LIMIT AND SHAKEDOWN ANALYSIS OF STRUCTURES WITH STOCHASTIC STRENGTHS

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SUMMARY

In recent papers the writers have presented, together with full proofs of the relevant theorems, a simple procedure for bounding on both sides the conditional probability of (plastic) collapse of any structure with stochastic "local" strengths, subjected to one- and multi-parameter "static" loads: the bounds could be made closer and closer by successive approximations. The probability of collapse (and of survival) under loads of given distributions could also be bounded; see:

G. Augusti, A. Baratta: "Limit Analysis of Structures with Stochastic Strength Variations". *J. Struct. Mech.*, Vol. 1 (No. 1), pp. 43-62; 1972.

G. Augusti, A. Baratta: "Theory of Plasticity and Limit Analysis of Structures under Multi-parameter Loading". *Foundations of Plasticity*, International Symposium 1972 (A. Sawczuck, Ed.), pp. 347-364; Noordhoff International Publ., Groningen.

In this paper, the theorems and procedure are further extended to cover the case of repeated and/or alternate loads, possibly including thermal and dynamic effects. Special consideration is paid to the possibility of "incremental collapse" under variable repeated loads, which is a well-known phenomenon in "classical" limit analysis, but, to the writers' knowledge, has never been examined before for structures with random strengths, although a few recent papers have been devoted to probabilistic studies of plastic structures subjected to repeated loads.

To this aim, the probabilities can be calculated of finding, among a number of displacement and stress fields, a *kinematically sufficient* and a *statically admissible* field, respectively, under loads of given maximum values that can combine and alternate in arbitrary cycles on the structure: because of theorems analogous to those presented by G. Augusti and A. Baratta, these probabilities bound the probability that the structure "shakes down" (i.e. stabilizes) under infinite cycles of the set loads, provided that the (stochastic) yield limits remain constant throughout the loading history.

The same type of procedure applies when dynamic forces cannot be neglected, as shown by appropriate extension of the theorem formulated by G. Ceradini.

Numerical example(s) illustrate the described procedures.

It is finally shown that, when also the randomness of loads and load cycles must be taken into consideration, in both cases of stochastic and deterministic strengths, the more significant results are obtained via the "static" procedure, i.e. through consideration of statically admissible stress fields rather than kinematically sufficient displacement fields.

1) Introduction - Summary of previous results

Limit analysis of elastic-perfectly plastic bodies becomes a formidable task when a safety assessment is required under uncertainty of the limit strength parameters and of the acting loads.

The difficulties in load forecasting are tied to both the scatter in loads experienced by many structures, and to the still scarce information available as to load statistics and their most efficient formulation, as it will be briefly discussed in Sec. 4. Therefore, limit analysis under stochastic uncertainties is best separated into two problems: first, the determination of the conditional probability of survival under given loads (say, defined by a n-dimensional vector $\underline{W}$); second, the combination of this probability with the stochastic distribution of the loads $\underline{W}$ in order to get the overall reliability [1,2].

It is soon to be noted that uncertainties in the limit strength parameters, and in their variation throughout the structure, may assign a definite non-zero probability of occurrence to any admissible failure mechanism (i.e. to any mechanism such that the power of the applied loads is larger than zero) and to any stress field in equilibrium with the given loads.

Therefore, in general it is not sufficient, even under given loads, to consider only one failure mechanism and one stress field as the ultimate situation for the structure. Rather, probability laws must be defined and found on the class "A" of all admissible mechanisms, and on the class "B" of all the equilibrated stress fields.

For a deterministic structure and monotonically increasing load parameters, collapse occurs according to the mechanism for which the ratio of the dissipated plastic power D_a over the load power D is a minimum, as soon as this minimum ratio equals unity.

