'NONSAP' — A GENERAL FINITE ELEMENT PROGRAM FOR NONLINEAR DYNAMIC ANALYSIS OF COMPLEX STRUCTURES

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SUMMARY

In this paper the recently developed very effective computer program NONSAP for the nonlinear static and dynamic analysis of complex structures is presented. The program is very flexible and was designed to be extended and modified by the user. In particular, realizing that active research is still being carried out in the field of nonlinear dynamic analysis, the program can easily be modified to use a different formulation of the equations of motions, different (new) time integration operators and other additional options. The program, therefore, is an excellent research tool, but, in particular, with the specific formulation and time integration operator currently used, it has proven to be very powerful in a variety of practical analyses.

In the first part of the paper, the incremental formulation currently used in the program is briefly reviewed. Specifically, recent experiences with problems of accuracy and stability of the formulation are discussed. It is realized that, in a dynamic analysis, an equilibrium iteration can be very important. Attention is also briefly directed to the current possibilities of representing the stress-strain relationships of various materials.

The general program organization is presented next. The program is an in-core solver. The capacity of the program is essentially determined by the total number of degrees of freedom in the system. However, there is practically no limit on the total number of finite elements used. In the solution, low-speed storage is used to store all information pertaining to the finite elements, which may be updated in each time step, and to store the constant structure matrices. To obtain maximum program capacity, the finite elements are processed in blocks according to their type and whether they are linear or nonlinear elements. The linear structure matrices (stiffness, mass and damping) are assembled in sequence using the same high speed storage before the time integration is performed. In each time step the effective stiffness matrix and the effective load vector are updated for the nonlinearities present in the system. All matrices are stored in compacted form, i.e. only nonzero elements are processed, resulting into maximum system capacity and solution efficiency. The specific scheme used to assemble the matrices in compacted form and to solve the resulting system equations is briefly described in the paper.

Program NONSAP has application in the nonlinear analysis of many different structures. Various sample solutions are presented to show typical analyses and solution times.

1. Introduction

The endeavor to perform nonlinear analyses has steadily increased in recent years [1], [2]. The safety of a structure may be increased and the cost reduced if a nonlinear analysis can be carried out. Primarily, nonlinear analyses of complex structures have become possible through the use of electronic digital computers operating on discrete representations of the actual structure. A very effective discretization procedure has proven to be the finite element method [3]. Based on this method, various large-scale general purpose computer programs with nonlinear capabilities are now in use [4].

The development of a nonlinear finite element analysis program requires knowledge in various scientific areas. The proper formulation of the nonlinear problem and its idealization to a representative finite element system demands a modern background in structural mechanics. For the solution of the equilibrium equations in space and time, stable and efficient numerical techniques need be employed. The efficiency of a nonlinear program depends largely on the optimum usage of computer hardware and software where, specifically, the appropriate allocation of high- and low-speed storage is important.

The earliest attempts to obtain nonlinear analysis programs essentially involved simple modifications of established programs for linear analysis, much in the same way as the linear structural theory was modified to account for nonlinearities. However, to analyze systems with large geometrical and material nonlinearities, the program should be designed specifically for the required iteration process and not merely be an extension of a linear analysis program. Naturally, a linear analysis program should be flexible, easy to modify and extend; however, this applies even more to a nonlinear analysis program. In particular, it should be realized that a great deal of research is still required and currently pursued in the nonlinear static and dynamic analysis of complex structures. Therefore, unless the general nonlinear analysis code is easy to modify, it may be obsolete within a few years after completion.

The nonlinear analysis program NONSAP presented in this paper is designed with two primary objectives. The first aim is the efficient solution of a variety of practical problems with the current capabilities of nonlinear analysis procedures and computer equipment. The second objective is to have a program which can be used effectively in the various research areas pertaining to nonlinear analysis. Because of continuous improvements in nonlinear analysis procedures, both objectives are attained simultaneously by the development of an efficient, modular, and easily modifiable general analysis code. The program is designed for a general incremental solution of nonlinear problems, but, naturally, can also be used for linear analyses.

