FINITE ELEMENT ANALYSIS OF A THICK WALLED MICROPOLAR CYLINDER LOADED AXISYMMETRICALLY

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SUMMARY

Numerous researchers have used the technique of the finite element method in obtaining the stress and displacement field in various bodies subjected to loadings and based on the theory of classical elasticity. In recent years, there has been a revived interest in the field of microelasticity, with Eringen and his co-workers and Mindlin formulating the theory of micromorphic solids independently.

Subsequently, Eringen formulated the theory of micropolar elasticity which involves a microdisplacement field in addition to the macrodisplacement field of classical elasticity. It appears that the theory of micropolar elasticity better simulates the behavior of some structural materials as opposed to the classical theory of elasticity. Based on this concept, the authors proceed to the application of the micropolar theory, in the form of a finite element approach, to an infinitely long thick walled concrete cylinder subjected to an internal pressure.

The problem considered herein is in a state of plane strain and use is made of a stiffness matrix for triangular elements formulated earlier by Baluch and his co-workers for micropolar materials in a state of plane strain. Micropolar materials in a state of plane strain have four physical, or material, constants which include the two Lame' constants, λ and μ, and two additional constants, χ and γ, due to the microstructure of the medium. Commonly used values for the modulus of elasticity E and Poisson ratio ν for concrete are used to establish the Lame' constants and experimental results of Bradfield, Pursey and Kroner are used to determine χ and γ. The stress field is determined for a range of values of the physical constants and the effects of the micropolar structure on the behavior of the thick walled concrete cylinder is studied by comparing the results with those as obtained by using the classical theory of elasticity.

1. Introduction

This paper is concerned with the analysis of a long thick walled concrete cylinder subjected to an internal pressure based on the hypothesis that concrete may be viewed as a micropolar material.

The initial study of oriented or directed media is credited to Duhem [1], with subsequent extensions by E. and F. Cosserat [2] and Ericksen and Truesdell [3]. Recent years have witnessed a great revival of interest in this field, motivated primarily by the works of Eringen and Suhubi [4] and Mindlin [5], dealing with a general theory of micro elastic solids. Later Eringen [6] by himself formulated the theory of micropolar materials wherein the microdisplacement field is restricted to one of pure rotations.

The boundary value problems describing most states of microelasticity have been acknowledged by many of the researchers in the area of micromechanics to be extremely involved and complex. This motivated Baluch, Goldberg and Koh [7] to devise a finite element scheme for the solution of problems in the area of plane microelasticity. It is this technique that is used in solving the case of a micropolar cylinder subjected to internal pressure, a problem of practical significance in reactor technology.

The hypothesis that concrete may be viewed as a micropolar material is based on the nature of the constituents used in the formation of concrete. One can view the coarse aggregate as being homogeneously distributed in the concrete mass as defined by cement and fine aggregate and possessing the capacity to undergo microdeformations independent of the "macromass." If these microdeformations of the aggregate are restricted to rotations, then one arrives at the micropolar theory of elasticity. If the microdeformations are extended to include stretching of the aggregate in addition to rotation, then the state would be described by the theory of micropolar materials with stretch.

The objectives of this work were to investigate the effects of the micropolar characteristics on the stress and strain displacement field in the cylinder. Treating it as a micropolar body in a state of plane strain, one obtains in addition to the classical stress and strain components, namely, t_{xx}, t_{yy}, t_{xy}, e_{xx}, e_{yy} and $2e_{xy}$, six more components of the stress and strain vectors, namely, $t_{yx} - t_{xy}$, m_{xz}, m_{yz}, ε_{xy}, Λ_{zx} and Λ_{zy}. One may recall that the Cauchy stress tensor of classical elasticity has $t_{yx} = t_{xy}$ but which is not so in micropolar elasticity. The stress $t_{yx} - t_{xy}$ is referred to as a microstress and m_{xz} and m_{yz} as couple stresses. Λ_{zx} and Λ_{zy} are gradients of the plane micro-rotation, ϕ_z, and ε_{xy} is a "mixed" strain component, involving both the macro and micro displacements.

