline{F} \qquad (22)$$

$$\bar{\underline{\varphi}} = \tilde{\bar{\underline{N}}}\ \bar{\underline{S}}\ \underline{C}\cdot\Delta\underline{u} - (\bar{\underline{H}}+\tilde{\bar{\underline{N}}}\bar{\underline{S}}\bar{\underline{N}})\cdot\Delta\underline{\lambda} + \bar{\underline{\varphi}}_{h-1} \qquad (23)$$

where: $$\underline{K} = \tilde{\underline{C}}\ \bar{\underline{S}}\ \underline{C} \qquad (24)$$

If the compatibility matrix $\underline{C}$ has rank n (no rigid body motions in the structure) the elastic stiffness matrix K of the assembled structure is non-singular. Then Eq.(22) can be solved with respect to $\Delta\underline{u}$ and substituted into Eq.(23):

$$\bar{\underline{\varphi}} = \tilde{\bar{\underline{N}}}\ \Delta\bar{\underline{Q}}^{e} - \underline{A}\cdot\Delta\underline{\lambda} + \bar{\underline{\varphi}}_{h-1} \qquad (25)$$

where: $\Delta\bar{\underline{Q}}^{e} \equiv \bar{\underline{S}}\ \underline{C}\ \underline{K}^{-1}\Delta\underline{F}$, $\underline{A} \equiv \bar{\underline{H}} - \tilde{\bar{\underline{N}}}\ \underline{Z}\ \bar{\underline{N}}$, with

$$\underline{Z} \equiv \bar{\underline{S}}\ \underline{C}\ \underline{K}^{-1}\tilde{\underline{C}}\ \bar{\underline{S}} - \bar{\underline{S}} \qquad (26)$$

$\Delta\bar{\underline{Q}}^{e}$ represents the linear elastic stress response to $\Delta\underline{F}$. The symmetric matrix $\underline{Z}$ which appears in Eq.(26) can be recognized, from Eqs.(11)(17)(18), as the matrix which transformes inelastic strains into selfstresses:

$$\Delta\bar{\underline{Q}}^{s} = \underline{Z}\cdot\Delta\bar{\underline{p}} \qquad (27)$$

The relation set (21) (22) (23) in the unknown vector $\Delta\underline{u}$, $\Delta\bar{\underline{\lambda}}$, $\Delta\bar{\underline{\varphi}}$ and the relation set (21)(25) in the unknown vectors $\Delta\bar{\underline{\lambda}}$, $\Delta\bar{\underline{\varphi}}$ represent alternative formulations of the analysis problem for a single loading stage. The latter exhibits the mathematical structure of a linear complementarity problem (LCP): two sign-constrained orthogonal vectors, one expressed as a linear mapping of the other [10] . The former relation set can be reduced to the L C P form through trivial artifices.

4. Quadratic programming formulations.

It is well known and easy to prove [11] , that a LCP with symmetric positive semi-definite matrix can be interpreted as Kuhn-Tucker conditions of a convex quadratic programming (QP) problem. This problem, therefore, is equivalent to the LCP.

The QP formulation that can be derived in this way from the LCP (21)(25) reads:

$$\min\ \frac{1}{2}\ \Delta\tilde{\bar{\underline{\lambda}}}\ \underline{A}\ \Delta\bar{\underline{\lambda}} - \Delta\tilde{\bar{\underline{\lambda}}}(\tilde{\bar{\underline{N}}}\cdot\Delta\bar{\underline{Q}}^{e} + \bar{\underline{\varphi}}_{h-1}) \qquad (27)$$

$$\text{subject to:} \qquad \Delta\bar{\underline{\lambda}} \geqslant \underline{0} \qquad (28)$$

The convex QP problem which follows through a parallel derivation from Eq.(21)(22)(23), can be expressed in the form:

$$\min\ \frac{1}{2}\Delta\tilde{\bar{\underline{\lambda}}}\bar{\underline{H}}\ \Delta\bar{\underline{\lambda}} + \frac{1}{2}\Delta\tilde{\bar{\underline{e}}}\ \bar{\underline{S}}\cdot\Delta\bar{\underline{e}} - \tilde{\bar{\underline{\varphi}}}_{h-1}\Delta\bar{\underline{\lambda}} - \Delta\tilde{\underline{F}}\ \Delta\underline{u} \qquad (29)$$

$$\text{subject to:} \quad \Delta\bar{\underline{e}} = \underline{C}\ \Delta\underline{u} - \bar{\underline{N}}\Delta\bar{\underline{\lambda}}\ , \quad \Delta\bar{\underline{\lambda}} \geqslant \underline{0} \qquad (30)$$

