

A METHOD OF ANALYSIS FOR THE CREEP BUCKLING OF TUBES UNDER EXTERNAL PRESSURE

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ABSTRACT

A numerical method for the creep buckling of long circular thin-walled cylinders which are initially out-of-round and subjected to uniform external pressure is presented. It is assumed that the initial out-of-roundness and subsequent deflections are independent of the axial coordinate and that the initial stress state is elastic. The method eliminates many of the major assumptions previously used and can handle any combination of the strain or time hardening laws and the Mises or Tresca flow rules. The method is easily computerized. A number of specific examples are studied for 15% cold worked Zircaloy-2. The results indicate that the calculated buckling times are highly dependent upon the hardening law and flow rule used.

NOMENCLATURE

θ, z, r	= tangential, axial, and radial coordinates
y	= radial distance from midsurface
t	= time coordinate
Δ	= change during an incremental time period
$\sigma_{\theta}, \sigma_z, \sigma_r$	= stress at any point of tube wall
σ_e	= effective stress
$\epsilon_{\theta}, \epsilon_z, \epsilon_r$	= total strains at any point of tube wall
$\epsilon'_{\theta}, \epsilon'_z, \epsilon'_r$	= elastic strains at any point of tube wall
$\epsilon''_{\theta}, \epsilon''_z, \epsilon''_r$	= creep strain at any point of tube wall
$\dot{\epsilon}''_{\theta}, \dot{\epsilon}''_z, \dot{\epsilon}''_r$	= creep strain rates at any point of tube wall
ϵ''_e	= effective creep strain
$\dot{\epsilon}''_e$	= effective creep strain rate
W_1	= initial deflection of center line of tube wall from a perfect circle (under zero external pressure)
W	= deflection of center line of tube wall circle caused by bending

- W_t = total deflection of center line of tube wall from a perfect circle
 ($W_t = W + W_1$)
 W_{cA} = deflection of center line of tube wall due to average tangential strain
 W_{10}, W_{0}, W_{t0} = values of W_1 , W , and W_t at $\theta = 0$
 M = tangential bending moment in tube wall
 M_0 = tangential bending moment in tube wall at $\theta = 0$
 m^2 = $1 + pr_m^3/D$
 κ = change in curvature of center line of tube wall due to bending
 κ_0 = change in curvature at $\theta = 0$
 T = temperature at any point in tube wall
 p = external pressure

$$p_{cr} = \frac{E}{4(1-\nu^2)} \left(\frac{h}{r_m}\right)^3 - \text{critical external pressure assuming elastic conditions and small } h/r_m$$

- N_0 = p/p_{cr}
 h = thickness of tube wall
 r_0 = outside radius of tube wall
 r_m = mean radius of tube wall
 E = modulus of elasticity
 ν = Poisson's ratio
 α = coefficient linear thermal expansion
 G = shear modulus
 λ = $\frac{\nu E}{(1+\nu)(1-2\nu)}$ - Lamé's constant
 I = $h^3/12$ - moment of inertia of unit axial length of wall thickness
 D = $\frac{Eh^3}{12(1-\nu^2)}$ - rigidity of unit axial length of wall thickness
 A = $2\pi r_m h$ - cross sectional area of tube

1. INTRODUCTION

A method of analysis for estimating the stress-strain history and creep-buckling time of long circular tubes (thin-walled cylinders) which are initially out-of-round and subjected to uniform external pressure is described here. The method is a numerical one which eliminates the need for major simplifying assumptions and is easily computerized. Since the method is numerical in nature various types of flow rules, Mises or Tresca, and creep laws, strain hardening or creep hardening, can be used in the analysis. As will

be shown in the examples considered the influence of the flow rules and creep laws used has a significant effect on the calculated creep buckling time. In the following sections the basic assumptions involved will be stated and the general method will be described.