There exist four constitutive coefficients in the theory of plane microelasticity, namely, the two Lame' constants λ and μ, which may be related to the modulus of elasticity E and Poisson ratio ν, and two additional constants, χ and γ, due to the microstructure of the medium. Results presented were obtained by using commonly accepted values for E and ν for concrete and a range of values for χ and γ.

2. The Basic Theory

In the case of the generalized three-dimensional micropolar elasticity problem, the displacement field consists of (a) the macrodisplacement vector $\vec{u} = u_x \vec{e}_1 + u_y \vec{e}_2 + u_z \vec{e}_3$ and (b) the microrotation vector $\vec{\phi} = \phi_x \vec{e}_1 + \phi_y \vec{e}_2 + \phi_z \vec{e}_3$ where $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are the unit base vectors. Using the notation that repeated Greek indices represent summation over range (x, y, z) and indices following a comma indicate the partial differentiation with respect to those indices (i.e. $u_{x,y} = \frac{\partial u_x}{\partial y}$), the strain and microstrain tensors are then defined as

$$e_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha})$$

$$\varepsilon_{\alpha\beta} = u_{\alpha,\beta} + \kappa_{\alpha\beta\gamma} \phi_\gamma$$

where

$\alpha, \beta = x, y, z$

and $\kappa_{\alpha\beta\gamma}$ is the three dimensional permutation symbol, i.e., $\kappa_{xyz} = \kappa_{yzx} = \kappa_{zxy} = -\kappa_{yxz} = -\kappa_{xzy} = -\kappa_{zyx} = 1$ with all other components being zero.

Also defining the components of the rotation vector in conventional fashion,

$$r_\alpha = \frac{1}{2} \kappa_{\alpha\beta\gamma} u_{\gamma,\beta}$$

we write the strain energy density according to Reference [6] as

$$V = \frac{1}{2} \{t_{\alpha\beta} [e_{\alpha\beta} + \kappa_{\alpha\beta\gamma} (r_\gamma - \phi_\gamma)] + m_{\alpha\beta} \phi_{\beta,\alpha}\}$$

where $t_{\alpha\beta}$ is the stress tensor and $m_{\alpha\beta}$ is the couple-stress tensor.

The constitutive equations for the generalized three-dimensional micropolar elasticity problem may be written as (Reference [6])

$$t_{\alpha\beta} = \lambda e_{\gamma\gamma} \delta_{\alpha\beta} + 2(\mu + \chi) e_{\alpha\beta} - \chi\varepsilon_{\alpha\beta}$$

$$m_{\alpha\beta} = \alpha\phi_{\gamma,\gamma} \delta_{\alpha\beta} + \beta\phi_{\alpha,\beta} + \gamma\phi_{\beta,\alpha}$$

where λ, μ are the classical Lame' constants and χ, α, β, γ are additional elastic constants due to the microstructure of the medium. The symbol $\delta_{\alpha\beta}$ is the Kronecker delta

$$\delta_{\alpha\beta} = 1 \quad \text{if } \alpha = \beta$$

$$\delta_{\alpha\beta} = 0 \quad \text{if } \alpha \neq \beta$$

Since the problem considered in this work is one in a state of plane strain, the three dimensional micropolar elasticity equations have to be reduced to an appropriate two dimensional state, with the displacement field being restricted to the two in plane components of the vector $\vec{u}$ (u_x and u_y) and one component of the vector $\vec{\phi}$ (ϕ_z). According to Reference [7], the expression for the strain energy density, V, and the constitutive equations, in scalar form, for the plane strain case may be written as

$$V = \frac{1}{2} [t_{xx}\, e_{xx} + t_{yy}\, e_{yy} + t_{xy}\, (2e_{xy}) + (t_{yx} - t_{xy})\, \varepsilon_{xy} + m_{xz}\, \Lambda_{zx} + m_{yz}\, \Lambda_{zy}]$$