Both QP problems (27)(28) and (29)(30) may be dualized so that two further characterizations of the holonomic stage solutions can be obtained. All four QP formulations thus achieved can be regarded as the counterparts for the single-stage analysis with stepwise holonomic laws, of the extremum theorems established by Maier for the analysis problem with piecewise linear holonomic laws, [3][5][6].

5. Main features of the multistage procedure.

The multistage method of elastoplastic analysis consists of the following phases: (a) determination of the entities $\underline{N}$,$\underline{K}$ and $\underline{H}$ which define the assumed piecewise linear approximation of the actual constitutive law; (b) approximation of the given, generally nonproportional, loading history by a sequence of individually proportional stages in accordance with Eq.(16), where $\underline{F}(0)$ will be the load condition at the onset of yielding ("elastic limit"); (c) solution in sequence of the LCP or QP problems for the various stage responses under the hypothesis of stepwise holonomic constitutive laws, updating vector $\bar{\underline{\varphi}}_{h-1}$ from stage to stage; (d) calculation of the total quantities by adding the stage contributions, e.g.: $\underline{u} = \sum_{1}^{s} {}_h \Delta \underline{u}_h$.

The following circumstances are worth noting.

(α) Besides the discretization and the approximate description of the material behavior, the only source of errors intrinsic in the procedure is represented by the possibility of a yielding-unloading sequence within a single loading stage. However, this possibility, which is common to all finite-step methods, turns out to be rather remote, inasmuch each stage is a proportional, monotonous loading process; moreover its consequences on the overall response are in general small, and can be reduced by reducing the step amplitude. On the other hand the step amplitude is not determined by the activation of a new yield plane, as in other methods (see e.g. [12]).

(β) The QP formulations of the nonlinear stage problem are more interesting than the LCP formulations; conceptually, because they express extremal properties of the solution; computationally, because QP algorithms seem to be in general more efficient than those dealing directly with LCP formulations. Among the latter class of techniques, generalizations of the Gauss-Seidel asymptotic-iterative procedure (Hildreth-D'Esopo [11], Cryer [13]) are particularly promising. Within the former class of QP algorithms, the generalizations of the "simplex" method of linear programming (e.g. Beale's 11) terminate in a finite number of pivotal transformations and are implemented in a number of commercial codes.

(γ) The formulations involving vector $\Delta \underline{u}$, in contrast to those which do not involve it, do not require any matrix inversion in forming the data of the LCP or QP problem, but are less economical to solve because of the larger number of variables. A remarkable economy has been achieved by solving the QP problem (27) (28) in $\Delta\bar{\lambda}$ only by a modified conjugate algorithm without forming and storing matrix $\underline{Z}$, Eq.(26), but calculating directly a selfstress distribution, Eq.(27), in order to form at each optimization step the gradient of the objective function (27) (De Donato and Franchi, [14]).

(δ) The size of the LCP or QP problem to be solved for each stage can be drastically reduced by ignoring in the formulation the yield planes which are unlikely to be activated during that stage, say the h-th. These planes can be singled out as those for which $\varphi_{h-1} < \tau$, τ being a tolerance to select on the basis of the elastic stress increments $\Delta\bar{\underline{Q}}^e$ or of an estimate of them. Obvious checks and possible reconsideration of modified problems, rule out any error from this size reduction.

(ε) The extension of the present approach to allow for large displacements (but small strains) can be carried out by adding in the l.h.s. of the equilibrium equation (18) the term $\underline{G}\,\Delta\underline{u}$, $\underline{G}$ being the geometric stiffess matrix depending on the stress state $\bar{\underline{Q}}_{h-1}$ at the beginning of the stage h. From stage to stage both $\underline{G}$ and $\underline{C}$ must be updated to take account of the geometry changes. Clearly, the use of equilibrium equations linear in both $\Delta\bar{\underline{Q}}$ and $\Delta\underline{u}$ ("second order" effects) is acceptable only for sufficiently small load steps. Attention has to be paid to the fact that, in this generalized context, the matrices of the LCP formulations are no longer guaranteed to be positive semidefinite (see [15]).

