

BUCKLING OF DEEP SHELLS

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SUMMARY

The reliable estimate of buckling loads for the shell type structures is of vital importance to the structural design engineer. Significant international research effort is presently being spent on investigating the stability behaviour of arbitrary shells by using the finite element techniques. It has been observed that the critical loads calculated by using the classical linear theory differ by a wide margin as compared with those obtained by experimental methods. This discrepancy may be attributed to the inaccurate theoretical formulation of the stability phenomenon of curved structures as well as to the inaccurate experimental simulation of idealised structural parameters. For shell type structures, a consistent nonlinear theory (Ref.: Vol'mir, Mushtari, etc.) must be employed to investigate the stability behaviour. The pre- and post-buckling states should be studied in details since a structure may remain stable ever after undergoing buckling or large displacements.

In this study, a finite element formulation is presented to study pre- and post-buckling behaviour of deep thin shells undergoing large displacements. The Newton-Raphson method is employed to solve the nonlinear system of equations. The nonlinear strain displacement relations are based on the deep shell theory. The material properties are assumed linear. A new method has been presented to initialize the unsymmetrical bifurcation paths for symmetrically loaded structures. This is achieved by introducing a small unsymmetrical load perturbation (0.1% of real load) over the symmetrical structure. If an unsymmetrical state exists nearby, the Newton Raphson method will converge to that point. The complete path is then traced in a routine manner.

Illustrative examples include arbitrary deep shells (spherical, cylindrical) under various load and boundary conditions. The displacement and stresses during the pre- and post-buckling states are studied in details. Wherever possible, these results are compared with those obtained by other investigators. The unsymmetrical bifurcation paths are also traced for a number of shells. It may be stressed that no significant work has been reported so far for analysing pre- and post-buckling states of arbitrary deep shells, (according to the authors' knowledge).

The current safety requirements are demanding sophisticated analysis for the structural design of nuclear reactors. This trend is attributed to the development of powerful finite element techniques supported by high speed computing facilities. The influence of creep, physical and geometrical nonlinearities is to be considered in the design analysis of various structural parts of nuclear reactors.

In this paper, the influence of geometrical nonlinearities on the buckling behaviour of arbitrary shallow and deep shells is investigated. The curved members form an important part of structures encountered in nuclear reactors. Upto 1962, classical analysis techniques have been employed to investigate stability of shells. Among many, the works of Von Karman and Tsien [1], Fung, Weitschke, Budiansky and Thurston etc. may be mentioned as reported in the Collected Papers on Instability of Thin Shells [2]. The unsymmetrical bifurcations have been studied by Huang [3], Tillman [4] and Bushnell [5]. Leicester [6] has reported a large number of shallow shells including unsymmetrical bifurcations.

Since 1962, the finite element method is applied by various researchers for analysing nonlinear stability behaviour of shells. Recently, Gallagher [7] has given a detailed survey of finite element applications to geometrically nonlinear problems. Brebbia and Connor [8] have studied elastic shallow shells undergoing large displacements by using a rectangular element. Dhatt [9,10] has studied the pre- and post-buckling behaviour of a number of arbitrary shallow shells by using a triangular element. The reader may refer to Gallagher [7] for other references. Unsymmetrical bifurcation paths are traced in details in references [11,12] for shallow and deep arches by using the finite element method. No significant work is reported for studying the complete unsymmetrical paths for arbitrary shells. So far, very little work is available on the nonlinear stability behaviour of deep shells by the finite element method.

In this paper, a finite element formulation for analysing deep shells undergoing large displacements is presented. A curved triangular element (based on the discrete Kirchhoff assumptions) with 27 degrees of freedom is employed to analyse the geometrical nonlinear behaviour including unsymmetrical bifurcations of thin deep shells. The nonlinear equations are solved by the Newton Raphson method. The role of independent and dependent variables may be interchanged in order to trace the multisolution paths in a routine manner. The unsymmetrical bifurcations are initialised by introducing a nonsymmetrical perturbing force. Later on, this force is removed and the bifurcation paths with single or double non symmetry are traced if they exist. This technique has been tested in references [11,12] for arches. The works of Haisler et al. [13] may be consulted for a general survey of various solution techniques for nonlinear problems.

A number of examples are presented to demonstrate the capability of the deep shell element developed herein for tracing the pre- and post-buckling displacement and stress histories of shells. Unsymmetrical paths are also studied in details. The results are compared with those obtained by other researchers experimentally or analytically.

1. Formulation

Novozhilov [14] has given a general formulation for nonlinear theory of elasticity. The nonlinear strain displacement relations for a triangular deep shell element are defined as follows.