Corresponding to the three main areas which form the basis of the development of a program, this compact is divided into three parts; namely, the formulation of the incremental equations of motion, the numerical solution of the equations, and the computer program implementation in NONSAP. Various representative sample solutions will be given at the conference presentation of the paper and in its later publication.

2. Formulation of Equations of Motion

2.1 Continuum Mechanics Formulation

The incremental formulation of the equations of motion need be general, because the material shall be allowed to undergo large strains, large displacements and be subjected to

any constitutive laws. Esentially, two different formulations can be used, namely the total Lagrangian formulation and the updated Lagrangian formulation [1],[2]. We summarize here the updated Lagrangian formulation because it is the usual formulation used in NONSAP.

Consider the motion of a body in a Cartesian coordinate system. At time t the coordinates of a particle of the body are (a_1, a_2, a_3) and at time t+Δt the same particle has coordinates (x_1, x_2, x_3) where $x_i = a_i + u_i$, i = 1,2,3, and the u_i are the displacement increments. The coordinates are always measured along the Cartesian coordinate axes.

The stresses and strains of the body at any time are also defined in the fixed Cartesian coordinate system. Our objective is to solve for the displacements, velocities, accelerations, and stresses of the body as it undergoes its motion. Assume that we have solved for the equilibrium position of the body at time t and that we require the solution for time t+Δt where Δt is an increment in time. We develop equations of virtual work from which the solution at time t+Δt can be calculated.

At any discrete time t we consider the following tensor and vector components, all of which are defined in the fixed Cartesian coordinate system [5]:

τ_{ik}^{t} = component of the Cauchy stress tensor defined per unit area of the body at time t. These are physical stresses. The superscript refers to the time at which the quantities are measured.

t_k^t = component of the surface force vector defined per unit area of the body.

f_k^t = component of the body force vector defined per unit mass of the body.

ρ^t = mass density of the body

u_k^t, $\dot{u}_k^t$, $\ddot{u}_k^t$ = components of displacements, velocities, and accelerations of the body.

Considering the body at time t+Δt, the virtual work of the traction and body forces when a small virtual displacement field is imposed can be written as

$$\delta W^{t+\Delta t} = \int_{S^{t+\Delta t}} t_k^{t+\Delta t}\, \delta u_k\, ds^{t+\Delta t} + \int_{V^{t+\Delta t}} \rho^{t+\Delta t}\, f_k^{t+\Delta t}\, \delta u_k\, dv^{t+\Delta t} \tag{1}$$

where the δu_k are small virtual displacements, $S^{t+\Delta t}$ is the surface area of the body, and $V^{t+\Delta t}$ is the volume of the body at time t+Δt. In eq. (1) and the equations to follow, the summation convention of tensor notation is implied. The equilibrium of the body in its interior and at its boundary requires, respectively, that

$$\frac{\partial \tau_{ik}^{t+\Delta t}}{\partial x_i} + \rho^{t+\Delta t}\, f_k^{t+\Delta t} = \rho^{t+\Delta t}\, \ddot{u}_k^{t+\Delta t} \tag{2}$$

and

$$t_k^{t+\Delta t} = \tau_{ik}^{t+\Delta t}\, n_i^{t+\Delta t} \tag{3}$$

where $n_i^{t+\Delta t}$ are the components of the normal to the body surface. Using eqs. (2) and (3) to substitute into eq. (1) for $\rho^{t+\Delta t}\, f_k^{t+\Delta t}$ and $t_k^{t+\Delta t}$, and transforming the area integral into a volume integral using the Gauss theorem, we obtain

$$\delta W^{t+\Delta t} = \int_{V^{t+\Delta t}} \rho^{t+\Delta t}\, \ddot{u}_k^{t+\Delta t}\, \delta u_k\, dv^{t+\Delta t} + \int_{V^{t+\Delta t}} \tau_{ik}^{t+\Delta t}\, \delta e_{ik}^{t+\Delta t}\, dv^{t+\Delta t} \tag{4}$$

where $\delta e_{ik}^{t+\Delta t}$ are virtual small strains corresponding to δu_k, i.e., $\delta e_{ik}^{t+\Delta t} = \delta u_{k,i}$.