6. Examples

The plane -stress structure represented in fig.1 (previously studied under proportional loads in [14] [15]) has been analysed under cyclic loads. Isotropic linear elasticity with E=1000 kg/mm^2, ν=0, and Mises' yield criterion have been assumed. Fig.2 shows the piecewise linearization of the original yield surface $\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y+3\tau_{xy}^2=\sigma_o^2$, with σ_o=32 kg/mm^2. The discretization adopted (158 constant-strain finite elements, 97 nodes) appears in fig.3. The vertical load W distributed parabolically on AD, see fig.1, has been varied as indicated in figs.4 and 5, where the abscissae are the vertical displacements at node O, the ordinates represent the load factor k defined as the ratio between the moment M=WL at the section BC and the limit moment of that section. Three hardening rules have been considered, as specified in figs.4-5.

The amplitude of the load cycle was subdivided in three stages (15 for the whole path),marked by circles in fig.4 and 5. The QP problems were solved by the modified conjugate gradient technique of Ref.[14],using the size reduction (δ) mentioned in Sec.5 and ending the iterations when $\underline{\tilde{\bar{\varphi}}}\,\Delta\underline{\bar{\lambda}}\,(\mathrm{mod}\,\underline{\bar{\varphi}}\cdot\mathrm{mod}\,\Delta\underline{\bar{\lambda}})^{-1} < 10^{-6}$. The average step number in a single optimization was 12, the average CPU time on a computer UNIVAC 1108 was 33 sec. per stage. Fig.6 shows the stresses on the section BC in various situations(labeled in the same way in figs.4-5). Excellent agreement was found by comparing the curve A for ideal plasticity to a result available in [16].

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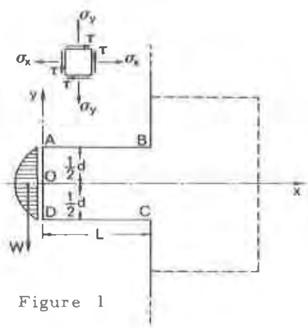


