APPLICATION OF SIMPLIFIED HYBRID DISPLACEMENT METHOD TO PLATE AND SHELL PROBLEMS

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SUMMARY

Finite element analyses of plates and shells are of great practical significance in various branches of engineering, but the compatible displacement method is not necessarily well suited for this purpose because of complexity of available conforming shape functions. To overcome the shortcomings, the authors proposed a simplified hybrid displacement method and have applied it to various problems of plates and shells. Furthermore, some theoretical studies have been also performed based on modern numerical analysis. Thus, the method has turned out to be effective to practical purposes due to its simplicity in computational processes and error estimations.

This paper summarizes the formulation and some applications of this approach to more complicated problems of plates and shells. First, the formulation for general shells in curvilinear coordinates is presented, and then the method is applied to the following sample problems:

(1) linear analysis of shells, which includes problems for both shallow and deep shells under static loading,
(2) nonlinear analysis of plates and shells with both material and geometrical nonlinearities included,
(3) dynamic analysis of shells.

Of course, vibration and buckling analysis of plates and shells may be conducted with ease by using eigenvalue techniques as special cases of the above items.

First, shallow shells of elliptic or hyperbolic paraboloid, or of circular cylinder shape are analyzed with the results compared with the double Fourier series solutions. A problem of unsupported deep cylindrical shell under a pair of pinching loads is also analyzed. Then nonlinear analysis is performed with respect to plates and shallow shells. Special emphasis is put on snap-through buckling of shallow shells under uniform external pressure. The general path of nonlinear equilibrium is usually followed by the piecewise linear procedure combined with the present hybrid approach, while the critical point of bifurcation is predicted by piecewise eigenvalue techniques. The dynamic analysis of snap-through phenomena is also discussed with inertia and damping forces included and a special type of lumped basis employed.

In this way, the effectiveness of the simplified hybrid displacement method is investigated by numerical examples including not only linear but also nonlinear and dynamic problems of plates and shells. As a result, it turns out that this method is almost as simple as the compatible method in formulation and much more convenient in practical applications.

1. Introduction

Finite element analysis of plates and shells are of great practical significance in various branches of structural engineering, but the familiar compatible models are not necessarily well suited for this purpose on account of complexity of available conforming shape functions. This is mainly attributed to the difficulty in achieving the conforming conditions imposed on the assumed lateral deflection field. (Zienkiewicz [1]).

To overcome such shortcomings of conventional methods, the present authors proposed a simplified hybrid displacement method as a simplest modification of compatible method and have applied it to various problems of plates and shells (Kikuchi and Ando [2]-[4]). Furthermore, some convergence studies have been also conducted for linear problems with the aid of functional analysis (Kikuchi and Ando [5],[6]). As a result, the method has turned out to be effective to various kinds of practical problems.

This paper reports the formulation and some applications of the above method to various problems of plates and shells. First the formulation is presented in the case of linear analysis of general shells and then the method is employed to the analysis of both linear and nonlinear problems of plates and shells. Especially, snap-through phenomena of shallow shells are discussed based on both static and dynamic considerations.

Since the number of available pages is quite limited, the details of formulation and computational method for nonlinear problems are entirely omitted in the following.

2. Variational functional for linear analysis of general shell

This section is to deal briefly with variational formulations of general isotropic shells in orthogonal curvilinear coordinates.

Let us consider a shell with smoothly varying geometric and material properties. We will choose orthogonal coordinates (a,b) on the middle surface of the shell whose coordinate directions coincide with the principal directions of curvature. The principal curvatures in a- and b- directions are respectively designated as $1/R_a$ and $1/R_b$. We also take z-axis as the normal of the middle surface, thus obtaining three dimensional coordinates (a,b,z). The displacements of the middle surface of the shell in a-, b- and z-directions are denoted as u, v and w, and the distributed forces per unit area of the middle surface in the corresponding directions as $\overline{X}$, $\overline{Y}$ and $\overline{p}$ (see Fig. 1).