The rotational β_x and β_y are assumed small so that:

$$\frac{1}{2}\beta_x^2, \frac{1}{2}\beta_y^2, |\beta_x \beta_y| \ll 1$$

The geometry of the element is always assumed shallow, such that:

$$z_{,x}^2, z_{,y}^2, |z_{,x} z_{,y}| \ll 1$$

where $z_{,x} = \frac{\partial z}{\partial x}$, $z_{,y} = \frac{\partial z}{\partial y}$ and $z = z(x,y)$ is the surface of the element defined in the local coordinate system.

Nonlinear Membrane Strain-Displacement Relations

$$\begin{bmatrix} e_x \\ e_y \\ e_{xy} \end{bmatrix} = \begin{bmatrix} u_{,x} & -rw + \frac{1}{2}\beta_x^2 \\ v_{,y} & -tw + \frac{1}{2}\beta_y^2 \\ u_{,y} + v_{,x} - 2sw + \beta_x \beta_y \end{bmatrix} \quad (1)$$

u, v, w are the displacements along x, y, z directions of the orthogonal surface coordinates. r, t, s are the curvatures which are assumed constant over the surface of an element.

Bending Strain Displacement Relations

$$\begin{bmatrix} \zeta k_x \\ \zeta k_y \\ \zeta k_{xy} \end{bmatrix} = \begin{bmatrix} \zeta \beta_{x,x} \\ \zeta \beta_{y,y} \\ \zeta (\beta_{x,y} + \beta_{y,x}) \end{bmatrix} \quad (2)$$

where ζ is the thickness coordinate such that $-\frac{t}{2} \leq \zeta \leq \frac{t}{2}$ where "t" is the thickness of the shell element.

Shear Strain Displacement Relations

$$\begin{bmatrix} e_{xz} \\ e_{yz} \end{bmatrix} = \begin{bmatrix} \beta_x + w_{,x} + ru + sv \\ \beta_y + w_{,y} + tv + su \end{bmatrix} \quad (3)$$

These relations are defined for a deep shell element. The discrete Kirchhoff assumptions are introduced at a number of points on the element. For details, references [9,15] may be consulted.

Stress Strain Relations

These relations are defined by: $\sigma_z = 0$

$$\begin{bmatrix} \sigma_{mx} \\ \sigma_{my} \\ \sigma_{mxy} \end{bmatrix} = D_m \begin{bmatrix} e_x \\ e_y \\ e_{xy} \end{bmatrix} \quad \text{(Membrane)} \quad (4a)$$

$$\begin{bmatrix} \sigma_{kx} \\ \sigma_{ky} \\ \sigma_{kxy} \end{bmatrix} = \frac{t}{12} D_m \begin{bmatrix} k_x \\ k_y \\ k_{xy} \end{bmatrix} \quad \text{(Bending)} \quad (4b)$$

where $D_m = \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & \nu & \frac{1-\nu}{2} \end{bmatrix}$

E = Modulus of elasticity

ν = Poisson's Ratio.

Energy of an Element

The total potential energy is defined by:

$$\mathcal{U}_e = \int_A (\frac{1}{2} \epsilon^T D \epsilon - d^T P) dA \quad (5)$$

where $\epsilon^T = (e_x \ e_y \ e_{xy} \ k_x \ k_y \ k_{xy})$

$d^T = (u \ v \ w)$

$$D = \begin{bmatrix} D_m & 0 \\ 0 & \frac{t}{12} D_m \end{bmatrix}$$

and P is the applied load along u, v, w directions.

Formulation of Rigidity Matrix

The displacements u, v, w are interpolated by 9 term cubic polynomials. β_x and β_y are approximated by complete quadratic polynomials. A discrete version of Kirchhoff assumptions is introduced over the element. A 27 degrees of freedom element is finally obtained, 9 degrees of freedom ($u, u_x, u_y, v, v_x, v_y, w, w_x, w_y$) assigned with each of three corner nodes of the triangle. For brevity, details of the formulation are omitted. Interested readers may consult references [9,15] for details.

2. Newton Raphson Method

The first and second variation of the energy of a complete shell may be defined by:

$$\delta \Pi = \Sigma \delta \Pi_e = \delta D^T (K D - \lambda F) \tag{6}$$

$$\delta^2 \Pi = \Sigma \delta^2 \Pi_e = \delta D^T (KN \delta D - \delta \lambda F) \tag{7}$$

where D = vector represents the assembled unknowns

λ = load factor

F = assembled load

K = $K(D)$, nonlinear matrix corresponding to equilibrium relations of the problem.