Figure 1

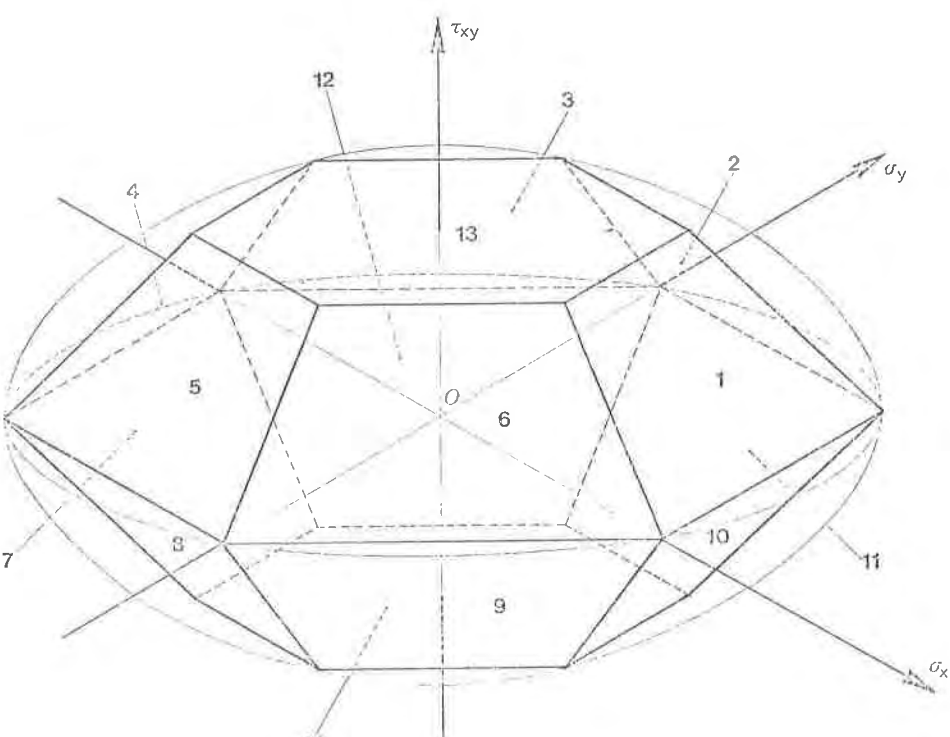


Figure 2

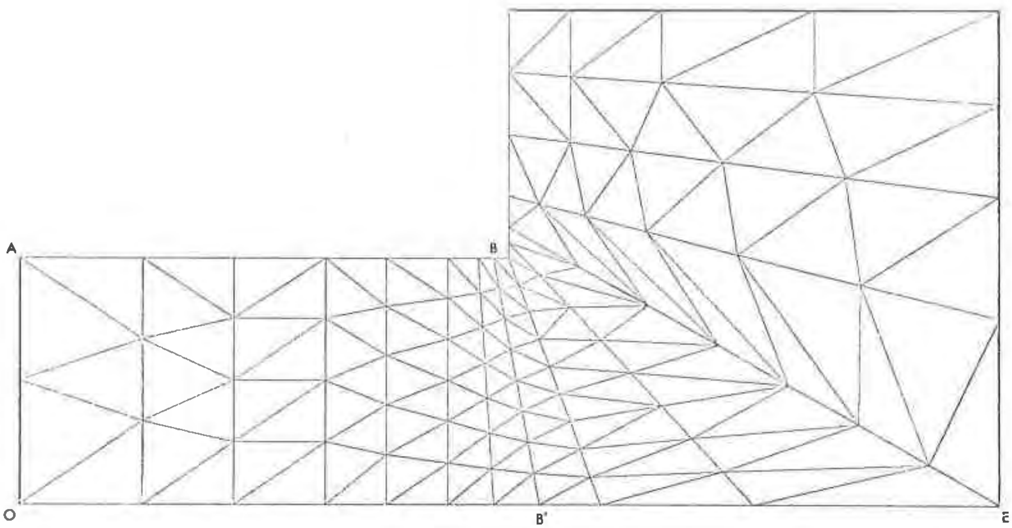
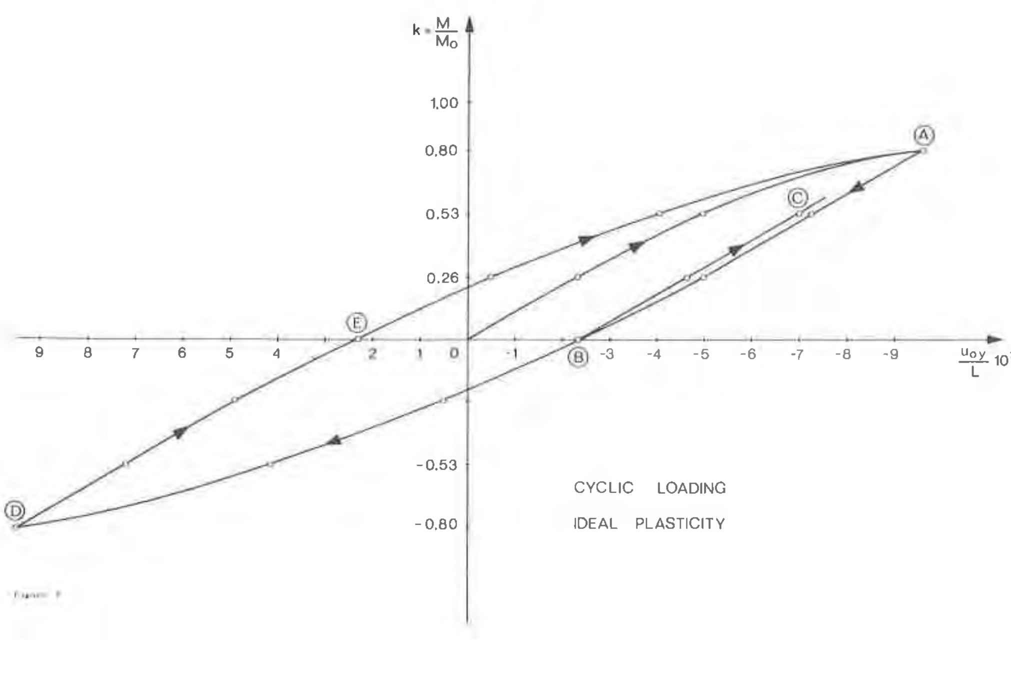