The generalized strain-displacement relations of the shell derived by Novozhilov [7] are

$$e_a = \frac{1}{A}\frac{\partial u}{\partial a} + \frac{1}{AB}\frac{\partial A}{\partial b} v + \frac{w}{R_a}, \qquad e_b = \frac{1}{B}\frac{\partial v}{\partial b} + \frac{1}{AB}\frac{\partial B}{\partial a} u + \frac{w}{R_b},$$

$$e_{ab} = \frac{B}{A}\frac{\partial}{\partial a}\left(\frac{v}{B}\right) + \frac{A}{B}\frac{\partial}{\partial b}\left(\frac{u}{A}\right),$$

$$- k_a = \frac{1}{A}\frac{\partial}{\partial a}\left(\frac{1}{A}\frac{\partial w}{\partial a} - \frac{u}{R_a}\right) + \frac{1}{AB}\frac{\partial A}{\partial b}\left(\frac{1}{B}\frac{\partial w}{\partial b} - \frac{v}{R_b}\right) , \tag{1}$$

$$- k_b = \frac{1}{B}\frac{\partial}{\partial b}\left(\frac{1}{B}\frac{\partial w}{\partial b} - \frac{v}{R_b}\right) + \frac{1}{AB}\frac{\partial B}{\partial a}\left(\frac{1}{A}\frac{\partial w}{\partial a} - \frac{u}{R_a}\right)$$

$$- k_{ab} = \frac{1}{AB}\left(\frac{\partial^2 w}{\partial a \partial b} - \frac{1}{A}\frac{\partial A}{\partial b}\frac{\partial w}{\partial a} - \frac{1}{B}\frac{\partial B}{\partial a}\frac{\partial w}{\partial b}\right) - \frac{1}{R_a}\left(\frac{1}{B}\frac{\partial u}{\partial b} - \frac{1}{AB}\frac{\partial A}{\partial b} u\right)$$

$$- \frac{1}{R_b}\left(\frac{1}{A}\frac{\partial v}{\partial a} - \frac{1}{AB}\frac{\partial B}{\partial a} v\right) ,$$

where e_a, e_b and e_{ab} are strain components at the middle surface, k_a, k_b and k_{ab} are curvature or twist changes there, and A and B are Lame's parameters.

The generalized stress-strain relations for isotropic shells are

$$N_a = \frac{Eh}{1 - \nu^2}(e_a + \nu e_b) , \qquad N_b = \frac{Eh}{1 - \nu^2}(e_b + \nu e_a) ,$$

$$N_{ab} = \frac{Eh}{2(1+\nu)}\left(e_{ab} + \frac{h^2}{6R_b} k_{ab}\right), \qquad N_{ba} = \frac{Eh}{2(1+\nu)}\left(e_{ab} + \frac{h^2}{6R_a} k_{ab}\right) \tag{2}$$

$$M_a = D(k_a + \nu k_b) , \qquad M_b = D(k_b + \nu k_a) ,$$

$$M_{ab} = M_{ba} = D(1 - \nu)k_{ab} ,$$

in which E is Young's modulus, h shell thickness, ν Poisson's ratio, D bending rigidity, and N and M respectively denote membrane stress resultants and moments. The boundary conditions to be treated are assumed to be homogeneous for simplicity.

Then a variational functional Π_1 for compatible finite element models of shells can be given by

$$\Pi_1 = \sum_m \iint_{S_m} \Big[\frac{1}{2}\frac{Eh}{1 - \nu^2}\left(e_a^2 + e_b^2 + 2\nu e_a e_b + \frac{1 - \nu}{2} e_{ab}^2\right) \tag{3}$$

$$+ \frac{1}{2} D\left(k_a^2 + k_b^2 + 2\nu k_a k_b + 2(1-\nu)k_{ab}^2\right) - \bar{X}u - \bar{Y}v - \bar{p}w\Big] AB\, da\, db,$$

where S_m is m-th finite element, and all the strain components satisfy eq.(1). The equilibrium equations of the shell, together with kinematic boundary conditions, can be derived by taking the variation of Π_1 with respect to u, v and w. As usual, the above integration is performed element by element, and then assembled to obtain the total potential energy of the system.