The equilibrium equation to be satisfied at time $t+\Delta t$ is therefore

$$\int_{S^{t+\Delta t}} {}_t\tau_k^{t+\Delta t}\, \delta u_k\, dS^{t+\Delta t} + \int_{V^{t+\Delta t}} \rho^{t+\Delta t}\, f_k^{t+\Delta t}\, \delta u_k\, dV^{t+\Delta t}$$

$$- \int_{V^{t+\Delta t}} \rho^{t+\Delta t}\, \ddot{u}_k^{t+\Delta t}\, \delta u_k\, dV^{t+\Delta t} - \int_{V^{t+\Delta t}} \tau_{ik}^{t+\Delta t}\, \delta e_{ik}^{t+\Delta t}\, dV^{t+\Delta t} = 0 \qquad (5)$$

In general, eq. (5) cannot be solved directly for the unknown static and kinematic variables at time $t+\Delta t$. The solution is obtained by using an equation in which the equilibrium at time $t+\Delta t$ is expressed with reference to the known configuration of the body at time t. In this equation we consider the following tensor and vector components acting at time $t+\Delta t$, all of which are also defined in the fixed Cartesian coordinate system:

S_{ik}^t = component of the second Piola Kirchhoff stress tensor defined per unit area of the body at time t. Note that the superscript t implies that the stresses are referred to the configuration of the body at time t.

T_k^t = component of the surface force vector defined per unit surface area of the body at time t.

F_k^t = component of the body force vector defined per unit mass of the body at time t.

Assuming that $\rho^t\, dV^t = \rho^{t+\Delta t}\, dV^{t+\Delta t}$, eqs. (1) to (3) with the variables referred to the known configuration of the body at time t are

$$\delta W^{t+\Delta t} = \int_{S^t} T_k^t\, \delta u_k\, dS^t + \int_{V^t} \rho^t F_k^t\, \delta u_k\, dV^t \qquad (6)$$

$$\frac{\partial}{\partial a_i}\left(S_{ij}^t \frac{\partial x_k}{\partial a_j}\right) + \rho^t F_k^t = \rho^t\, \ddot{u}_k^{t+\Delta t} \qquad (7)$$

$$T_k^t = \left(S_{ij}^t \frac{\partial x_k}{\partial a_j}\right) n_i^t \qquad (8)$$

Substituting into eq. (6) for $\rho^t F_k^t$ and T_k^t from eqs. (7) and (8), respectively, and transforming the area integral to a volume integral using the Gauss theorem, we have

$$\delta W^{t+\Delta t} = \int_{V^t} \rho^t\, \ddot{u}_k^{t+\Delta t}\, \delta u_k\, dV^t + \int_{V^t} S_{ik}^t\, \delta \varepsilon_{ik}^t\, dV^t$$

where $\delta \varepsilon_{ik}^t$ is a virtual variation in the total strain increment ε_{ik}^t,

$$\varepsilon_{ik}^t = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_k} + \frac{\partial u_k}{\partial a_i} + \frac{\partial u_j}{\partial a_k}\frac{\partial u_j}{\partial a_i}\right) \qquad (9)$$

The equilibrium equation for the body at time $t+\Delta t$ is therefore

$$\int_{S^t} T_k^t\, \delta u_k\, dS^t + \int_{V^t} \rho^t F_k^t\, \delta u_k\, dV^t = \int_{V^t} \rho^t\, \ddot{u}_k^{t+\Delta t}\, \delta u_k\, dV^t + \int_{V^t} S_{ik}^t\, \delta \varepsilon_{ik}^t\, dV^t \qquad (10)$$

For solution we define an increment in stress σ^t_{ik} measured per unit area of the body at time t, i.e.

$$S^t_{ik} = \tau^t_{ik} + \sigma^t_{ik} \tag{11}$$

and separate the linear and nonlinear strain components in eq. (9) as follows

$$\varepsilon^t_{ik} = e^t_{ik} + \eta^t_{ik} \quad ; \quad e^t_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_k} + \frac{\partial u_k}{\partial a_i}\right) \quad ; \quad \eta^t_{ik} = \frac{1}{2}\left(\frac{\partial u_j}{\partial a_k}\frac{\partial u_j}{\partial a_i}\right) \tag{12}$$