KN = $KN(D)$, nonlinear variational (tangent) matrix

The residue vector is defined by: $R = K D - \lambda F$.

The Newton Raphson iteration scheme is obtained as follows:

For an i 'th cycle:

$$R^{(i)} = K(D^{(i)}) D^{(i)} - \lambda^{(i)} F \tag{8}$$

$$R^{(i+1)} = R^{(i)} + \frac{\partial R^{(i)}}{\partial D} \delta D^{(i)} + \frac{\partial R^{(i)}}{\partial \lambda} \delta \lambda^{(i)} = 0$$

or $-R^{(i)} = KN(D)^{(i)} \delta D^{(i)} - F \delta \lambda^{(i)} \tag{9}$

Equation (9) is solved to get $\delta D^{(i)}$ and $\delta \lambda^{(i)}$ and one has

$$D^{(i+1)} = D^{(i)} + \delta D^{(i)}$$

$$\lambda^{(i+1)} = \lambda^{(i)} + \delta \lambda^{(i)}$$

Iterations are terminated when the norm $\|n\|$ becomes less than the permissible value Δ .

$$\|n\| = \left[(D^{(i+1)})^T D^{(i+1)} - D^{(i)T} D^{(i)} \right] / D^{(i+1)T} D^{(i+1)} \Bigg]^{\frac{1}{2}} < \Delta$$

The value of Δ is assumed to be equal to 0.1 in this study.

Unsymmetrical Bifurcation

A simple numerical scheme is employed to initialise the unsymmetrical bifurcations for a symmetrically loaded shell. A symmetrical nonlinear load displacement path is first traced. In order to obtain unsymmetrical bifurcation, the complete structure corresponding to the desired unsymmetry (may be single or double) is then studied. A perturbing non symmetrical force is applied. The corresponding unsymmetrical nonlinear displacement configuration is obtained as usual. Then the perturbation force is removed. If the nonsymmetrical path exists, the solution obtained by the Newton Raphson method will possibly converge to the unsymmetrical path which has been already initialised. If not, then in that region no unsymmetrical paths exist.

3. Illustrative Examples

A linear analysis of a deep shell element is presented to demonstrate the efficiency of the deep shell triangular element.

3.1. Pinched Cylinder

A pinched cylinder as shown in Fig.1 is analysed. This problem has been previously studied by Bogner et al. [16] and Cantin and Clough [17] by the finite element method. The results presented herein are obtained with two types of triangular elements. The element called KCS [15] is based on the shallow shell relations and the element "KCD", developed here, is based on the deep shell relations of section 1. It may be concluded from Fig. 1 that the element KCS is not suitable for analysing deep shells. It is interesting to note that the element KCD gives significantly superior convergence as compared with the elements employed in references [16,17].

3.2. Nonlinear Analysis

Pre- and post-buckling behaviour of large number of arbitrary shells has been recently studied in details in reference [20]. The nonlinear behaviour including unsymmetrical bifurcations of two types of shells is studied herein. These examples are chosen so that the results may be compared with those reported in references [6,19].

3.2.1. Unsymmetrical Bifurcations

A hinged spherical shell, Fig. 2, with a rectangular planform subject to normal pressure is studied in details. The symmetrical pre- and post-buckling path is traced by studying $\frac{1}{4}$ of the complete shell. These results are shown in Figs. 2 and 3 for 3x3 and 6x6 mesh sizes. The displacement w_{av} in Fig. 2 is the average of the normal displacements "w" over all the nodes. This average is obtained by the Simpson's rule. The

symmetrical configuration (S-S) obtained herein compares well with that obtained by Leicester [6] as may be seen in Figs. 2 and 3.

The unsymmetrical bifurcations are initialised by applying an unsymmetrical perturbation load. For single unsymmetry U-S, $\frac{1}{2}$ of the shell is studied with 8x4 mesh size. A load of 1.30 times the symmetrical nodal load is applied at a node adjacent to "C" in Fig. 2 along the symmetry line. This load has been applied at the stage where unsymmetrical bifurcation is assumed to exist. In this study, this choice has been influenced by Leicester's results. Once a point on the unsymmetrical path U-S is obtained, the perturbing load is removed. The unsymmetrical path U-S, as shown in Figs. 2 and 3, is thereby traced in a routine manner. In the initial stages of the path, about 3-4 iterations are required. Later on 2 iterations are sufficient.