We define the following quantities on ∂S_m, the boundary of S_m :

$$M_n = M_a \tilde{l}^2 + M_b m^2 + 2M_{ab} \tilde{l} m \quad , \quad \frac{\partial w}{\partial n} = \frac{1}{A}\frac{\partial w}{\partial a}\tilde{l} + \frac{1}{B}\frac{\partial w}{\partial b} m \, , \qquad (4)$$

where $\tilde{l}$ and m are direction cosines of outward normal on ∂S_m.

In the usual compatible models of plates and shells, continuity is required not only for u, v, and w, but also for $\partial w/\partial n$. On the other hand, the above severe conditions can be relaxed to a considerable extent by merely introducing an auxiliary function ϕ on ∂S_m, which is taken to be common to interconnecting elements. In this procedure, called simplified hybrid displacement method (Kikuchi and Ando [2],[3]), the continuity of $\partial w/\partial n$ across interelement boundaries is no longer needed, and Π_1 is modified as

$$\Pi_2 = \sum_m \iint_{S_m} [\frac{1}{2} \frac{Eh}{1-\nu^2} (e_a^2 + e_b^2 + 2\nu e_a e_b + \frac{1-\nu}{2} e_{ab}^2)$$

$$+ \frac{1}{2} D (k_a^2 + k_b^2 + 2\nu k_a k_b + 2(1-\nu) k_{ab}^2) - \bar{X} u - \bar{Y} v - \bar{p} w] \, AB \, da \, db$$

$$+ \sum_m \int_{S_m} (\frac{\partial w}{\partial n} - \phi) M_n \, ds \quad , \qquad (5)$$

in which ds is line element on ∂S_m, and M_n is calculated from u, v and w in S_m utilizing eqs. (2) and (4). Usually ϕ is made equal to $\partial w/\partial n$ at nodal points.

In the case of shallow shells (and flat plate), some considerable simplifications become available. That is, the coordinates (a,b) can be approximately replaced with the usual (x,y) coordinates, and the strain components are merely

$$e_x = \frac{\partial u}{\partial x} + \frac{w}{R_x} \, , \quad e_y = \frac{\partial v}{\partial y} + \frac{w}{R_y} \, , \quad e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \, ,$$
$$- k_x = \frac{\partial^2 w}{\partial x^2} \, , \quad -k_y = \frac{\partial^2 w}{\partial y^2} \, , \quad -k_{xy} = \frac{\partial^2 w}{\partial x \partial y} \qquad (6)$$

where R_x and R_y are principal radii of curvatures in x- and y-directions, respectively. An alternative formulation (but an essentially the same one) is given in Ref. [8] by Washizu.

The nonlinear analysis of shells becomes possible if the e_x, e_y and e_{xy} in eq. (6) are replaced by the corresponding nonlinear expressions, and the constitutive equations by nonlinear ones. Incremental expressions of the variational functional are well suited to plastic analysis : as for such a formulation, the readers should be refered to Ref. [4] (kikuchi and Ando).

The dynamic analysis of shells may be also performed if the inertia force and damping force are introduced into the above functional. A lumped basis for plate bending given in Ref. [9] (Kikuchi and Ando) may be conveniently employed for such purposes.

3. Triangular finite elements employed in the numerical analysis

The following three types of triangular finite elements are developed based on the present approach.

Model-1 Shallow shell element with 27 degrees of freedom : The distribution of u, v and w are approximated by complete cubic polynomials in each element and ϕ is linear along each side of element. The nodal variables are

$$(u_i, \frac{\partial u}{\partial x}\Big|_i, \frac{\partial u}{\partial y}\Big|_i, v_i, \frac{\partial v}{\partial x}\Big|_i, \frac{\partial v}{\partial y}\Big|_i, w_i, \frac{\partial w}{\partial x}\Big|_i, \frac{\partial w}{\partial y}\Big|_i) \quad , \quad (i = 1,2,3), \qquad (7)$$

at node i, and the other three unknowns are eliminated by the static condensation. In the case of cylindrical shells, this model gives identical results to those derived from Donnell's theory (Novozhilov [7]).