Substituting the expressions in eqs. (11) and (12) into eq. (10) and writing all known quantities as contributions to the loads, we have

$$\int_{S^t} \sigma^t_{ik}\,\delta e^t_{ik}\,dv^t + \int_{V^t} \tau^t_{ik}\,\delta\eta^t_{ik}\,dv^t + \int_{V^t} \sigma^t_{ik}\,\delta\eta^t_{ik}\,dv^t + \int_{V^t} \rho^t\,\ddot{u}^{t+\Delta t}_k\,\delta u_k\,dv^t$$

$$= \int_{S^t} T^t_k\,\delta u_k\,ds^t + \int_{V^t} \rho^t\,F^t_k\,\delta u_k\,dv^t - \int_{V^t} \tau^t_{ik}\,\delta e^t_{ik}\,dv^t \tag{13}$$

To obtain the solution for time $t+\Delta t$, eq. (13) is linearized by assuming that

$$\int_{V^t} \sigma^t_{ik}\,\delta\eta^t_{ik}\,dv^t = 0 \tag{14}$$

and using tangent material properties to express the stress increment σ^t_{ik} in terms of the linear strain increment e^t_{ik}, i.e.

$$\sigma^t_{ik} = D^t e^t_{ik} \tag{15}$$

where D^t is the stress strain matrix at time t with its elements, in general, depending on the solution history.

2.2 Finite Element Formulation .

In NONSAP isoparametric finite element discretization is used [3], in which for a typical element

$$u_i = \sum_{j=1}^{q} h_j\,u^j_i \tag{16}$$

$$a_i = \sum_{j=1}^{q} h_j\,a^j_i \tag{17}$$

where u^j_i and a^j_i, $i = 1,2,3$ and $j = 1,\ldots,q$, are, respectively, the global displacements and coordinates of the q element nodes, and the h_j are interpolation functions.

Consider the finite element solution of eq. (13) for the kinematic and static variables at time $t+\Delta t$. For convenience of writing we omit in this section superscripts indicating time.

The finite element discretization of eq. (13) yields

$$M\,\ddot{u}^{t+\Delta t} + (K_L + K_{NL})\,u = R - F^e \tag{18}$$

where M is the mass matrix, K_L is the linear strain or incremental stiffness matrix, K_{NL} is the nonlinear strain or initial stress stiffness matrix, R is the externally applied load vector, and F^e are the loads already balanced by the stresses in the system. Using a summation sign to indicate the direct stiffness procedure [6], we have

$$\begin{aligned}
M &= \sum_m M^m \quad ; \quad M^m = \int_{V^m} H^{m^T}\,\rho^m\,H^m\,dv^m \\
K_L &= \sum_m K_L^m \quad ; \quad K_L^m = \int_{V^m} B_L^{m^T}\,D^m\,B_L^m\,dv^m \\
K_{NL} &= \sum_m K_{NL}^m \quad ; \quad K_{NL}^m = \int_{V^m} B_{NL}^{m^T}\,T^m\,B_{NL}^m\,dv^m \\
R &= \sum_m R^m \quad ; \quad R^m = \int_{S^m} \tilde{H}^{m^T}\,T^m\,ds^m + \int_{V^m} H^{m^T}\,F^m\,dv^m \\
F^e &= \sum_m F^{e^m} \quad ; \quad F^{e^m} = \int_{V^m} B_L^{m^T}\,T^m\,dv^m
\end{aligned} \tag{19}$$

where H and $\tilde{H}$ are, respectively, body and surface displacement interpolation matrices, B_L and B_{NL} are the linear and nonlinear strain displacement matrices, T is a matrix of the stress components τ_{ij}, T and F are the applied surface and body forces and m indicates a typical element. It should be noted that in eq. (19) all integrations are performed over the volumes and loaded surfaces of the finite elements in their configuration at time t.

The solution of eq. (18) yields displacements, velocities, and accelerations at time $t+\Delta t$ from which the element strains, material laws and stresses at time $t+\Delta t$ are evaluated. This may complete the solution for this time step.

