SPARE PARTS RESERVATION IN THE CASE OF COMPONENTS
SUBJECTED WEAR-OUT AND/OR FATIGUE ACCORDING TO
A WEIBULL DISTRIBUTION

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SUMMARY

In its simplest form, the problem of the reservation of spare parts can be presented in
the following manner: at \( t = 0 \) the first spare part begins its work-period, it fails at a random
time \( t = \tau_1 \), and is replaced by the second spare part, which, in its turn, fails at \( t = \tau_1 + \tau_2 \)
and is replaced by the third spare part, and so forth. By assuming that the time necessary
to carry out the replacement can be neglected, and that the random times (life lengths of
the spare parts) \( \tau_1, \tau_2, \tau_3, \ldots \) are statistically independent and belong to the same distribution
function, say \( F(x) \), the sequence \( \{\tau_i\} \) constitutes an ordinary renewal process. (*)

The question to be asked is: "How many spare parts are necessary in order to assure
with a probability \( \geq W \) (say \( \geq 0.9 \)) an uninterrupted operation over the time interval \( (0, T) \)?", or,
in the inverse formulation: "How large is the probability that \( n \) spare parts will suffice
to assure an uninterrupted operation over the time interval \( (0, T) \)?"
The answer is easily found, and given by the expression
\[
W = W(n, T) = \Pr \{\tau_1 + \tau_2 + \tau_3 + \ldots + \tau_n > T\}
= 1 - \Pr \{\tau_1 + \tau_2 + \tau_3 + \ldots + \tau_n \leq T\}
= 1 - F_n(T) = 1 - \int_0^T F_{n-1}(T-x) \, dF(x)
\]
with \( F_i(x) = F(x) = \Pr \{\tau_i \leq x\} \), \( i = 1, 2, 3, \ldots \) (1)

It can be shown that the reservation of \( n = T/MTTF \) spare parts, where \( MTTF = E \{\tau_i\} \)
is the expected value (mean) of the life lengths of each spare part, will assure the uninterrupted
operation over \( (0, T) \) in only approx. 50% of all cases.

In this contribution, the expression (1) will be studied for the case of a Weibull distribution
\[
F(x) = 1 - \exp(-{(\rho x)^\beta}), \quad x \geq 0; \rho, \beta > 0
\]
with the parameter \( \beta \geq 1 \). For \( \beta > 1 \) the failure rate
\[
\lambda(x) = \frac{F'(x)}{1-F(x)} = \rho(\rho x)^{\beta-1}
\]
increases monotonely; this fact explains the frequent occurrence of this distribution function
in the case of components subjected to wear-out and/or fatigue. For \( \beta=1 \) one has the
exponential distribution function.

For both exponential and Gamma distribution functions the solution of (1) leads to
the \( \chi^2 \)-distribution. However, the case of the Weibull distribution must be treated completely
numerically (even for \( \beta=2 \)).

After a brief exposition of the model charts of the solution of (1) for \( W = 0.99, 0.95, 0.9, \)
0.8 and \( n \leq 10 \) are given. For large \( n \), an approximate solution is derived with the help of
the central limit theorem. Some applications are discussed.

(*) If the replacement times can not be neglected, one must simply operate with an alternating
renewal process (which, in this particular problem can be treated like a modified renewal process).