Model-2 Cylindrical shell element based on Novozhilov's theory : The distribution of displacements is the same as that of Model-1, but the strain-displacement relation of eq. (1) is employed instead of eq.(6).

Model-3 Shallow shell element for nonlinear and dynamic analysis with 15 degrees of freedom : In this element, u and v are approximated by the piecewise linear shape functions, while w and ϕ are incompletely cubic and linear, respectively. This is identical to the one developed in Ref. [4] except in the point that w is interpolated by the picewise linear shape functions in the determination of membrane and geometric stiffness characters. The nodal parameters in this model are

$$(u_i, v_i, w_i, \frac{\partial w}{\partial x}\Big|_i, \frac{\partial w}{\partial y}\Big|_i) \quad (i = 1,2,3) \ . \qquad (8)$$

4. Numerical examples

Some numerical results obtained by the present theory are given below with brief explanations. Fig. 2 shows examples of mesh patterns employed in the analysis of shallow shells with square planform. In the present examples, simply supported edge condition implies that $u = v = w = M_n = 0$ along the considered edge.

(1) Linear analysis of shallow shells with square planform under uniform pressure (Model-1 and Model-2)

As a first example, we consider shallow shells with initial deflections (i) $w_0 = y^2/2R$, (ii) $w_0 = (x^2+y^2)/2R$, (iii) $w_0 = (x^2- y^2)/2R$, which respectively represent portions of (i) circular cylindrical shell, (ii) spherical shell, and (iii) hyperbolic-paraboloidal shell. Table 1 shows numerical results of central deflection with necessary data, and Fig. 3 does distribution of lateral deflection w along x-axis in the case of (ii). Exact solutions based on double Fourier expansion are also included in the table (Timoshenko et al. [10]). It is apparent that remarkable accuracy is obtained in every case.

(2) Pinched circular cylindrical shell with free edges (Fig,4, Model-1 & -2)

Table 2 denotes the maximum deflections obtained by Novozhilov's and Donnell's type elements for several mesh patterns. Considerable difference may be found between these results since the shell is deep. The results by

Model-2 are in good coincidence with Cantin's solution (Ref.[11]).

(3) Large deflection analysis of square plates under uniform pressure (Model-3)

Figures 5 and 6 are respectively deflection and stress σ_x at center of clamped and simply supported plates under uniform pressure. The results are obtained with 5 x 5 A mesh and coincides reasonably with Rushton's solution based on dynamic relaxation method (Ref.[12]).

(4) Large deflection of simply supported square plate under uniform uniaxial edge displacement (Model-3, 5 x 5 A mesh)

Figure 7 shows post-buckling behavior of square plates with small initial deflection. The definition of $\overline{T}$ is given in Fig. 2, and a comparative solution by Coan (Ref.[13]) is also included.

(5) Large deflection analysis of clamped shallow circular cylindrical shell under uniform external pressure (Model-3, 5 x 5 A mesh)

This example is the same as dealt by Brebbia and Conner (Ref.[14]), in which the necessary data are :

$R = 100$ in., $L = 20$ in., $E = 450{,}000$ psi, $\nu = 0.3$, $h = 0.125$ in.

Figure 8 shows pressure-deflection curve at center.

(6) Dynamic analysis of snap-through buckling of shallow shell with simply supported edges (Model-3, 5 x 5 A mesh)

Figure 9 illustrates an example of dynamic analysis of snap-through of shallow shell with square planform. Although the dynamic path deviates considerably from the static one near the critical pressure, it approaches rather quickly the static equilibrium state if the damping term is suitably chosen. Fairly good agreement may be seen between the present static solution and the one by Kawai (Ref.[15]).