2. METHOD OF ANALYSIS

The basic technique used in this method of creep analysis of out-of-round tubes is similar to that of Mendelson et.al. [1], Lin [2], and Wilson and Davis [3]. The method is a numerical one in which the creep time is divided up into a large number of small time increments. Each time increment is taken small enough such that the stress can be considered to be constant during the increment and to increase as a step function at the end of the increment. The stresses at zero time are taken as the elastic stresses. By considering the stresses constant during a particular time increment the change in creep strains during that increment can be determined. Knowing the changes in the creep strains which occurred, the changes in total strains can then be determined from equations developed from the conditions of equilibrium and compatibility. Finally, knowing the change in creep strain and total strain, the change in elastic strain and thus stress can be calculated. The newly calculated stresses are then held constant during the next time increment. And so the procedure described above continues in a cyclic manner until collapse (gross deformation) occurs. As the time increments become infinitesimally small, the calculated stresses, strains, and deflections approach the exact solution for the assumptions involved. The equations which are involved in the cyclic process will now be developed. First the assumptions involved will be discussed.

2.1 Assumptions

(a) The cylinder considered in this analysis is long enough such that the end effects can be neglected. The initial out-of-roundness and subsequent deflections of the cylinder are not functions of the axial coordinate.

(b) The initial (t=0) stresses, strains, and deflections of the cylinder are elastic. This analysis could be extended to consider the effects of initial plastic flow.

(c) Planes originally perpendicular to the axial direction, remain plane and perpendicular to the axial direction. That is, the total axial strain is a function of time and not the radial or tangential coordinates.

(d) Planes initially perpendicular to the tangential coordinate remain plane. That is, total tangential strains vary linearly with radius at each tangential cross section.

(e) Assumed radial stress and temperature distributions are functions of the radial coordinate only.

2.2 Basic Relations

The conditions of equilibrium and the strain-displacement relations must be satisfied at all times. There are three equilibrium conditions which must be satisfied.

(a) Axial force equilibrium (the coordinate system is shown in Fig. 1):

$$r_o^2 p = - \int_0^{2\pi} \int_{-h/2}^{+h/2} \sigma_z(r_m + y) dy d\theta \tag{1}$$

(b) Tangential force equilibrium: $+h/2$

$$pr_o = - \int_{-h/2}^{+h/2} \sigma_\theta dy \quad (2)$$

(c) Equilibrium of tangential moments:

$$M = \int_{-h/2}^{+h/2} \sigma_\theta y dy \quad (3)$$

where the moment M is a function of radial displacement,

$$M = M_o + pr_m (W_t - W_{to}) \quad (4)$$

The strain-displacement relations and strain conditions which must be satisfied are:

(a) Linear relation for total tangential strain:

$$\epsilon_\theta = \epsilon_A + \kappa y \quad (5)$$

where ϵ_A is the strain at the center of the cylinder wall and κ is the change in the curvature of the tube wall due to bending.

(b) Curvature-deflection relationship:

$$\kappa = - \frac{1}{r_m} \left[W + \frac{d^2 W}{d\epsilon^2} \right] \quad (6)$$

This expression is derived by Timoshenko [4].

The stress-elastic strain relationships must also be satisfied. The relationships can be written in terms of total and creep strains.

$$\sigma_\theta = \lambda(\epsilon_\theta + \epsilon_r + \epsilon_z) + 2G(\epsilon_\theta - \epsilon_\theta'') - \frac{\lambda ET}{1-2\nu} \quad (7)$$

$$\sigma_z = \lambda(\epsilon_\theta + \epsilon_r + \epsilon_z) + 2G(\epsilon_z - \epsilon_z'') - \frac{\lambda ET}{1-2\nu} \quad (8)$$

$$\sigma_r = \lambda(\epsilon_\theta + \epsilon_r + \epsilon_z) + 2G(\epsilon_r - \epsilon_r'') - \frac{\lambda ET}{1-2\nu} \quad (9)$$

In eqs. (7), (8), and (9) use is made of the incompressibility of creep strains.

$$\epsilon_\theta'' + \epsilon_z'' + \epsilon_r'' = 0 \quad (10)$$

Equations (7) through (10) can be combined to give more convenient expressions for ϵ_θ and σ_z .

$$\sigma_\theta = \frac{E}{(1-\nu^2)} [\epsilon_\theta + \nu\epsilon_z - (\epsilon_\theta'' + \nu\epsilon_z'') - (1+\nu)\alpha T + \frac{(1+\nu)}{E} \sigma_r] \quad (11)$$

$$\sigma_z = \frac{E}{(1-\nu^2)} [\epsilon_z + \nu\epsilon_\theta - (\epsilon_z'' + \nu\epsilon_\theta'') - (1 + \nu)\alpha T + \frac{(1+\nu)}{E} \sigma_r] \quad (12)$$