If the limit strength parameters are not known, but are only described by some joint probability law, for each mechanism "a" and given load parameters the plastic power D_a is a random variable: the conditional probability of survival, S_c, with respect to plastic flow collapse, can be expressed by:

$$S_c = \text{Prob}\left\{ \frac{Da}{D_w} > 1 \quad \forall a \in A;\ \ \underline{W} \text{ given} \right\} \tag{1.1}$$

and the whole class A should be considered for application of eq. (1.1).

Similarly, from the "static" point of view, the structure does not collapse if an element "b" of the class B exists such that the yield condition is satisfied in every point (synthetically $\phi(b) < 0$). This leads to the alternative expression

$$S_c = \text{Prob}\left\{ \exists\, b \in B:\ \phi(b) < 0;\ \ \underline{W} \text{ given} \right\} \tag{1.2}$$

in which the whole class B should be considered.

The kinematic and static theorems of limit analysis ensure the equivalence of eqs. (1.1) and (1.2); they can be stated in the following way, which allows their immediate application:

1.a) Kinematic Theorem: If under given loads, $\underline{W}$, an element "a" of A exists such that

$$\frac{Da}{D_w} < 1 \tag{1.3}$$

the structure collapses.

1.b) Static Theorem: If under given loads, $\underline{W}$, an element "b" of B exists such that, with the previous notation:

$$\phi(b) < 0 \tag{1.4}$$

the structure does not collapse.

The previous theorems of limit analysis find a counterpart in the analogous theorems of probabilistic limit analysis [1,2]:

2.a) Kinematic Theorems: For any real number $S_1 \varepsilon (0,1)$, if a subclass A_1 of A exists such that, under the given loads $\underline{W}$,

$$S_{c\gamma} = \text{Prob}\left\{ \frac{Da}{D_w} > 1 \ \forall a \ \varepsilon \ A_1 \right\} \leq S_1 \tag{1.5}$$

then:

$$S_c \leq S_1 \tag{1.6}$$

2.b) Static Theorem: For any real number $S_2 \varepsilon (0,1)$, if a subclass B_1 of B exists such that, under the given loads $\underline{W}$,

$$S_{c\psi} = \text{Prob}\left\{ \exists \ b \ \varepsilon \ B_1 : \phi(b) < 0 \right\} \geq S_2 \tag{1.7}$$

then:

$$S_c \geq S_2 \tag{1.8}$$

The two theorems can also be given a unified formulation, that is:

$$S_{c\psi} \leq S_c \leq S_{c\gamma} \tag{1.9}$$

for any subclass A_1 of A and for any subclass B_1 of B. An inequality similar to eq. (1.9) holds for the conditional probability of failure $P_c = 1 - S_c$:

$$P_{c\psi} = 1 - S_{c\psi} \geq P_c \geq 1 - S_{c\gamma} = P_{c\gamma} \tag{1.10}$$

2) Shakedown in Stochastic Limit Analysis

Assume now that a structure has to carry loads $\underline{W}$ varying so that different loading arrangements follow each other randomly both in intensity and succession: the components of $\underline{W}$ may include thermal and dynamical loads.

It is well known that in this case, even if the $\underline{W}$ are not able to cause collapse by themselves, a process may be set up in which plastic work is dissipated in each loading cycle, leading either to increasing permanent displacements or to rupture by alternating plasticity in one or more points (incremental collapse). If the total plastic work is bounded even for an infinite number of load repetitions, the structure is said to shakedown.

Kinematic and static theorems can be formulated for this type of plastic collapse too [6,7,8].

In particular, the kinematic theorem states that incremental collapse takes place if any admissible plastic mechanism "a" can be found such that the work L_w developed by the external loads in any cycle of loading can exceed the plastic work L_a dissipated by the plastic deformations consequent to the activation of the mechanism.