(7) Snap-through buckling of clamped shallow spherical shell under uniform external pressure (Model-3, Figs. 10 & 11)

As a popular problem of nonlinear analysis of shells, the above one is treated by the present approach. A typical pressure-deflection curve at apex is illustrated in Fig. 12 togehter with a comparative result by Kikuchi et al. (Ref. [17]). The buckling pressures shown in Fig. 3 together with Weinitschke's solutions does not appear to be accurate enough probably because of insufficient number of employed finite elements.

(8) Large deflection analysis of elastic-plastic plates and shallow shells with square planform (Model-3, 5 x 5 A mesh)

Figure 14 is pressure-central deflection curves for a flat plate and shallow shells under uniform pressure. The initial deflection of the shells is $w_0 = 10.0(1-(x/100)^2)(1-(y/100)^2)$ mm ($-100 \leq x,y \leq 100$), and other employed data are : $L = 200$ mm, $h = 10$ mm, $E = 2000$ kg/mm^2, $\nu = 0.3$, σ_y (yield point) $= 30$ kg/mm^2, H' (work hardening rate) = 200 kg/mm^2. The present constitutive equation in plastic range is due to Yamada (Ref.[18]).

5. Concluding remarks

The simplified hybrid displacement method has been applied to various problems of plates and shells. The details of formulation, numerical methods and results will be published elsewhere in due course.

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Table 1 Central deflection at center of shallow shells under uniform pressure

Shell theory	Shallow shell theory [7]			Novozhilov [7]
Shell \ Mesh	Circular cylinder	Sphere	Hyperbolic paraboloid	Circular cylinder
3 X 3 A	123.5	39.80	331.1	124.9
3 x 3 B	124.4	40.39	331.4	125.8
5 x 5 A	123.6	39.87	330.6	125.0
5 x 5 B	123.9	40.09	330.8	125.3
Exact	123.6	39.92	330.3	-----

Data : $E = 10.92$, $h = 0.1$, $\nu = 0.3$, $R = 6.0$,
$L = 3.0$, $\bar{p} = 1.0$

Table 2 Displacement under the load for the pinched circular cylinder

Shell Theory \ Mesh	Novozhilov	Donnell
3 x 4 A	0.1073	0.06382
3 x 4 B	0.1069	0.06374
5 x 5 A	0.1101	0.06541
5 x 5 B	0.1099	0.06535
5 x 8 A	0.1120	0.06639
5 x 8 B	0.1119	0.06633
Exact	0.1139 *)	-------

*) From Cantin [11].