Flow rules of some type must be used to obtain the triaxial creep rates from uniaxial creep data. The von Mises flow rule and the Tresca flow rule will be considered. Other rules can just as easily be used with the method of solution presented here. The von Mises flow rules are defined by the following equations:

$$\dot{\epsilon}_z'' = \frac{\dot{\epsilon}_e''(\tau_e, \epsilon_e'')}{\tau_e} \left(\sigma_z - \frac{\sigma_z}{2} - \frac{\sigma_r}{2} \right) \quad (13)$$

$$\dot{\epsilon}_r'' = \frac{\dot{\epsilon}_e''(\tau_e, \epsilon_e'')}{\sigma_e} \left(\sigma_z - \frac{\sigma_z}{2} - \frac{\sigma_r}{2} \right) \quad (14)$$

$$\dot{\epsilon}_r'' = -\dot{\epsilon}_z'' - \dot{\epsilon}_z''$$

where

$$\tau_e = \frac{1}{\sqrt{2}} [(\tau_z - \tau_r)^2 + (\sigma_z - \sigma_r)^2 + (\sigma_z - \sigma_z)^2]^{1/2} \quad (15)$$

$$\epsilon_e'' = \sqrt{\frac{2}{3}} [(\epsilon_r'')^2 + (\epsilon_z'')^2]^{1/2} \quad (16)$$

$$\dot{\epsilon}_e'' = F_1(\tau_e, \epsilon_e'') \text{ or } \dot{\epsilon}_e'' = F_2(\tau_e, t) \quad (17)$$

The first equation of eq. (17) is the form of the equation of state for a strain hardening creep and the second equation is applicable for a time hardening creep. The functions F_1 and F_2 are determined from uniaxial creep data.

When the Tresca flow rule is used eq. (17) is combined with the following:

$$\dot{\epsilon}_1'' = \frac{\dot{\epsilon}_e''}{\tau_e} (\tau_1 - \tau_3) \quad (18)$$

$$\dot{\epsilon}_2'' = 0 \quad (19)$$

$$\dot{\epsilon}_3'' = -\dot{\epsilon}_1'' \quad (20)$$

where

$$\tau_e = \tau_1 - \tau_3 \quad (21)$$

$$\epsilon_e'' = \frac{\epsilon_1'' - \epsilon_2''}{2} \quad (22)$$

and $\tau_1 = \sigma_2 = \sigma_3$. The stresses σ_1 , σ_2 , and σ_3 are related to σ_θ , σ_z , and σ_r by the above inequalities. Usually, initial elastic conditions result in $\sigma_r > \sigma_z > \sigma_\theta$, but as time increases the relative magnitude of these stresses will shift at some points in the tube wall.

By properly combining these given relations, the equations needed for carrying out the incremental time step procedure outlined above can be obtained. In obtaining these equations it is assumed that the initial out-of-roundness can be expressed as $W_1 = W_{10} \cos 2\theta$. As will be evident, the method described here is equally applicable to any out-of-roundness which can be expressed by terms of a Fourier series.

2.3 Initial Elastic Solution

By combining eqs. (1) through (12) the following initial elastic relations can be obtained.

$$\begin{aligned} \epsilon_z = & - (1/2) \frac{r_o}{r_m} - \nu) \frac{r_o p}{Eh} + \frac{\alpha}{h} \int_{-h/2}^{h/2} T \, dy - \frac{\nu}{Eh} \int_{-h/2}^{h/2} \sigma_r \, dy \\ & + \frac{\alpha}{r_m h (1-\nu)} \int_{-h/2}^{h/2} T y \, dy - \frac{\nu}{E r_m h (1-\nu)} \int_{-h/2}^{h/2} \sigma_r y \, dy \end{aligned} \quad (23)$$

$$\begin{aligned} \epsilon_A = & - \frac{p r_o (1-\nu^2)}{Eh} - \nu \epsilon_z + \frac{(1+\nu)\alpha}{h} \int_{-h/2}^{h/2} T \, dy \\ & - \frac{\nu(1+\nu)}{Eh} \int_{-h/2}^{h/2} \sigma_r \, dy \end{aligned} \quad (24)$$