where $\Lambda_{zx} = \frac{\partial \phi_z}{\partial x}$, $\Lambda_{zy} = \frac{\partial \phi_z}{\partial y}$

and

$$t_{xx} = (\lambda + 2\mu + \chi)\, e_{xx} + \lambda e_{yy}$$

$$t_{yy} = \lambda e_{xx} + (\lambda + 2\mu + \chi)\, e_{yy}$$

$$t_{xy} = 2(\mu + \chi)\, e_{xy} - \chi\, \varepsilon_{xy}$$

$$t_{yx} - t_{xy} = 2\chi\, (\varepsilon_{xy} - e_{xy})$$

$$m_{xz} = \gamma\, \Lambda_{zx}$$

$$m_{yz} = \gamma\, \Lambda_{zy}$$

In order that a finite element formulation of the plane micropolar elasticity problem be possible, appropriate stress and strain vectors have to be defined. According to Reference [7], they are taken as

$$\sigma = \{t_{xx} \;\; t_{yy} \;\; t_{xy} \;\; (t_{yx} - t_{xy}) \;\; m_{xz} \;\; m_{yz}\}$$

$$e = \{e_{xx} \;\; e_{yy} \;\; 2e_{xy} \;\; \varepsilon_{xy} \;\; \Lambda_{zx} \;\; \Lambda_{zy}\}$$

Such a description of the stress and strain vectors enables one to write the strain energy density and the constitutive equations in matrix form as

$$V = \frac{1}{2} e^T \sigma$$

and

$$\sigma = De$$

where

e^T is the transpose of e and D is the 6*6 symmetric matrix

$$\begin{bmatrix} \lambda + 2\mu + \chi & \lambda & 0 & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu + \chi & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu + \chi & -\chi & 0 & 0 \\ 0 & 0 & -\chi & 2\chi & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix}$$

The element stiffness matrix as derived in Reference [7] is based upon linear displacement fields assumed as

$$u_x = c_1 x + c_2 y + c_3$$

$$u_y = c_4 x + c_5 y + c_6$$

$$\phi_z = c_7 x + c_8 y + c_9$$

The element stiffness matrix turns out to be a 9*9 symmetric matrix and its details may be found in Reference [7].

3. Numerical Results and Conclusions

The problem considered in this work is that of a long, thick walled, micropolar cylinder subjected to an internal pressure of 10 psi as shown in Figure 1. The internal radius was taken as 120 inches and the external radius as 180 inches. Since the problem considered is of an axisymmetric nature, a wedge isolated by two radial lines five degrees apart was chosen for analysis. This wedge was subdivided into 80 elements, resulting in 62 nodes and as shown in Figure 2.

The Lame' constants λ and μ are related to the modulus of elasticity E and Poisson's Ratio ν by the following relationships

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}$$

$$2\mu + \chi = \frac{E}{(1 + \nu)}$$

Commonly used values for E and ν in concrete analysis were taken as $E = 3.0 * 10^6$ psi and $\nu = 0.2$. An approximation for χ was obtained on experimental evidence as reported by Bradfield and Pursey in Reference [8] in terms of a ratio $\frac{d}{c}$, where

$$d^2 = \frac{\gamma}{2(2\mu + \chi)}$$

$$c^2 = \frac{\gamma(\mu + \chi)}{\chi(2\mu + \chi)}$$

The material constant γ is established based on Kroner's work in Reference [9] in terms of a ratio $\frac{t}{c}$, where t is the element thickness (assumed as unity) and c is as defined earlier.

The computer program was written so as to obtain the following quantities at each node:

Strains — e_{xx}, e_{yy}, $2e_{xy}$, ε_{xy}, Λ_{zx} and Λ_{zy}

Stresses — t_{xx}, t_{yy}, t_{xy}, $t_{yx} - t_{xy}$, m_{xz} and m_{yz}

The results are shown in Tables I and II corresponding to ratios of $\frac{d}{c}$ = 0·10 and 0·20. Each case of $\frac{d}{c}$ was also run for three values of $\frac{t}{c}$ in order to investigate the effects of γ on the relevant quantities. The results in Table I correspond to node 25 and those in Table II to node 37. In order to make a meaningful study of the micropolar effect on the stresses and strains in the cylinder, the results for the micropolar problem were compared to those as obtained for the classical elasticity case based on the same finite element model, merely by using zeroes for the micropolar constants.