It need be noted, however, that the above solution can only be accurate if eqs. (14) and (15) hold. Experience shows that accurate results can be obtained if the material properties do not change appreciably, the nonlinear strain increments are small and, in particular, only a few time steps are considered. However, errors accumulate in the solution and, depending on the material model used to establish the matrix D in eq. (15) it may be necessary to perform an equilibrium iteration in each or some time steps. In the equilibrium iteration, the error in the solution is evaluated using the finite element discretization form of eq. (5). This equation yields, with the static and kinematic variables for time $t+\Delta t$ obtained above, instead of zero a vector of out-of-balance loads R^r, where

$$R^r = R - M\,\ddot{u}^{t+\Delta t} - F^e \tag{20}$$

and

$$R = \sum R^m \quad ; \quad R^m = \int_{V^m} H^{m^T}\,f^m\,dv^m + \int_{S^m} \tilde{H}^{m^T}\,t^m\,ds^m$$

$$M = \sum M^m \quad ; \quad M^m = \int_{V^m} H^{m^T} \rho^m H^m \, dv^m \tag{21}$$

$$F^e = \sum F^{e^m} \quad ; \quad F^{e^m} = \int_{V^m} B_L^{m^T} T^m \, dv^m$$

In eq. (21) the matrices H and $\tilde{H}$ are as before displacement interpolation matrices and B_L is the linear strain displacement matrix, but all matrices are now defined in the t+Δt - updated configuration of the elements. In particular, in eq. (21) ρ is the mass density, t and f are the surface and body forces and T is the stress matrix in the updated configuration, i.e. T is calculated directly using the material law in this configuration.

The out-of-balance loads in eq. (20) are now used as a load vector in eq. (18) to evaluate corrections to the displacements and accelerations. This iteration process is continued until convergence is obtained, i.e., until eq. (5) is satisfied within a convergence tolerance.

The process of equilibrium iteration is recognized as the search for the solution of eq. (5) with the constant stiffness matrix and mass matrix established in eq. (18). This iteration scheme can be interpreted as a modified Newton iteration because a conventional Newton iteration would be performed if new stiffness and mass matrices were computed in each iteration step [2]. In the equilibrium iteration an initial triangular factorization of an effective stiffness matrix is carried out and the iteration then involves only vector forward reductions and back-substitutions. Computationally, the triangular factorization can be considerably more expensive than the calculation of corrections to displacements, velocities and accelerations. Therefore, it may be desirable to find the solution over a number of time steps using the same mass matrix and stiffness matrix and iterating in each time step until eq. (5) is sufficiently satisfied. Convergence in this iteration is, of course, dependent on the nonlinearities in the system.

2.3 Material Identification

A great deal of research is still required in the identification of material properties for large strains [5]. Nearly all practical nonlinear material laws, such as the isotropic and kinematic hardening models, the cap model, the variable tangent moduli model and the Mohr-Coulomb model, are strictly only applicable to small strains, although often they are also used when the material undergoes large strains. In addition, it is difficult to take the anisotropy of the material into consideration [7] [8].

It need be noted that the procedure for the solution of the system equilibrium equations, eqs. (5) and (13), should also be chosen according to the material model used, i.e., depending on the material law, equilibrium iteration may be required or need not be performed.

3. Numerical Time Integration

In program NONSAP it is currently assumed that the mass matrix M in eq. (18) is constant, i.e., only stiffness nonlinearities are considered. The solution of eq. (18) is obtained using numerical integration. Various procedures could be used [9]. For the solution of linear problems, the accuracy and stability of the integration schemes have been studied. However, additional research is required to evaluate the accuracy and stability

of integration operators applied to nonlinear systems.

Although the properties of the Wilson θ-operator have strictly only been established for the analysis of linear problems, in practice many solutions in nonlinear analysis have been obtained. As the solution scheme is currently used in NONSAP a brief summary is given. In the θ-method a linear variation of acceleration over the time interval $\tau = \theta \Delta t$ is assumed, where for unconditional stability in the analysis of linear problems $\theta \geq 1.37$ [9]. Including now the effect of velocity dependent damping forces [10], the equilibrium equations considered are

$$M \ddot{u}^{t+\tau} + C \dot{u}^{t+\tau} + K^t \Delta u^t = R^t + \theta(R^{t+\Delta t} - R^t) - F^{e^t} \tag{22}$$