$$M_o = - \frac{\alpha E}{(1-\nu)} \int_{-h/2}^{h/2} T y \, dy + \frac{\nu}{(1-\nu)} \int_{-h/2}^{h/2} \sigma_r y \, dy + \frac{3 p r_m W_{10}}{(4-m^2)} \quad (25)$$

$$W = \frac{p r_m^3}{D(4-\lambda^2)} \cos 2\theta \quad (26)$$

$$\kappa = \frac{3 p r_m}{D(4-\lambda^2)} \cos 2\theta \quad (27)$$

Another expression of interest is the radial deflection due to the average tangential strain in the tube wall. This deflection is

$$W_{\epsilon_A} = r_m \epsilon_A \quad (28)$$

where the value of ϵ_A is obtained from eq. (24). The initial elastic stresses can be determined by substituting eqs. (23), (24), and (27) into eqs. (11) and (12) with $\epsilon_\theta'' = \epsilon_z'' = 0$.

2.4 Creep Analysis

As stated above, by taking very small time increments and assuming that the stress doesn't change during the increment, but increases as a step function at the end of the increment, the changes in creep strain during the increment can be determined. The corresponding changes in total strains and deflections can be related to these changes in creep strains by combining some of the basic relations previously given. The resulting equations are listed below.†

$$\Delta \epsilon_z = \frac{1}{A(1-\nu^2)} \int_0^{2\pi} \int_{-h/2}^{h/2} \left\{ (1-\nu^2) r_m \Delta \epsilon_z'' + (\Delta \epsilon_z'' + \nu \Delta \epsilon_\theta'') y \right\} dy d\theta \quad (29)$$

$$\Delta \epsilon_A = -\nu \Delta \epsilon_z + \frac{1}{h} \int_{-h/2}^{h/2} (\Delta \epsilon_\theta'' + \nu \Delta \epsilon_z'') dy \quad (30)$$

$$\frac{d^2(\Delta W)}{d\zeta^2} + m^2(\Delta W) = F(\theta) \quad (31)$$

where

$$F(\theta) = -\frac{r_m^2}{I} \int_{-h/2}^{h/2} (\Delta \epsilon_\theta'' + \nu \Delta \epsilon_z'') y dy + \frac{r_m^2}{D} (p r_m \Delta W_\theta - \Delta M_\theta) \quad (32)$$

By following Lin [2] and representing the righthand side of eq. (32) as a Fourier series, differential eq. (31) can be readily solved. The coefficients of the Fourier series

$$F(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos n\theta + b_n \sin n\theta \right\} \quad (33)$$

are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} F_\theta(\theta) \cos n\theta d\theta$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} F_\theta(\theta) \sin n\theta d\theta$$

†The derivation of these equations is described elsewhere by Wilson [5].

Now for the case of $W_1 = W_{10} \cos 2\theta$, it follows from the resulting symmetry that $b_n = 0$ for all n and $a_n = 0$ for odd values of n .

Using this Fourier series to represent the right side of differential eq. (31) and the boundary conditions, $d(\Delta W)/d\theta = 0$ at $\theta = 0$ and $\pi/2$, the solution of the equation becomes

$$\Delta W = - \sum_{n=2,4,6,\dots} \frac{a_n}{n^2 - m^2} \cos n\theta \quad (34)$$

$$\Delta \kappa = - \frac{1}{r_m} \sum_{n=2,4,6} \frac{n^2 - 1}{n^2 - m^2} a_n \cos n\theta \quad (35)$$

$$\Delta M_o = - \frac{2D}{\pi I} \int_0^{\pi/2 + h/2} \int_0^{-h/2} (\Delta \epsilon_{\theta}'' + \nu \Delta \epsilon_z'') y \, dy \, d\theta + p r_m \Delta W_o \quad (36)$$

where the a_n 's are determined from eqs. (32) and (33).

