INFLUENCE OF FINITE DURATION PRESSURE PULSE ON THE TRANSIENT RESPONSE OF ELASTIC-PLASTIC, STRAIN-RATE-SENSITIVE CYLINDRICAL AND SPHERICAL SHELLS

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ABSTRACT

Cylindrical and spherical shells, of interest as reactor containment vessels and piping, are subjected to uniform internal pressure pulse loadings that are rectangular in time and of large magnitude. The structures analyzed are constructed of elastic-plastic materials including strain hardening and linear strain rate sensitivity. It is assumed for the cylindrical shell that all points in the reference surface undergo uniform radial motion, and, for the spherical shell, that motion is spherically symmetric. The solutions, obtained in terms of elementary functions, are utilized to investigate the influence of the pressure pulse duration on the dynamic response of the structures considered. It is shown that, for a given pulse duration, peak deflections are more highly influenced by material strain rate sensitivity as the impulse level is increased. Further, it is found that for a given total impulse, the influence of strain rate sensitivity on peak deflections continually decreases as the pulse duration increases.

*This work was supported by the United States Atomic Energy Commission.
1. INTRODUCTION

Over the past twenty years, numerous analytical solutions for the plastic response of structures such as beams, cylinders, and plates subjected to dynamic loading have been developed. Many of these solutions were presented for the case in which the blast loading was idealized as an equivalent ideal impulse (See Bodner [1] and Wang [2], for example). In practice, the blast loading is of finite duration, and it is important to evaluate whether the ideal impulse approximation is a reasonable one, or whether details of the blast duration are significant.

Considering the influence of finite pulse duration time on plastic structural response, Florence [3] presents a bending solution for a rigid-plastic clamped circular plate subjected to a uniform pressure pulse over a circular central region. The pulse shape considered is rectangular in time. Results showed that, for a given total impulse, the central deflection increases with the magnitude of the applied pressure and corresponding decrease of pulse duration time, and is a maximum when the blast pulse approaches an ideal impulsive load. Including the influence of both membrane forces and bending moments, Jones [4] recently solved for the final deflection of a rigid-plastic annular plate also loaded by a rectangular pulse. Conclusions regarding the influence of the pulse duration for a given total impulse qualitatively agreed with those of Florence [3].

Jones [5] and Perrone [6] have shown that, at least for purely impulsive loading, material strain rate sensitivity is important and should be taken into account in structural analyses when the elastic limit of the material is exceeded. However, the writer is unaware of any attempts to evaluate the relative importance of strain rate sensitivity in the case of finite-duration pulse loading.

It is the purpose of this paper, then, to evaluate the influence of strain rate sensitivity on structures loaded with a finite-duration pulse. The structures chosen are cylindrical and spherical shells for which ideal impulsive solutions exist (Duffey and Krieg, [7], [8]). The shells are constructed of an elastic-plastic material exhibiting linear strain hardening and linear strain rate sensitivity, and are loaded by a uniform, internal finite duration pressure pulse that is rectangular in time. It is shown that if only uniform radial deformations are permitted, the set of transient solutions for both cylindrical and spherical shells considered takes on a similar form. The solutions, developed in terms of elementary functions, are then utilized to provide further information on the influence of finite pulse duration time on structural elements that are not restricted to be rigid-
perfectly plastic, such as by Nachbar [9], but exhibit strain hardening and strain rate sensitivity as well.

The solutions for the cylindrical and spherical shells are developed by partitioning the response into four phases; (1) initial elastic motion, before the stress state of any fibers in the shell cross section reach the strain-rate-dependent von Mises yield surface; (2) elastic-plastic motion as the plastic zone propagates across the shell thickness (later neglected); (3) fully plastic motion, and (4) elastic unloading. This partitioning of the response is permissible due to the restriction of uniform radial deformations.

First, the spherical shell analysis is presented, followed by the cylindrical shell. The influence of finite-duration loading on these solutions is then evaluated.

2. SPHERICAL SHELL ANALYSIS

A spherical shell segment is shown in fig. 1. For small, spherically symmetric displacements, the radial equation of motion of the shell, after Baker [10], is

\[ \rho h \frac{d^2 w}{dt^2} + \frac{2N}{a} = p(t) \]

(1)

where \( \rho \) is mass density, \( h \) is shell thickness, \( a \) is radius, \( p(t) \) is the pressure pulse, \( w \) is radial displacement, \( N \) denotes normal force per unit length, and \( \zeta \) represents distance from the reference surface.