Figure 3



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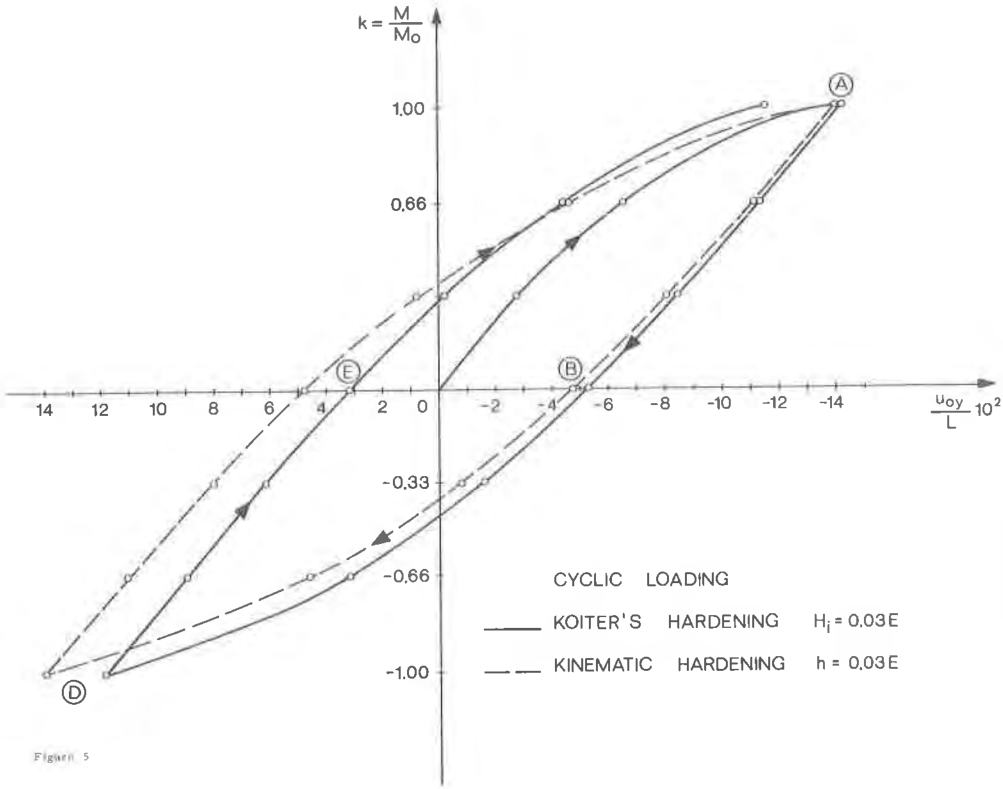

Figure 5

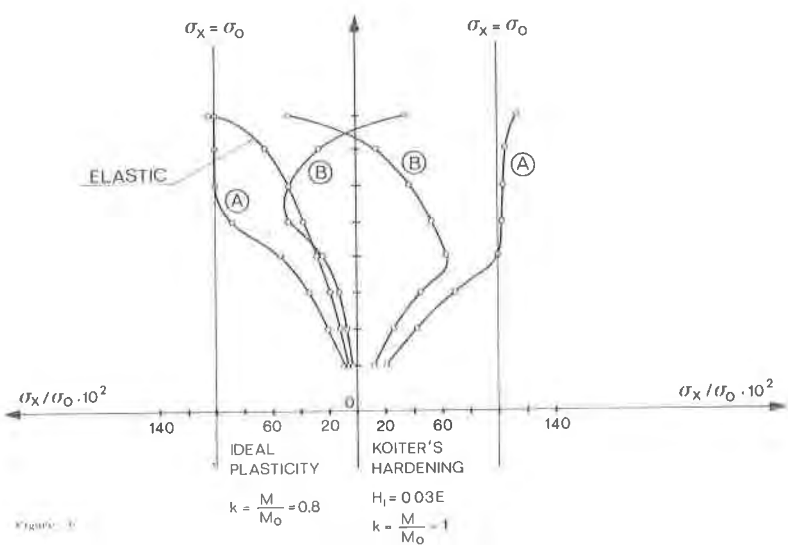

Figure 6