It becomes apparent on studying the results as shown in Tables I and II that while one of the two micropolar constants γ affects basically the micropolar quantities, ε_{xy}, Λ_{zx}, Λ_{zy}, $t_{yx} - t_{xy}$, m_{xz} and m_{yz}, the other constant χ affects the macropolar quantities, e_{xx}, e_{yy}, $2e_{xy}$, t_{xx}, t_{yy} and t_{xy}. This is as anticipated, since taking into account the effects of χ on the stress-strain field essentially amounts to changing the magnitude of the conventional shear modulus G.

On further examination of the results shown as well as those for other nodes not shown for reasons of brevity, one concludes that based on the hypothesis that concrete may be treated as a micropolar material, the problem of the infinitely long cylinder subjected to an internal pressure may be approximated for design purposes by the conventional classical elasticity theory since the micropolar effects on the behavior of the cylinder are not very pronounced. This must not hasten one to the conclusion that micro-elasticity effects may totally be ignored in all problems of concrete analysis. There exists a need to conduct investigations into changes in micropolar effects due to a variation in loading and boundary conditions and also a search for a more sophisticated micromorphic model to simulate concrete.

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TABLE I (Results for Node 25)

Quantities	$\frac{d}{c} = 0.2$			$\frac{d}{c} = 0.1$			Classical Elasticity Finite Element
	$\frac{t}{c} = 5$	10	15	$\frac{t}{c} = 5$	10	15	
e_{xx}	5.5180E-06*	5.5180E-06	5.5180E-06	5.4073E-06	5.4073E-06	5.4073E-06	5.3728E-06
e_{yy}	-2.6479E-06	-2.6479E-06	-2.6479E-06	-2.6198E-06	-2.6198E-06	-2.6198E-06	-2.6110E-06
$2e_{xy}$	-3.7415E-07	-3.7415E-07	-3.7415E-07	-3.6847E-07	-3.6847E-07	-3.6847E-07	-3.6670E-07
ϵ_{xy}	-1.7381E-07	-1.7416E-07	-1.7422E-07	-1.7112E-07	-1.7147E-07	-1.7154E-07	-4.3035E-20
Λ_{zx}	2.7436E-08	2.7491E-08	2.7502E-08	2.7011E-08	2.7067E-08	2.7077E-08	0
Λ_{zy}	-2.3341E-10	-2.3394E-10	-2.3403E-10	-2.2855E-10	-2.2908E-10	-2.2918E-10	0
t_{xx}	1.6187E+01	1.6187E+01	1.6187E+01	1.5841E+01	1.5841E+01	1.5841E+01	1.5734E+01
t_{yy}	-4.2278E+00	-4.2278E+00	-4.2278E+00	-4.2265E+00	-4.2265E+00	-4.2265E+00	-4.2260E+00
t_{xy}	-4.6907E-01	-4.6904E-01	-4.6903E-01	-4.6091E-01	-4.6091E-01	-4.6090E-01	-4.5837E-01
$t_{yx}-t_{xy}$	2.7640E-03	2.6911E-03	2.6676E-03	6.6249E-04	6.4456E-04	6.4123E-04	0
m_{xz}	2.1949E-04	5.4983E-05	2.4446E-05	5.4022E-05	1.3533E-05	6.0172E-06	0
m_{yz}	-1.8673E-06	-4.6787E-07	-2.0803E-07	-4.5710E-07	-1.1454E-07	-5.0929E-08	0

* $5.5180E\text{-}06 = 5.5180 * 10^{-6}$

TABLE II (Results for Node 37)