So, if Ω_a is the domain of variability of admissible loads $\underline{W}$, the probability of shakedown, when random variability of the strength parameters is introduced, can be defined:

$$S_d = \text{Prob} \{ \frac{L_a}{L_w} > 1 \ \forall a \in A, \forall \underline{W} \in \Omega_a \} \tag{2.1}$$

Now, let B_o be the class of stress fields b_o in equilibrium with zero acting loads, and B_e the set of elastic stress fields b_e associated with each admissible loading. The static (Bleich-Melan) theorem affirms that, if an element "b_o" of B_o exists such that the stress field $b = b_e + b_o$ satisfies the yield condition for any $b_e \in B_e$ and for any point $\underline{x}$, then the structure does not collapse; from this point of view, the probability of shakedown is given by:

$$S_d = \text{Prob} \{ \exists\, b_o \in B_o : \phi\,(b_o + b_e) \leq 0 \ \forall b_e \in B_e \} \tag{2.2}$$

Eqs. (2.1) and (2.2) are perfectly equivalent: their evident analogy with eqs. (1.1) (1.2) justifies immediately the following theorems of probabilistic shakedown:

1) Kinematic Theorem: For any real number $S_1 \in (0,1)$, if a subclass A_1 of A exists, such that, for a given domain of variability of loads

$$S_{d\gamma} = \text{Prob} \{ \frac{L_{a_1}}{L_w} > 1 \ \forall a_1 \in A_1 \} \leq S_1 \tag{2.3}$$

then:

$$S_d \leq S_1 \tag{2.4}$$

2) Static Theorem: For any real number $S_2 \in (0,1)$, if a subclass B_1 of B_o exists such that, for a given domain of variability of loads

$$S_{d\psi} = \text{Prob} \{ \exists\, b_1 \in B_1 : \phi\,(b_1 + b_e) < 0 \ \forall b_e \in B_e \} \geq S_2 \tag{2.5}$$

then:

$$S_d \geq S_2 \tag{2.6}$$

The two theorems can be given a unified formulation, that is:

$$S_{d\psi} \leq S_d \leq S_{d\gamma} \tag{2.7}$$

where $S_{d\psi}$ and $S_{d\gamma}$ are defined by eqs. (2.3) and (2.5). Also in this case an inequality similar to eq. (2.7) holds for the probability of incremental collapse, $P_d = 1-S_d$:

$$P_{d\psi} = 1 - S_{d\psi} \geq P_d \geq 1 - S_{d\gamma} = P_{d\gamma} \qquad (2.8)$$

It is worthwhile to remember explicitly that the considered loads may also include dynamic parameters [3,4]: in this case, the formulation of the static theorem does not vary: note only that B_o must include all stress fields consequent to elastic motions depending on all possible initial conditions, i.e. initial self-stresses, initial displacements, initial velocities, and calculated in absence of the forcing cause; while B_e must include all stress fields associated with the elastic motions consequent to the applied loading. Unfortunately, an equivalent kinematic theorem has not been found for the dynamic case, and therefore the kinematic theorem of probabilistic plastic shakedown cannot include dynamic loading.

It is interesting to note that, if the loads $\underline{W}$ are allowed to increase according to a factor k, $P_{d\psi}$ and $P_{d\gamma}$ can be visualized as the distribution probability laws of, respectively, a statically admissible factor $\tilde{\psi}_d$ [6], and of a kinematically sufficient factor $\tilde{\gamma}_d$. If $\tilde{s}_d$ is the actual safety factor, a random variable, it can be shown, in a way quite analogous to the argument led out in [1], that, denoting by $\bar{\psi}_d$, $\bar{s}_d$, $\bar{\gamma}_d$ the average values of $\tilde{\psi}_d$, $\tilde{s}_d$, $\tilde{\gamma}_d$ and by $\sigma_{d\psi}$, σ_{ds}, $\sigma_{d\gamma}$ the respective standard deviations, bounds can be obtained for these quantities formally from the same relations presented in [1] with reference to monotonic loading.

In particular, it turns out that the safety factor calculated for the deterministic structure whose strength parameters are considered equal to their respective average values (expected structure), is always larger than $\bar{s}$, the actual mean value of the safety factor.