The complete shell with 8x8 mesh is considered for double unsymmetry. The perturbing load of 1.5 symmetrical nodal load is applied at a node next to node "C" in Fig. 2 in the first quadrant. The complete U-U path is shown in Figs. 2 and 3. The displacement configurations and bending moment M_x along the symmetry lines x-x and y-y are given in Fig. 4 and Fig. 5a respectively for S-S, U-S, U-U cases at two load levels. The variations of the N_x, N_y at the center for various load levels are shown in Fig. 5. One may observe the variations of stresses and moments at different levels of loading and in the post-buckling stages. It may be noticed that w/t is less than 1 for the post-buckling region studied herein.

3.2.2. Nonlinear Analysis of an Elliptic Cylinder

This problem has been studied experimentally in reference [19]. The closed elliptic cylinder, with two edges membrane supported, is subject to normal pressure. A mesh of 4x4 is employed for 1/8 of the shell due to double symmetry. The nonlinear load-displacement relations at point D are shown in Fig. 6 for two types of boundary conditions. The experimental result is also shown along with. The critical load, calculated herein, compares well (10% superior) with that obtained by experimental solution

4. Conclusion

The efficiency of the deep shell element, presented herein, has been demonstrated as superior as compared with other elements for a pinched cylinder. Complete unsymmetrical paths have been traced in details by the numerical technique proposed in this study.

The results presented have shown the reliability of finite element solutions for tracing unsymmetrical paths and studying nonlinear behaviour of deep shells. It may be remarked that a very high amount of computation effort is required for studying unsymmetrical bifurcations for deep shells. It is necessary to devote more research effort in order to modify the existing numerical techniques so that the computation time for nonlinear solutions falls within an acceptable range.

5. References

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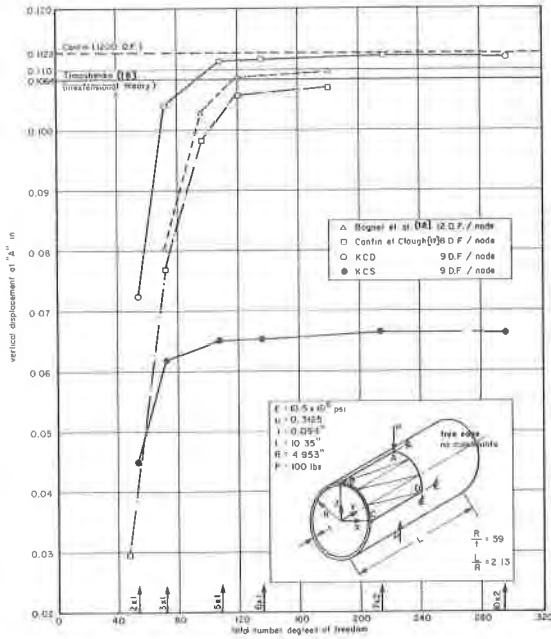


Fig. 1 : Pinched Cylinder : Comparison of Results.

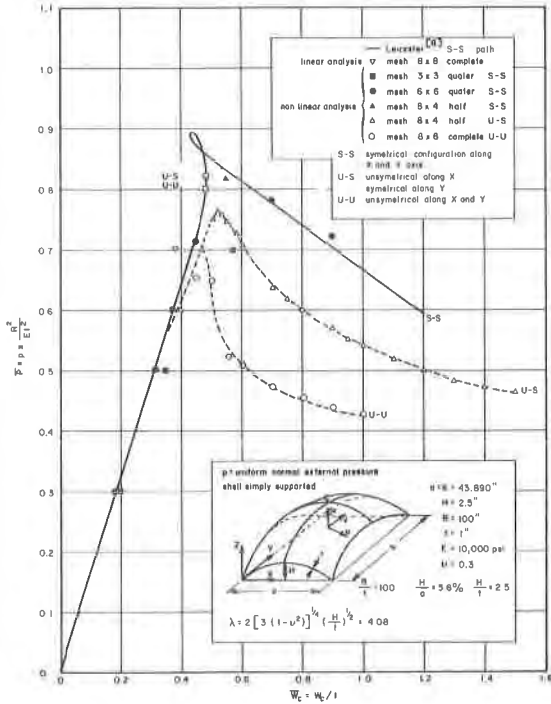


Fig. 2 : Hinged Spherical Shell with KCS Element. Deflection at the Center.