The change in the uniform radial deflection due to the change of the average tangential strain in the tube wall is

$$\Delta W_{\epsilon_A} = \frac{2r_m}{\pi} \int_0^{\pi/2} \Delta \epsilon_A \, d\theta \quad (37)$$

The procedure used to determine the changes in displacements, strains, and stresses during a particular time increment will now be described. Each time increment is taken small enough such that at each point in the tube wall the change in stress during the increment is a very small portion of the stress at the beginning of the interval. Therefore, the error produced by assuming that the stress is constant during a time increment is small. The error can be reduced as much as is desired by taking smaller and smaller time increments.

By holding the effective stresses constant during each time increment the changes in creep strains during the increment can be calculated at a finite number of points on the cross section of the cylinder from eqs. (13) and (14) or (18), by putting them in a revised form similar to the following equation:

$$\Delta \epsilon_{\theta}'' = \frac{\Delta \epsilon_e''}{\sigma_e} (\sigma_{\theta} - 1/2 \sigma_z - 1/2 \sigma_r) \quad (38)$$

The value of $\Delta \epsilon_e''$ is determined by first finding the point on the effective creep surface $F_3(\sigma_e, \epsilon_e'', t) = 0$ which corresponds to σ_e and either ϵ_e'' (strain hardening) or t (time

hardening) at the beginning of the time increment. Then by moving along the curve upon which σ_e is constant, the change in $\Delta \epsilon_e''$ during the time increment Δt can be determined.

Once the changes in creep strain are determined at a finite number of points on the cross-sectional plane of the cylinder, the integrals in eqs. (29), (30), (32), and (33) can be determined by numerical integration. Therefore, $\Delta \epsilon_z$, $\Delta \epsilon_o(\theta)$, $\Delta W(\theta)$, and $\Delta \kappa(\theta)$ can now be calculated. By knowing $\Delta \epsilon_o(\theta)$ and $\Delta \kappa(\theta)$, the change in total tangential strain $\Delta \epsilon_\theta(\theta)$ can be determined.

$$\Delta \epsilon_\theta = \Delta \epsilon_A + \Delta \kappa y \quad (39)$$

Once the changes in total strains have been determined during the Nth time increment, the changes in elastic strains can be evaluated by subtracting the changes in creep strains from the changes in total strains. Next the changes in stress can be calculated by use of the elastic stress-strain relations. At the end of the Nth time increment the stresses, strains, and displacements are

$$\begin{aligned} \sigma_{\theta N} &= \sigma_{\theta N-1} + \Delta \sigma_{\theta N} \\ \epsilon_{\theta N}'' &= \epsilon_{\theta N-1}'' + \Delta \epsilon_{\theta N}'' \end{aligned} \quad (40)$$

$$W_N(\theta) = W_{N-1}(\theta) + \Delta W_N(\theta)$$

For the next time increment (N+1) the newly determined stresses are held constant and the changes in creep strains are calculated. This cyclic process continues until the deflections become exceedingly large.

3. APPLICATION TO ZIRCALOY TUBES

A number of specific cases are investigated employing the method described above. In particular the influence of various combinations of flow rules and hardening laws are studied. Unless stated otherwise all cases considered have the same geometry ($r_m = 0.32$ in., $h = 0.035$ in., and $W_{10} = 0.0009$ in.) and external pressure ($p = 2000$ psi). Also the influences of temperature and radial stress are neglected. The material properties used are those for 15% cold worked Zircaloy-2 at 660°F ($E = 12 \times 10^6$ psi, $\nu = 0.3$). Creep data for this material was obtained by Pankaskie [6]. The data has been represented by an analytic function,[†] a plot of which is shown in Fig. 2.

In Fig. 3(a) the time-deflection curves are compared for the strain hardening and the time hardening creep laws. In both cases the Mises flow rule was used. As can be observed, the time hardening collapse time is approximately ten times larger than the strain hardening collapse time. It is generally believed that the strain hardening relation is more accurate.

[†] Form of creep functions given elsewhere by Wilson [5].

The time deflection curve obtained using the Tresca flow rule is compared with that obtained using the Mises flow rule in Fig. 3(b). For both cases the strain hardening creep law was imposed. The collapse time corresponding to the Tresca flow rule is less than one tenth the collapse time corresponding to the Mises flow rule.