3) A Specific Example

Numerical results as to the shakedown of the frame in Fig. 1, having 11 independent random limit moments and subjected to arbitrarily alternated vertical and horizontal forces, are presented in [9], to which the readers are referred for details of procedures and calculations.

The domain of admissibility for loading is defined by maximum and minimum load-components which are allowed to act on the structure (extreme load parameters), that is:

$$\begin{aligned} 0 \leq W_1 \leq W_{1m} \\ -W_{2m} \leq W_2 \leq W_{2m} \end{aligned} \qquad (3.1)$$

Therefore this domain only depends on two parameters, and it is possible to draw the level curves of the surface $P_{d\psi}$ $(\underline{W}_m)$ and $P_{d\gamma}$ $(\underline{W}_m)$ in the plane of the load components $(W_{1m}, W_{2m}) = \underline{W}_m$ (Fig. 2a,b). Each level line, say $P_{d\psi}$ $(\underline{W}_m) = P$ (or $P_{d\gamma}$ $(\underline{W}_m) = P$) encloses domain denoted by Ψ_{Pd} (or Γ_{Pd}). The following properties hold:

1) The level curves P_d $(\underline{W}_m) = P$ of the actual probability of incremental collapse are always in the band between the corresponding curves $P_{d\psi}$ and $P_{d\gamma}$.

2) Any loading defined by a point internal to the domain Ψ_{Pd} corresponds to a probability $P_d \leq P$. Therefore, if D_{Pd} is the domain of the extreme load parameters corresponding to $P_d \leq P$, one gets $D_{Pd} \supseteq \Psi_{Pd}$.

3) Any loading defined by a point external to Γ_{Pd} corresponds to a probability $P_d \geq P$. Therefore $D_{Pd} \subseteq \Gamma_{Pd}$.

4) Domain D_{Pd} has, for any P, the property (semiconvexity, cf. [2]), that its convex envelope is always contained in domain D_{2Pd}. This allows, if it is needed, to approach D_{Pd} by a convex domain, for sufficiently small P.

The same curves shown in Fig. 2a can be used to study the shakedown probability of the same frame subjected to a time-varying horizontal force (Fig. 3), taking into account dynamical effects. It is only necessary to transform the ordinates W_{2m} in the maximum value of the total shear force by means of a suitable law of the motion. In fact, if c is the reactive elastic shear force consequent to a unitary displacement, and u(t) is any chosen law of the motion, one gets

$$W_2(t) = c\,u(t) \tag{3.2}$$

and

$$W_{2m} = c\,u_{max} \tag{3.3}$$

If thermal stresses act on the frame (Fig. 4) besides the loads W_1 and W_2, the elastic bending moments are modified, and the number of the load parameters on which P_d depends increases. Fig. 5 shows a pair of curves in the plane Δt, W_{1m}, for a given ratio $\alpha = W_{2m}/W_{1m} = \bar{\alpha}$.

4) Considerations on Stochastic Load Applications

So far, the treatment of probabilistic limit analysis has not included any consideration on the actual history of load applications: also the procedures introduced in Sec. 2 and applied in Sec. 3 have safely introduced the probability of shakedown S_d (and of incremental collapse $P_d = 1 - S_d$) with respect to the possibility of infinite load repetitions.

A complete determination of the reliability (total probability of survival) of an elasto-plastic structure would involve the knowledge of the characteristics of the multivariate stochastic process defining the multidimensional loads applied on the structure. Quite apart from the practical impossibility of such statistics, the computations would become too lengthy and complicated to be justified in front of the simplifications inherent in limit analysis.

Assume, on the contrary, that only the statistics of the (independent) parameters defining the extreme loads applied on the structure, $\underline{W}_m$, are known.