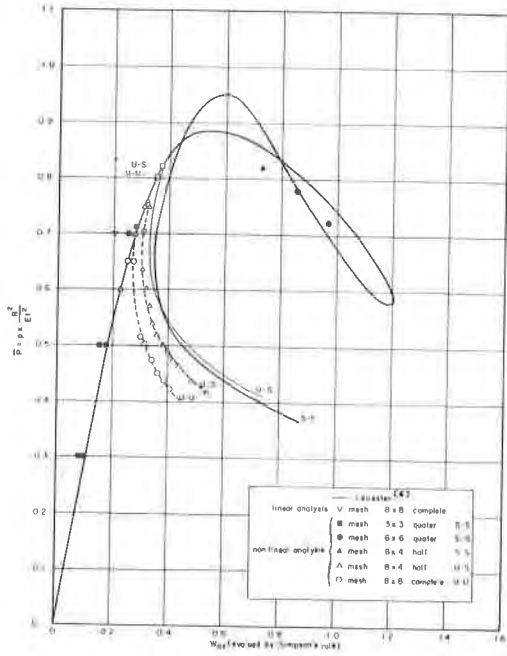


Fig. 3 : Hinged Spherical Shell with KCS Element. Average Normal Displacement.

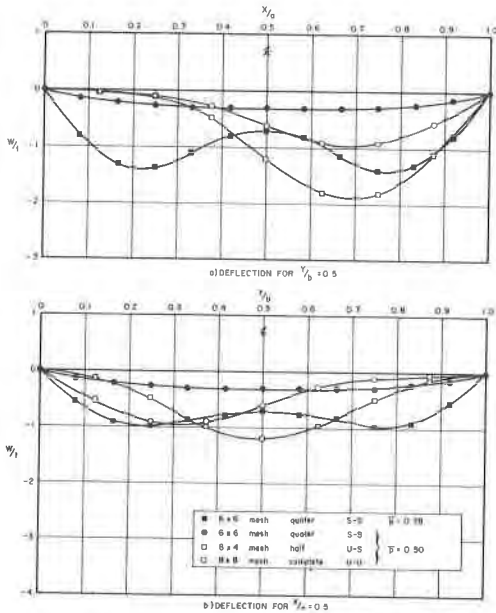


Fig. 4 : Spherical Shell. Typical Deflections.

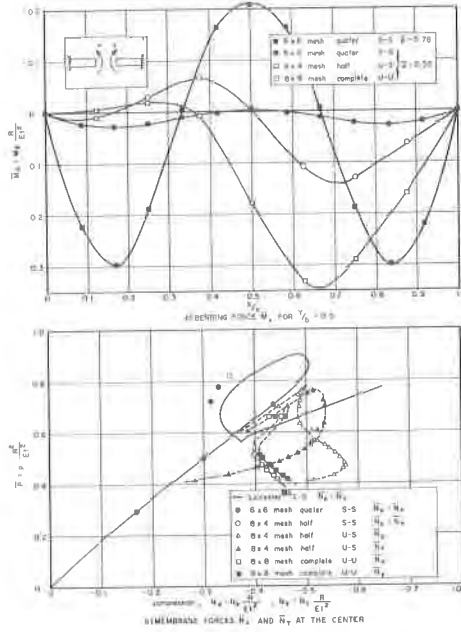


Fig. 5 : Spherical Shell. Typical Forces.

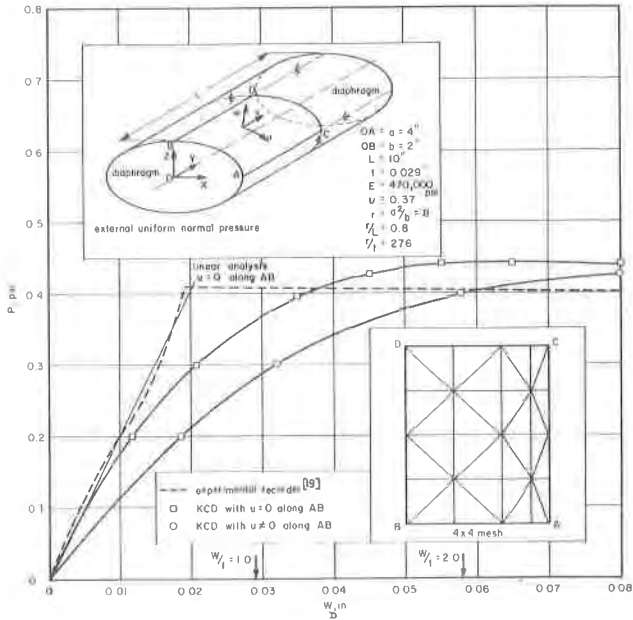


Fig. 6 : Elliptic Cylinder with KCD Element.