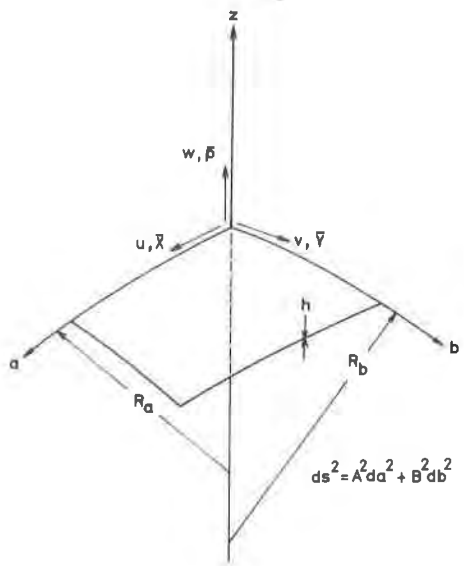

Fig. 1 Coordinate system of shell

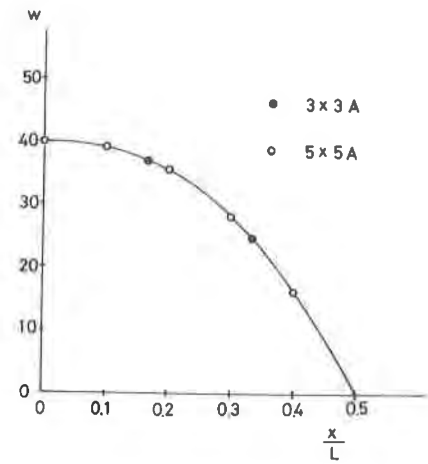

Fig. 3 Distribution of lateral deflection w of portion of spherical shell

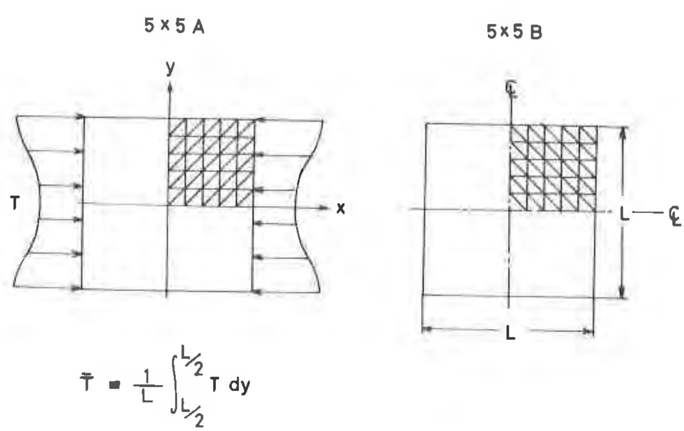

Fig. 2 Examples of mesh patterns for plate and shell with square planform

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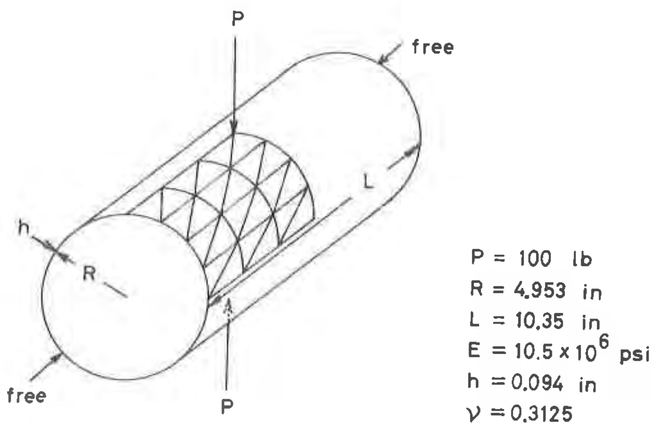

Fig. 4 Finite element representation of pinched cylinder (3 x 4 A mesh)