where C is a constant damping matrix, $K^t = K_L + K_{NL}$, F^{e^t} are nodal forces equivalent to the system stress distributions, and the superscripts indicate as before the time considered. For solution we use

$$\dot{u}^{t+\tau} = \dot{u}^t + \frac{\tau}{2}(\ddot{u}^{t+\tau} + \ddot{u}^t) \tag{23}$$

$$u^{t+\tau} = u^t + \tau \dot{u}^t + \frac{\tau^2}{6}(\ddot{u}^{t+\tau} + 2\ddot{u}^t) \tag{24}$$

which gives

$$\ddot{u}^{t+\tau} = \frac{6}{\tau^2} \Delta u^t - \frac{6}{\tau} \dot{u}^t - 2\ddot{u}^t \tag{25}$$

$$\dot{u}^{t+\tau} = \frac{3}{\tau} \Delta u^t - 2\dot{u}^t - \frac{\tau}{2} \ddot{u}^t \tag{26}$$

where

$$\Delta u^t = u^{t+\tau} - u^t \tag{27}$$

Substituting the relations for $\ddot{u}^{t+\tau}$ and $\dot{u}^{t+\tau}$ into eq. (22), an equation with Δu^t as the only unknown is obtained. Solving for Δu^t and using the linear acceleration assumption the required vectors $u^{t+\Delta t}$, $\dot{u}^{t+\Delta t}$ and $\ddot{u}^{t+\Delta t}$ are calculated. The complete solution algorithm with the option of equilibrium iteration is summarized in Table I.

4. Program Organization

The solution process is divided into three distinct phases, namely, the input phase, the matrix assemblage phase and the time integration. In each phase an optimum allocation of high speed storage is used. The program is an in-core solver; therefore, the maximum storage required in any one of the phases will govern the system size that can be solved. It should be noted, however, that with small modifications in the program, use can be made of extended core storage.

Throughout the solution the first MAXEST high speed storage locations are reserved for element data which are used in each of the three phases. The elements are divided into groups according to their type and according to whether they are linear or nonlinear elements. The data pertaining to each individual element group need fit into the MAXEST storage locations. Therefore, the minimum that MAXEST should be specified is equal to the locations needed to store the data pertaining to any one of the elements. It follows that the number of linear element groups and the number of nonlinear element groups is determined by the total number of linear and nonlinear elements which constitute the structure, the number of highspeed storage locations MAXEST allocated to element data storage and the number of high speed storage locations needed for each element.

The program has been written for general three-dimensional analysis. Currently, the three-dimensional truss element and two-dimensional plane-stress, plane strain and

axysymmetric isoparametric elements have been incorporated. Work is now initiated for the incorporation of three-dimensional isoparametric elements, although the practical use of general three-dimensional nonlinear analysis with current computer capabilities is still limited.

4.1 Input Phase

The first step in the input phase is to read and generate the restraint array, specifying which of the six possible degrees of freedom at each finite element node need be considered as unknown degrees of freedom. At the same time the x,y,z coordinate arrays of the nodes are read and generated, and the equation numbers are established [6]. Next, the externally applied load vectors for each time step are calculated and stored sequentially on secondary storage. In the last step of the input phase, sequentially, all element informations pertaining to the elements in each group are read and generated. Specifically, the element coordinates, the material properties and the element connection arrays are established. Also, working vectors which store required element strains, stresses and other variables are initialized. For each element group this information is processed in the first MAXEST high speed storage locations and then written together in one block on secondary storage. During the next phases of the solution, therefore, the required element data can be read in blocks, sequentially one block at a time, into the same MAXEST high speed storage locations. Regarding the total number of required high speed storage locations for solution, the system size is usually not governed by the storage locations required in this phase.

4.2 Matrix Assemblage Phase

During the input phase the user has specified if mass and damping effects shall be considered in the analysis, i.e., if a static or dynamic analysis shall be carried out. In this phase, sequentially, the required structure matrices K, $\tilde{K}$, M and C, see eq. (18) and Table I, are assembled and stored on secondary storage. It need be noted that the same high speed storage locations are used for each matrix, thus the maximum high speed storage locations required are essentially MAXEST + NEQ + NWK, where NWK is equal to the number of nonzero elements in K. All matrices are stored in column compacted form [11]. Therefore the variable NWK is also equal to the sum of active column lengths in K. The NEQ storage locations are needed to store the addresses of the diagonal elements in the matrices from which the address of any matrix element can be calculated.