Since the instability condition is being analyzed the difference in calculated collapse times for the various combinations of creep hardening laws and flow rules is not unexpected. The difference in creep rates between the various combinations which are small during the early stages of creep are very rapidly amplified with increasing time. The larger these differences become the more rapid the differences increase, resulting in substantial differences in collapse times.

The approximate influence of radial stress on the collapse time is considered by assuming a linear variation of σ_r through the wall thickness which satisfies the two normal traction conditions ($\sigma_r = 0$ at $y = -h/2$ and $\sigma_r = -p$ at $y = h/2$). The effect of radial stress appears explicitly in the equations for σ_θ , σ_x , and σ_a . For the Mises flow rule and strain hardening the resulting collapse time is 30% greater than that obtained when the radial stress was neglected.

Variations of some of the stress and strain distributions at $\theta = 0$ as a function of time have been plotted for the Mises flow rule and strain hardening combination. In Fig. 4 the tangential stress and total strain distributions are shown, and in Fig. 5 the tangential and axial creep strain distributions are shown. The axial stress distribution and axial total strain are shown in Fig. 6. As can be noted in Fig. 6(b) the axial total strain changes very little with time.

The size of time increments, number of differences used for the numerical integrations, and number of Fourier series terms used in obtaining the results presented above were selected such that any further refinement would have an insignificant influence on these results.

4. SUMMARY

A numerical method for the calculation of creep-buckling times of long thin-walled circular cylinders which are initially out-of-round and subjected to uniform external pressure has been presented. The initial out-of-roundness and subsequent deflections are not functions of the axial coordinate. The method eliminates the need for many of the major assumptions previously used. Any combination of either time hardening or strain hardening and either the Mises flow rule or the Tresca flow rule can be used. All of the pertinent equations for the method have been given. The method is easily computerized and requires very little computer time.

A number of specific cases were studied using the creep properties of 15% cold worked Zircaloy-2. The results indicate on extreme sensitivity of calculated collapse time to the type of flow rule and hardening law used. It is of interest to note that by using the modified Bailey flow rule proposed by Wahl [7] collapse times between that predicted by the Tresca flow rule and the Mises flow rule could be obtained.