For uniform radial deformations the strain-displacement relation is

\[ \varepsilon = \frac{w}{a + \zeta} \]

(2)

A spatially uniform, rectangular pressure pulse of magnitude \( P_0 \) and duration \( t_o \) is applied to the inner surface of the shell,

\[ p(t) = \begin{cases} P_0 & 0 < t < t_o \\ 0 & \text{elsewhere} \end{cases} \]
Material behavior is taken as elastic-plastic, linearly strain hardening, and linearly strain rate sensitive, as shown in fig. 2. Due to spherically symmetric motions, shell response can be partitioned into four phases; (1) initial elastic phase, before any plastic yielding occurs; (2) elastic-plastic phase, during which the plastic zone propagates across the sphere cross section (this phase is later neglected); (3) fully plastic phase; and (4) elastic unloading phase.

It is possible that the blast pulse may end during any one of the four above phases, and the various possibilities are taken into consideration as they appear.

Initial Elastic Phase

The following nondimensional equation of motion may be developed from the above equations and Hooke's law:

\[
\frac{d^2 u}{dr^2} + \beta^2 u = \ddot{a} p (1 - \nu_o) \quad \tau < \tau_o
\]  

where \( u = w/a, \beta^2 = 2 \frac{a}{h} \ln \left[ \frac{2(a/h) + 1}{2(a/h) - 1} \right], \tau = c_v \frac{t}{a}, c_v = \left( \frac{E_1}{\rho (1 - \nu_o)} \right) \frac{1}{k} \),

\( \ddot{a} = a/h, \ddot{p} = P_o/E_1, E_1 \) is the elastic modulus, \( \nu_o \) is Poisson's ratio, and \( \tau_o \) is the nondimensional pulse duration time corresponding to \( t_o \).

The solution of eq. (3) with initial conditions \( u(0) = 0 \) and \( \frac{du}{dr} (0) = 0 \) is

\[
u = \ddot{a} \ddot{p} (1 - \nu_o) (1 - \cos \beta \tau) / \beta^2 \quad \tau < \tau_o \]

Eq. (4) represents the transient displacement of the sphere until either the pressure pulse ends or yielding occurs.

If the pressure pulse ends before the elastic phase is completed (i.e., before yielding occurs) then the homogeneous portion of eq. (3) governs following \( \tau_o \). The conditions at the time the blast wave ends are then

\[
u(\tau_o) = u_p = \ddot{a} \ddot{p} (1 - \nu_o) (1 - \cos \beta \tau_o) / \beta^2
\]

\[
\frac{du}{dt} (\tau_o) = u'_p = \ddot{a} \ddot{p} (1 - \nu_o) \sin \beta \tau_o / \beta
\]

The elastic solution after the blast wave ends is

\[
u = u_p \cos \beta (\tau - \tau_o) + u'_p \sin \beta (\tau - \tau_o) / \beta \quad \tau > \tau_o
\]
Depending upon whether or not the pressure pulse ends in the elastic phase, eqs. (4) or (6) govern response until fibers in the sphere cross section reach the dynamic yield stress and begin to yield. The dynamic yield stress \( \sigma^D_Y \) with assumed linear dependence on rate of straining, may be written as

\[
\frac{\sigma^D_Y}{\sigma_Y} = 1 + k_V \frac{\text{d}e}{\text{d}t}
\]  

(7)

where \( k_V \) is a constant for a given material and geometry*, \( \sigma_Y \) represents the static yield stress, and \( \text{d}e/\text{d}t \) is the nondimensional strain rate. Fibers in the sphere begin to yield when they reach the strain rate dependent yield surface. Thus, yielding occurs when the stress calculated assuming an elastic path equals the dynamic yield stress given by eq. (7).

By combining eqs. (2) and (4), it can be shown that if pulse duration time, \( \tau_o \), is greater than the yielding time \( \tau_2 \), the strain rate for \( \tau < \tau_2 \) is

\[
\frac{\text{d}e}{\text{d}t}(\zeta, \tau) = \left( \frac{a}{a + \zeta} \right) \frac{\bar{e}}{\bar{p}} (1 - \nu_o) \sin \beta \tau / \beta
\]  

(8)

The stress assuming an elastic path may be equated to the rate-dependent yield stress using eqs. (7) and (8) to obtain

\[
\frac{\varepsilon_{E1}}{(1 - \nu_o)} = \sigma_Y \left[ 1 + k_V \left( \frac{a}{a + \zeta} \right) \frac{\bar{e}}{\bar{p}} (1 - \nu_o) \sin \beta \tau_2 / \beta \right]
\]  