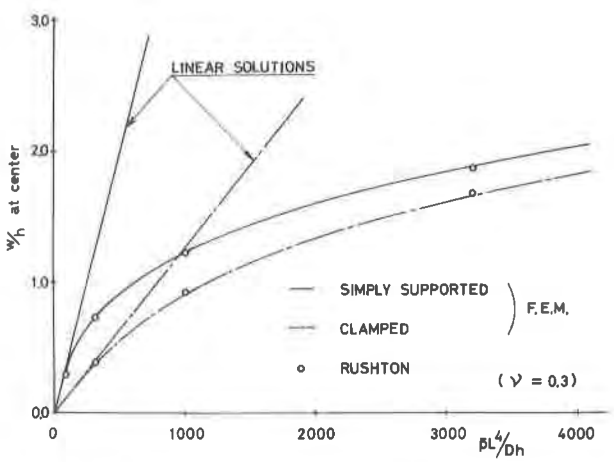

Fig. 5 Central deflection curves of square plates

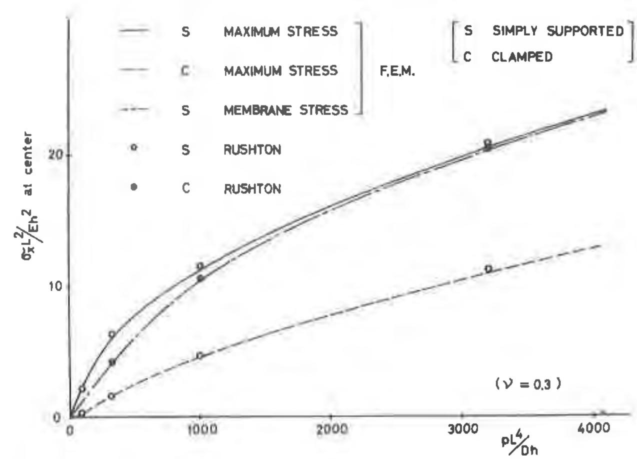

Fig. 6 Central stress-pressure curves of square plates

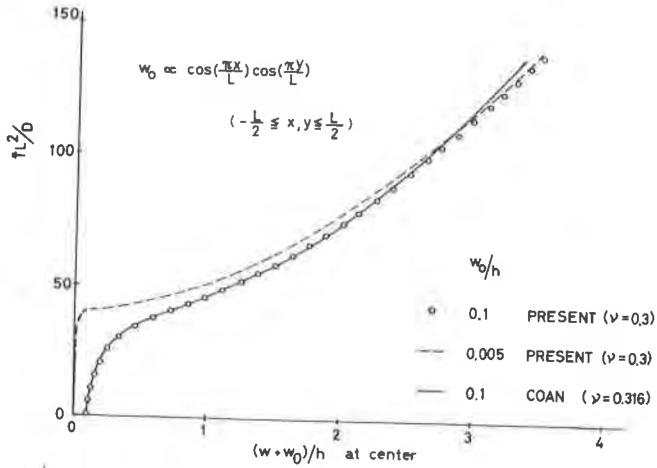

Fig. 7 Post-buckling behavior of simply supported square plate

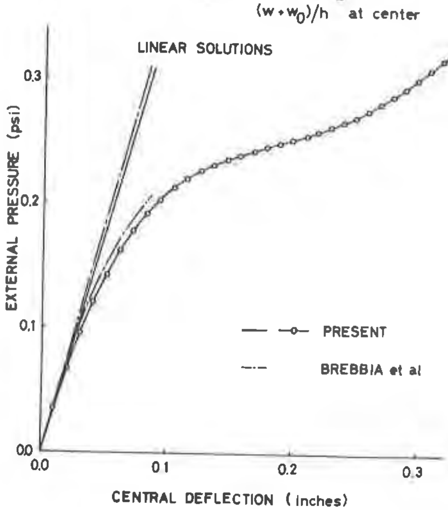

Fig. 8 Pressure-deflection curve at center of clamped shallow cylindrical shell with square planform