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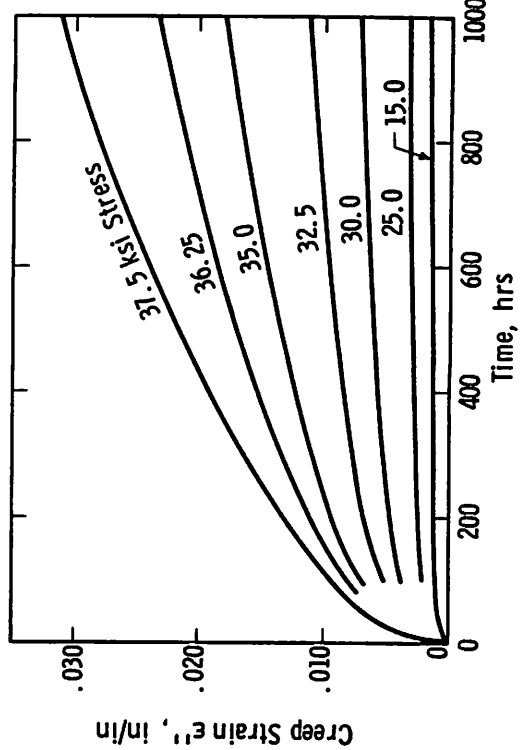
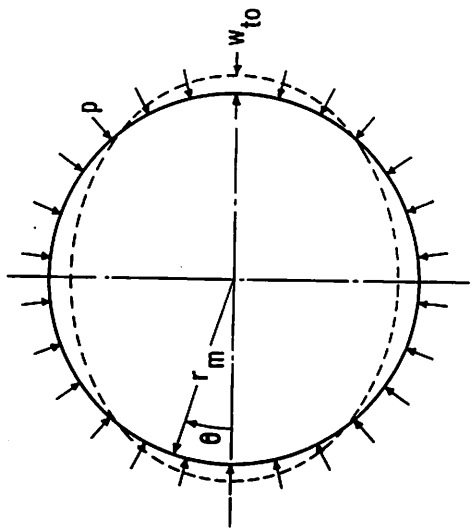
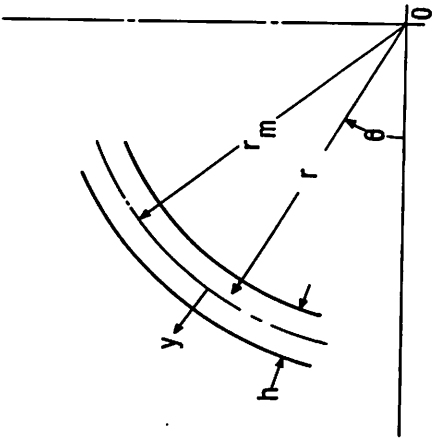
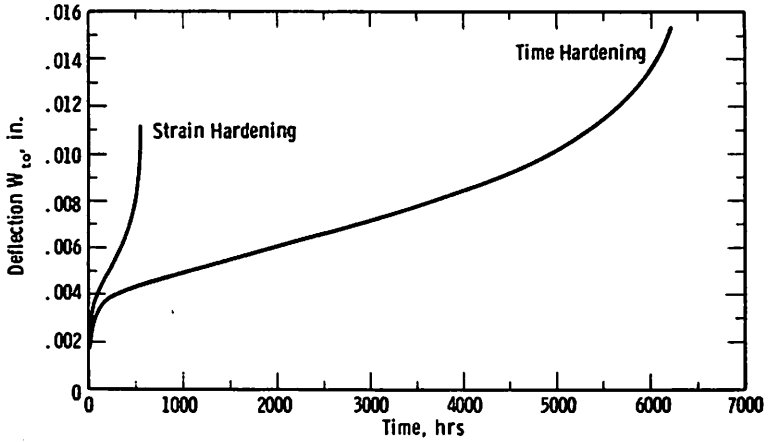
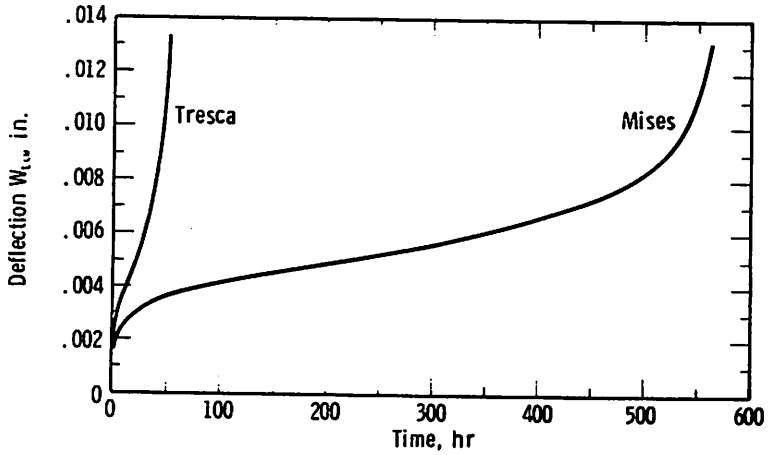


Fig. 2—Assumed creep curve for 15% cold worked zircaloy, 660°F

Fig. 1—Coordinate system for tube under external pressure



(a) Mises flow rule



(b) Strain hardening

Fig. 3—Comparison of time - deflection curves for various combinations of flow rules and hardening laws