(9)

Utilizing eqs. (2), (4), and (9), the time at which yielding first occurs at the shell midsurface may be determined as

\[
\tau_2 = \frac{1}{\beta} \arccos \left[ \frac{1}{\left( \beta \varepsilon_Y k_V (1 - \nu_o) \right)^2 + 1} \left\{ \frac{\beta^2 \varepsilon_Y}{\bar{p}} \right\} \right.
\]

\[
- \beta \varepsilon_Y k_V (1 - \nu_o) \left[ \left( \beta \varepsilon_Y k_V (1 - \nu_o) \right)^2 + 1 - \left( 1 - \frac{\beta^2 \varepsilon_Y}{\bar{p}} \right)^{\frac{1}{2}} \right] \left\{ \right. \]

\[
0 \leq \beta \tau_2 \leq \pi , \\
\tau_o > \tau_2
\]  

(10)

*The constant, \( k_V \), contains a "size effect" such that physically smaller shells exhibit a greater sensitivity to strain rate effects (see Duffey [8]).
where $\epsilon_y$ is the static uniaxial yield strain, $\sigma_y/E_1$.

Should the blast pulse end before yielding occurs, then eq. (6) governs the elastic response for $\tau > \tau_o$. This equation may be rewritten as

$$u = \left[ u_p^2 + \left( \frac{u_p'}{\beta} \right)^2 \right]^{\frac{1}{2}} \cos \left[ \beta (\tau - \tau_o) - \frac{1}{\beta} \right] - \frac{1}{\beta} \arctan \left( \frac{u_p'}{\beta u_p} \right)$$  (11)

In a manner analogous to that used to determine eq. (10), the time of yielding may be found for this condition as

$$\tau_2 = \tau_o + \frac{1}{\beta} \arctan \left( \frac{u_p'}{\beta u_p} \right) - \frac{1}{\beta} \arccos \left[ \frac{1}{1 + \left( k_0 \beta \epsilon_y (1 - \nu_o) \right)^2} \right]$$

$$\left[ \epsilon_y (1 - \nu_o) + k_0 \beta \epsilon_y (1 - \nu_o) \left( \left[ k_0 \beta \epsilon_y (1 - \nu_o) \right]^2 + 1 - \left[ \frac{\epsilon_y (1 - \nu_o)}{\delta} \right]^2 \right)^{\frac{1}{2}} \right]$$  (12)

where

$$\delta = \left[ u_p^2 + \left( \frac{u_p'}{\beta} \right)^2 \right]^{\frac{1}{2}}$$

**Elastic-Plastic Phase**

This phase, which continues as the yield point sweeps over the sphere cross section, has been analyzed in detail elsewhere by Duffey and Krieg [7] for a similar structure. The conclusions of that investigation are that the influence of this phase of the motion is negligible for thin shells. Consequently, this phase of the motion is ignored, and the fully plastic phase is assumed to govern motion after yielding reaches the shell midsurface.

**Plastic Phase**

The initial conditions on the motion in this phase are dependent upon whether or not the blast wave is still in effect after the elastic phase. For $\tau_o > \tau_2$:

$$u(\tau_2) = u_2 = \ddot{u}_p (1 - \nu_o) \left[ 1 - \cos \beta \tau_2 \right] / \beta^2$$

$$\frac{du}{dt} (\tau_2) = u'_2 = \ddot{u}_p (1 - \nu_o) \sin \beta \tau_2 / \theta$$  (13)
For $\tau_0 < \tau_2$:

$$u(\tau_2) = u_2 = u_p \cos \beta(\tau_2 - \tau_0) + u_p' \sin \beta(\tau_2 - \tau_0)/\beta$$

(14)

$$\frac{du}{d\tau}(\tau_2) = u_2' = -u_p \beta \sin \beta(\tau_2 - \tau_0) + u_p' \cos \beta(\tau_2 - \tau_0)$$

In either case, the normal force per unit length may be written

$$N = \int_{-h/2}^{h/2} \left\{ \sigma_y(\xi) + \frac{E_2}{(1-v^*)} \left[ \epsilon - \epsilon_D(1-v_o) \right] \right\} d\zeta$$