4.3 Time Integration

In this phase the calculations summarized in Table I are carried out. In each time step (which is a load step in a static analysis) the effective load vector is formed and the increments in displacements are calculated, from which the new displacements, velocities and accelerations are obtained. In the calculations sequentially the linear stiffness matrix K, which is used for the evaluation of the linear part of the elastic forces F^{et}, the mass matrix M and the damping matrix C, if required, are read into high speed core for the formation of the effective load vector. Then the linear effective stiffness matrix $\tilde{K}$, or the decomposition of the last formed effective stiffness matrix, is read into high speed core and a pass is made through all nonlinear element groups to update the matrix $\tilde{K}$, if applicable, and the effective load vector for the nonlinearities present in the system. As in the matrix assemblage phase, in these calculations and in the equilibrium iteration, the

same high speed storage locations are used for each structure matrix.

It should be noted that the program being written for the general problem of nonlinear dynamic analysis solves other possible problems without loss of efficiency. In a linear problem, the user does not specify nonlinear element groups, and in a static problem mass and damping effects are not considered. Correspondingly, in the program the respective matrix assemblages and calculations in the time integration are skipped. In nonlinear analysis the user can specify the intervals of effective stiffness reformation and equilibrium interation in order to reduce solution cost.

The high speed storage requirements during the time integration of a dynamic analysis are approximately MAXEST + 8*NEQ + NWK with slightly less storage required in a static analysis. Therefore, the maximum system size that can be analyzed on a given computer is usually determined by these high speed storage requirements. It need be noted, however, that also sufficient secondary storage to process the element informations and structure matrices must be available.

4.4 Calculation of Eigensystem

An important aspect is the selection of an appropriate time step size for the analysis. For the selection of the time step size and for a study of an optimum finite element mesh layout the program has an option to calculate the lowest frequencies of the structure in its unloaded configuration [12]. Also, in any time (or load) step a linearized buckling analysis can be carried out in order to check for the static stability of the system.

4.5 Definition of Material Properties

The element matrices in eqs. (19) and (21) are evaluated using Gauss numerical integration. Therefore, in the evaluation of the element stiffness matrices K_L^m and K_{NL}^m and the element nodal force vectors F^{em} the constitutive relations of the material at the Gauss integration points are required. In NONSAP an entry is provided for the user to supply a subroutine which calculates from the total current strains (or extensions) the tangent stress strain matrix D^m and the current stress matrix T^m. Since some history of strains, stresses and other quantities may be required, the user needs to specify the amount of high speed storage that is needed per integration point in his material specification subroutine. It may be noted that in an analysis using three-dimensional isoparametric elements, the storage required for the calculation of the material properties may be quite extensive, and will largely limit the system size that could be solved on a given computer.

5. Final Remarks

The solution of general nonlinear problems using the finite element method requires an efficient general purpose program, which is designed specifically for the iteration solution procedure. For overall program efficiency the specific aspects of nonlinear analyses and the available computer software and hardware need influence the design of the program.

In this compact the general nonlinear analysis program NONSAP has been presented together with the solution techniques currently used. Apart from being efficient in the solution of practical problems (some of which will be discussed at the conference presentation and in the final publication of the paper), the program is also very effectively used in the different research areas pertaining to nonlinear analyses.