Consider first a structure with deterministic strengths. In the $\underline{W}_m$ space, four regions can be in general individuated, namely the elastic region Δ_E, the shakedown region Δ_{SD}, the incremental collapse region Δ_{IC}, the plastic collapse region Δ_P. These four regions are qualitatively indicated in Fig. 6 with reference to the example described in Sec. 3. (Note that in this example the plastic collapse boundary (yield domain) is the same in the space of the extreme loads and in the space of the instantaneously acting loads: this is true whenever the latter domain is such that, taking any internal point $\underline{W}$, all points $\underline{W}'$: $0 \leq \underline{W}' \leq \underline{W}$ are also internal; note further that in a number of cases the incremental collapse region vanishes.)

If $\underline{W}_m$ falls either in the incremental collapse region Δ_{IC} or in the shakedown region Δ_{SD}, plastic work is dissipated in each load cycle: in the latter case ($\underline{W}_m \in \Delta_{SD}$) only, the total dissipated work (and related permanent displacement) is bounded irrespective of the number of cycles. However, if the vector $\underline{W}$ of the instantaneous loads falls in Δ_{IC} a finite number of times, the total dissipated work (and permanent displacements) remain limited: and this is always true, unless either the lifetime of the structure or the load frequency are considered infinite. Thus, upon elementary considerations on the load history, the clear-cut distinction between incremental collapse and shakedown appears to fade away even for a deterministic structure. When stochastic uncertainties may interest also the strength parameters, it can only be said that the conditional probability of survival S for the given $\underline{W}_m$ is not smaller than the probability of shakedown S_d defined in Section 2

$$S_d\left[\underline{W}_m\right] \leq S\left[\underline{W}_m\right] \tag{4.1}$$

the equality sign holding when infinite cycles of the loads $\underline{W}_m$ are applied.

On the other hand, the structure does collapse if the loads fall once or more in the plastic collapse region Δ_P; whence

$$S\left[\underline{W}_m\right] \leq S_c\left[\underline{W}_m\right] \tag{4.2}$$

Eqs. (4.1) and (4.2) bound the conditional probability of survival $S[\underline{W}]$, and therefore the reliability

$$R = \int_\Omega S\left[\underline{W}_m\right]\, p\left[\underline{W}_m\right]\, d\,\underline{W}_m \tag{4.3}$$

where $p\left[\underline{W}_m\right]$ is the joint probability density of the (extreme) load parameters and Ω its region of definition.

On the other hand, it has been shown in the previous sections that bounds on S_c and S_d can be calculated rather easily, while their exact determination would be very time-consuming. Combining the bounds, eqs. (1.9), (2.7), (4.1) and (4.2),

$$S_{d\psi}\left[\underline{W}_m\right] \leq S\left[\underline{W}_m\right] \leq S_{c\gamma}\left[\underline{W}_m\right] \tag{4.4}$$

$$\int_\Omega S_{d\psi}\left[\underline{W}_m\right]\, p\left[\underline{W}_m\right]\, d\underline{W}_m \leq R \leq \int_\Omega S_{c\gamma}\left[\underline{W}_m\right]\, p\left[\underline{W}_m\right]\, d\underline{W}_m \tag{4.5}$$

In other words, fully reliable bounds can be obtained by applying the kinematic approach with respect to the instantaneous plastic collapse, and the static approach with respect to incremental collapse. On the other hand, the possibility of the latter type of collapse should not be overlooked when the number of load applications is not negligible: the procedures outlined in this paper offer a comparatively easy way for taking it into account.

A reliability formula more precise than eq. (4.5) seems unjustified at the present state of knowledge of probabilistic limit analysis; while it is to be hoped for some treatment which could take a more detailed account of the actual load repetitions.