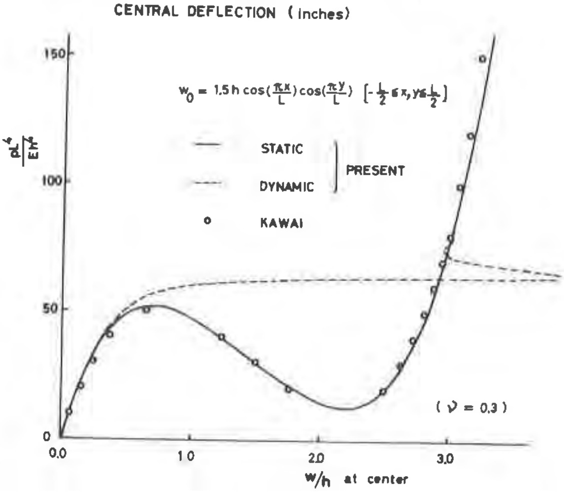

Fig. 9 Snap-through of shallow shell

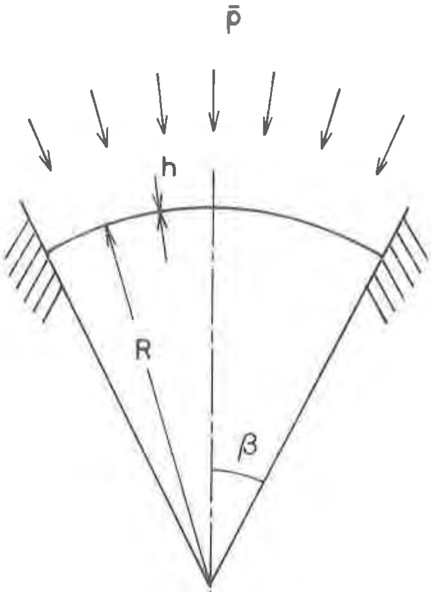

Fig. 10 Shallow clamped spherical shell under uniform external pressure

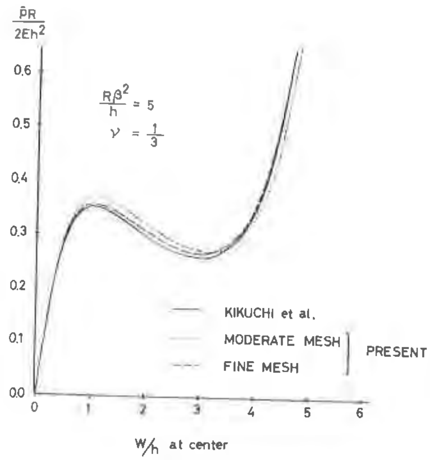

FIg. 12 Pressure-central deflection curve of shallow spherical shell

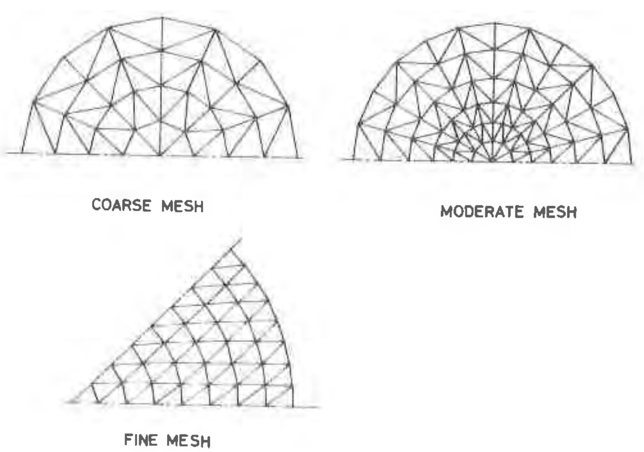

Fig. 11 Mesh patterns of shallow spherical shell

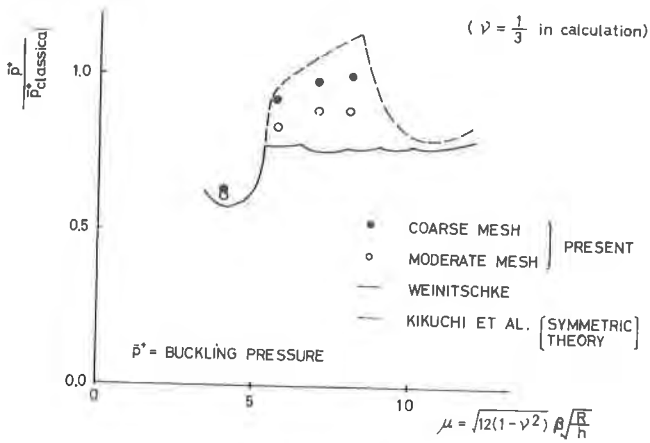

Fig. 13 Buckling pressure curve of shallow spherical shell

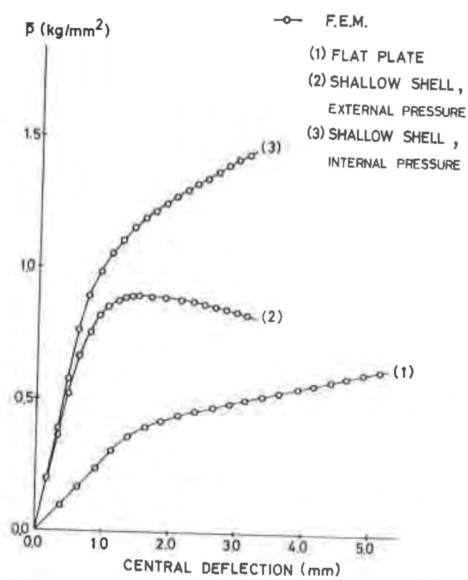

Fig. 14 Pressure-central deflection curves of plates and shallow shells