TABLE I SUMMARY OF STEP-BY-STEP INTEGRATION

- - - INITIAL CALCULATIONS - - -

1. Form linear stiffness matrix K, mass matrix M and damping matrix C; initialize u^o, $\dot{u}^o$, $\ddot{u}^o$

2. Calculate the following constants:

   in dynamic analysis $\theta \geq 1.37$, usually $\theta = 1.4$; in static analysis $\theta = 1$; $\tau = \theta \Delta t$;

   tol ≤ 0.01 ; nitem ≥ 3 $a_o = 6/\tau^2$ $a_1 = 3/\tau$ $a_2 = 2a_1$ $a_3 = \tau/2$ $a_4 = a_o/\theta$

   $a_5 = -a_2/\theta$ $a_6 = 1 - 3/\theta$ $a_7 = \Delta t/2$ $a_8 = \Delta t^2/6$

3. Form effective linear stiffness matrix: $\tilde{K} = K + a_o M + a_1 C$

4. In linear analysis triangularize $\tilde{K}$

- - - FOR EACH TIMESTEP - - -

A. IN LINEAR ANALYSIS

(i) Form effective loadvector:

$$\tilde{R}^t = R^t + \theta(R^{t+\Delta t} - R^t) + M(a_o u^t + a_2 \dot{u}^t + 2\ddot{u}^t) + C(a_1 u^t + 2\dot{u}^t + a_3 \ddot{u}^t)$$

(ii) Solve for displacement increments:

$$\tilde{K} u^{t+\tau} = \tilde{R}^t \quad ; \quad \Delta u^t = u^{t+\tau} - u^t$$

(iii) Go to C.

TABLE I (cont.)

B. IN NONLINEAR ANALYSIS

(i) If a new stiffness matrix is to be formed, update $\tilde{K}$ for nonlinear stiffness effects to obtain $\tilde{K}^t$; triangularize $\tilde{K}^t$:

$$\tilde{K}^t = LDL^T$$

(ii) Form effective loadvector:

$$\tilde{R}^t = R^t + \theta(R^{t+\Delta t} - R^t) + M(a_2\dot{u}^t + 2\ddot{u}^t) + C(2\dot{u}^t + a_3\ddot{u}^t) - F^{e^t}$$

(iii) Solve for displacement increments using latest D,L factors:

$$LDL^T \Delta u^t = \tilde{R}^t$$

(iv) If required, iterate for dynamic equilibrium; then initialize $\Delta u^t_{(1)} = \Delta u^t$, $i = 0$

(a) $i = i + 1$

(b) Calculate i'th approximation to accelerations, velocities, and displacements:

$$\ddot{u}^{t+\tau}_{(i)} = a_0 \Delta u^t_{(i)} - a_2\dot{u}^t - 2\ddot{u}^t \quad ; \quad \dot{u}^{t+\tau}_{(i)} = a_1 \Delta u^t_{(i)} - 2\dot{u}^t - a_3\ddot{u}^t \quad ; \quad u^{t+\tau}_{(i)} = u^t + \Delta u^t_{(i)}$$

(c) Calculate i'th out-of-balance loads: $R^{r^{t+\tau}}_{(i)} = R^t + \theta(R^{t+\Delta t} - R^t) - M\ddot{u}^{t+\tau}_{(i)} - C\dot{u}^{t+\tau}_{(i)} - F^{e^{t+\tau}}_{(i)}$

(d) Solve for i'th correction to displacement increments: $LDL^T \Delta\Delta u^t_{(i)} = R^{r^{t+\tau}}_{(i)}$

(e) Calculate new displacement increments: $\Delta u^t_{(i+1)} = \Delta u^t_{(i)} + \Delta\Delta u^t_{(i)}$

(f) Iteration convergence if $\|\Delta\Delta u^t_{(i)}\|_2 / \|\Delta u^t_{(i+1)} + u^t\|_2 < tol$

If convergence: $\Delta u^t = \Delta u^t_{(i+1)}$ and go to C;

If no convergence and $i <$ nitem: go to (a); otherwise restart using new stiffness matrix and/or a smaller time step size.

C. CALCULATE NEW ACCELERATIONS, VELOCITIES, AND DISPLACEMENTS

$$\ddot{u}^{t+\Delta t} = a_4 \Delta u^t + a_5\dot{u}^t + a_6\ddot{u}^t \quad ; \quad \dot{u}^{t+\Delta t} = \dot{u}^t + a_7(\ddot{u}^{t+\Delta t} + \ddot{u}^t) \quad ; \quad u^{t+\Delta t} = u^t + \Delta t\dot{u}^t + a_8(\ddot{u}^{t+\Delta t} + 2\ddot{u}^t)$$

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