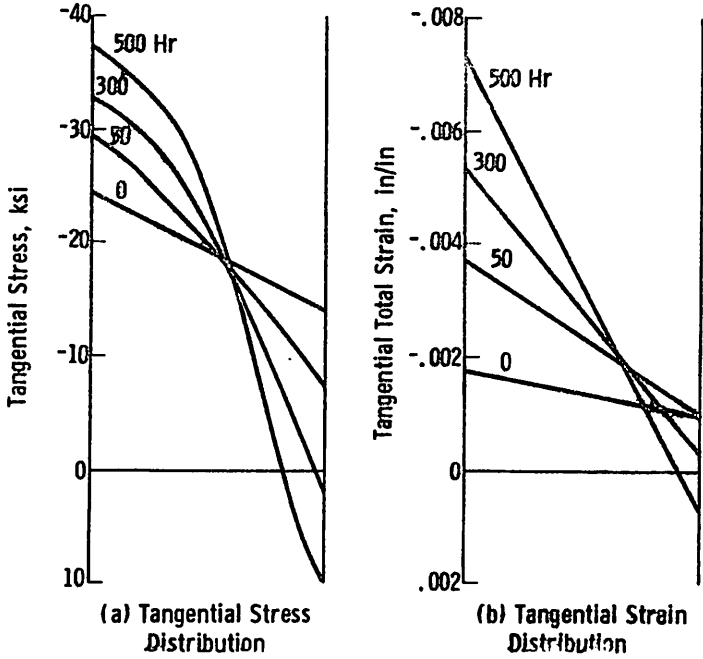


Fig. 4—Tangential stress and total strain distribution on the $\theta = 0$ plane at various times for the Mises-strain hardening case

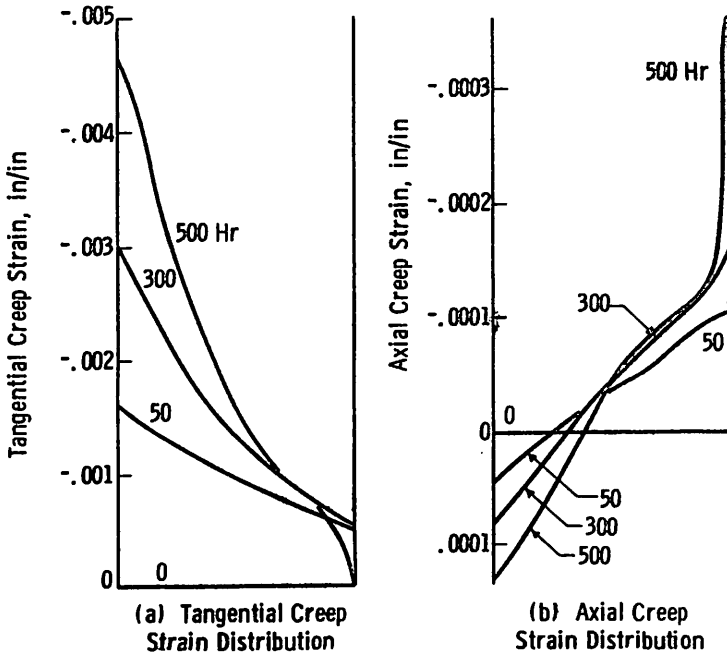
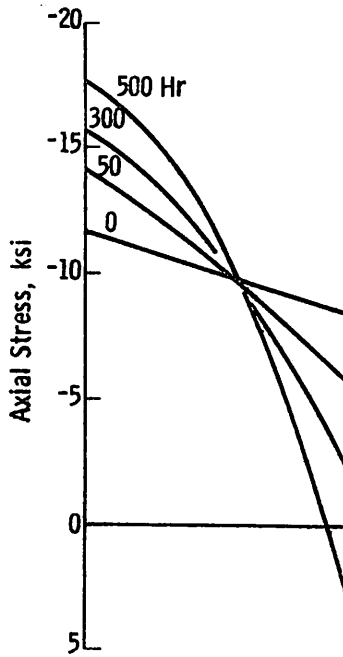
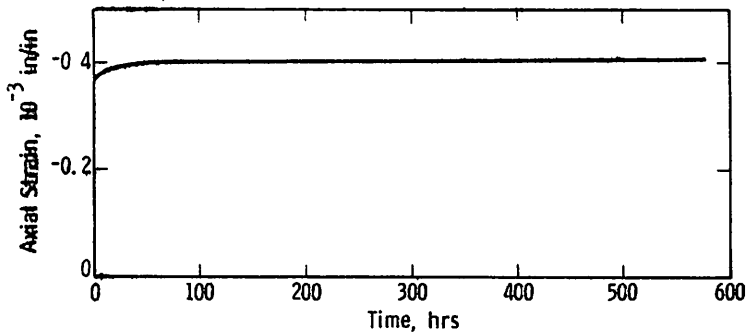


Fig. 5—Tangential creep strain and axial creep strain distribution on the $\theta = 0$ plane at various times for the Mises-strain hardening case



(a) Axial stress distribution



(b) Axial total strain

Fig. 6—Axial stress distribution on the $\theta = 0$ plane and axial total strain as function of time for Mises - strain hardening case