Quantities	$\frac{d}{c} = 0.2$			$\frac{d}{c} = 0.1$			Classical Elasticity Finite Element
	$\frac{t}{c} = 5$	10	15	$\frac{t}{c} = 5$	10	15	
e_{xx}	4.9237E-06	4.9237E-06	4.9237E-06	4.8235E-06	4.8235E-06	4.8235E-06	4.7922E-06
e_{yy}	-1.9623E-06	-1.9623E-06	-1.9623E-06	-1.9370E-06	-1.9370E-06	-1.9370E-06	-1.9291E-06
$2e_{xy}$	-3.1362E-07	-3.1362E-07	-3.1362E-07	-3.0839E-07	-3.0839E-07	-3.0839E-07	-3.0677E-07
ε_{xy}	-1.7667E-07	-1.7702E-07	-1.7709E-07	-1.7392E-07	-1.7428E-07	-1.7435E-07	-3.6230E-20
Λ_{zx}	2.5766E-08	2.5818E-08	2.5827E-08	2.5364E-08	2.5417E-08	2.5427E-08	0
Λ_{zy}	-2.5372E-10	-2.5427E-10	-2.5437E-10	-2.4890E-10	-2.4946E-10	-2.4957E-10	0
t_{xx}	1.4777E+01	1.4777E+01	1.4777E+01	1.4464E+01	1.4464E+01	1.4464E+01	1.4367E+01
t_{yy}	-2.4380E+00	-2.4380E+00	-2.4380E+00	-2.4371E+00	-2.4371E+00	-2.4371E+00	-2.4368E+00
t_{xy}	-3.8996E-01	-3.8992E-01	-3.8991E-01	-3.8499E-01	-3.8499E-01	-3.8498E-01	-3.8346E-01
$t_{yx} - t_{xy}$	-4.1370E-03	-4.2112E-03	-4.2250E-03	-9.9600E-04	-1.0143E-03	-1.0177E-03	0
m_{xz}	2.0612E-04	5.1635E-05	2.2958E-05	5.0729E-05	1.2709E-05	5.6504E-06	0
m_{yz}	-2.0298E-06	-5.0854-07	-2.2611E-07	-4.9781E-07	-1.2473E-07	-5.5459E-08	0

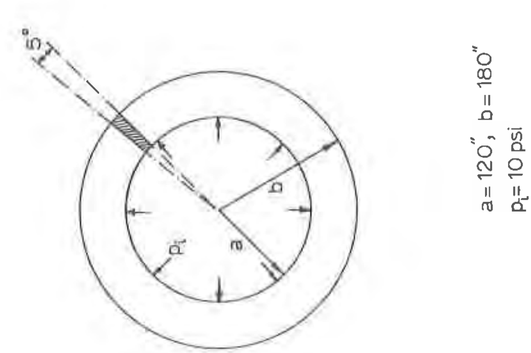

FIGURE 1 GEOMETRY OF THE CYLINDER

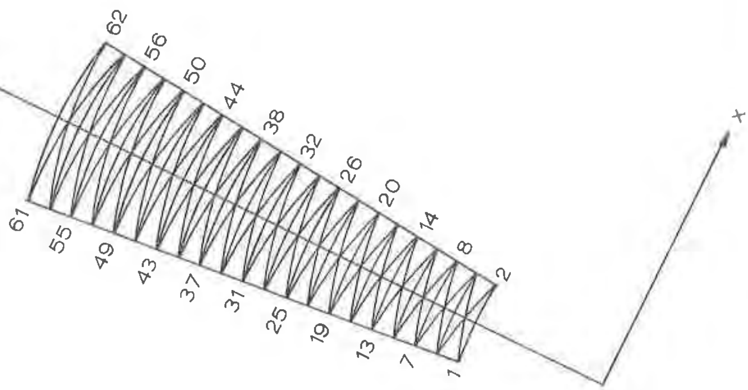

FIGURE 2 GRID DISTRIBUTION ON ISOLATED WEDGE