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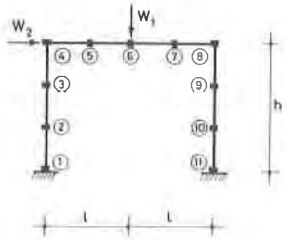

Fig 1

Fig. 1: Example frame, with sections of possible yielding

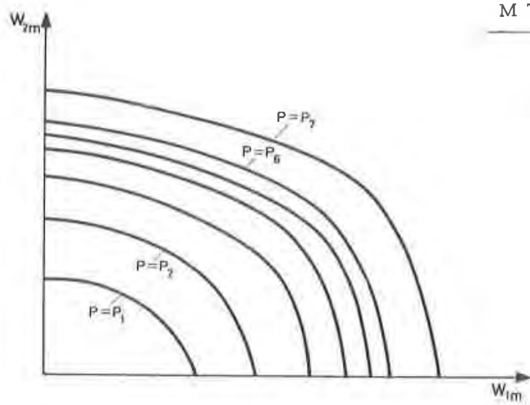

Fig.2,a: Ψ_{Pd} Domains

Fig. 2a,b: Inner and outer bounds to the level curves of the incremental collapse probability

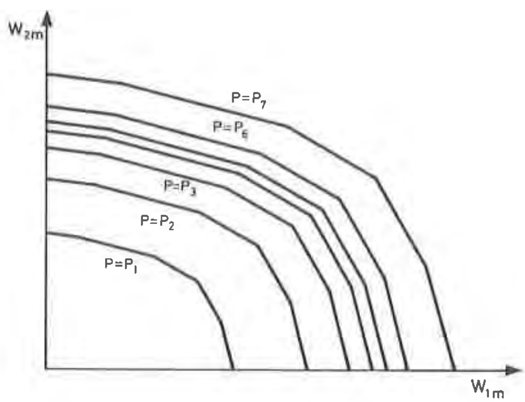

Fig.2,b: Γ_{Pd} Domains

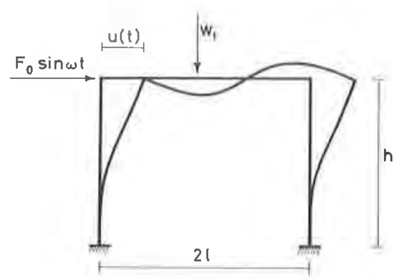

Fig.3

Fig. 3: Frame with time-varying horizontal force

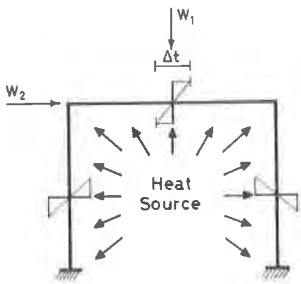

Fig. 4

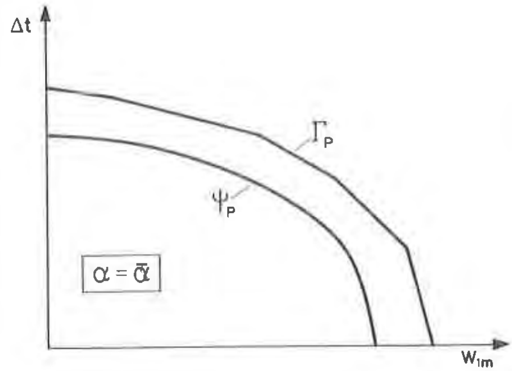

Fig 5

Fig. 4: Frame with heat source

Fig. 5: Shakedown curves with variable temperature

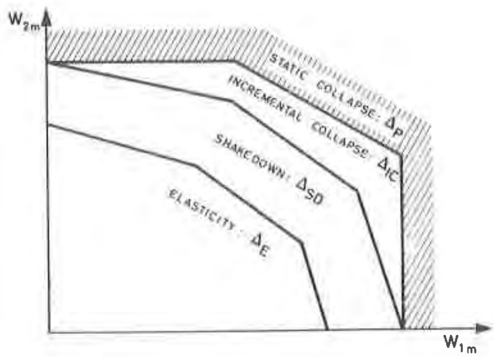

Fig. 6

Fig. 6: Deterministic frame; response regions