(15)

where

$$v^* = \frac{E_2}{E_1} (v_o - 1/2) + 1/2$$

a strain hardening Poisson's ratio, $E_2$ is the tangent modulus, as shown in fig. 2, and where the current dynamic yield strain is determined from eq. (7) as

$$\epsilon_D = \frac{\sigma_y}{E_1} [1 + k_v \left( \frac{a}{a + \zeta} \right) \frac{du}{d\tau}]$$

(16)

Eqs. (1), (15), and (16) are combined to obtain the following equation of motion:

$$\frac{d^2u}{d\tau^2} + \epsilon_y k_v (1-v_o) \beta^2 \left[ 1 - \lambda \left( \frac{1-v_o}{1-v^*} \right) \right] \frac{du}{d\tau} + \lambda \beta^2 \left( \frac{1-v_o}{1-v^*} \right) u$$

$$+ 2(1-v_o) \epsilon_y \left[ 1 - \lambda \left( \frac{1-v_o}{1-v^*} \right) \right] = \tilde{a}(1-v_o)$$

(17)

$$\tau < \tau_0$$

where $\lambda = E_2/E_1$, a strain hardening parameter.

For the case in which the pressure pulse extends into the plastic response phase ($\tau_0 > \tau_2$), the response during the plastic phase is given by the solutions to eq. (17) until $\tau_0$. There are three such solutions to this equation with initial conditions, eq. (13). The solutions depend upon the relative magnitudes of the parameters $\alpha$ and $\gamma$ defined as

$$\alpha = \epsilon_y k_v (1-v_o) \beta^2 \left[ 1 - \lambda \left( \frac{1-v_o}{1-v^*} \right) \right] /2 ; \quad \gamma = \beta \left[ \lambda \left( \frac{1-v_o}{1-v^*} \right) \right]^k$$

(18)
The following three solutions to eq. (17) may be interpreted as the underdamped, critically damped, and overdamped solutions, respectively, damping provided by the viscoplastic material behavior.

The solution for \( \alpha^2 < \gamma^2 \) is

\[
\begin{align*}
\alpha (\tau - \tau_2) \\
L \cos (\gamma^2 - \alpha^2)^{\frac{1}{2}} (\tau - \tau_2) \\
M \sin (\gamma^2 - \alpha^2)^{\frac{1}{2}} (\tau - \tau_2)
\end{align*}
\]

\[
+ S
\]

\[
\text{where}
\]

\[
L = u_2 + \frac{2\varepsilon_y(1 - \nu_0)}{\gamma^2} \left[ 1 - \lambda \left( \frac{1 - \nu_0}{1 - \nu} \right) / \alpha - \bar{p}_a(1 - \nu_0) \right]
\]

\[
M = \left( \alpha L + \frac{\text{d}u_2}{\text{d}\tau} \right) / \left( \gamma^2 - \alpha^2 \right)^{\frac{1}{2}}
\]

\[
S = L - u_2
\]

The solution for \( \alpha = \gamma \) is

\[
\begin{align*}
\alpha (\tau - \tau_2) \\
\left\{ \frac{\text{d}u_2}{\text{d}\tau} + u_2 \alpha \right. \\
2\varepsilon_y(1 - \nu_0) \left[ 1 - \lambda \left( \frac{1 - \nu_0}{1 - \nu} \right) / \alpha - \bar{p}_a(1 - \nu_0) \right] / \alpha + u_2 \\
+ S
\end{align*}
\]

\[
\text{The solution for } \alpha^2 > \gamma^2 \text{ is}
\]

\[
\begin{align*}
\alpha (\tau - \tau_2) \\
\left[ \alpha + (\alpha^2 - \gamma^2)^{\frac{1}{2}} \right] (\tau - \tau_2) + Qe \left[ \alpha + (\alpha^2 - \gamma^2)^{\frac{1}{2}} \right] (\tau - \tau_2) - S
\end{align*}
\]

\[
\text{where}
\]

\[
P = u_2^2 / 2 + \frac{\text{d}u_2}{\text{d}\tau} + a u_2^{\frac{1}{2}} + \frac{\varepsilon_y(1 - \nu_0) \left[ 1 - \lambda \left( \frac{1 - \nu_0}{1 - \nu} \right) / \alpha - \bar{p}_a(1 - \nu_0) / 2 \right]}{(\alpha^2 - \gamma^2)^{\frac{1}{2}} \alpha - (\alpha^2 - \gamma^2)^{\frac{1}{2}}}
\]
and
\[
Q = \frac{u_2}{2} - \frac{\frac{du_2}{d\tau} + au_2}{2(a^2 - \gamma^2)^{\frac{1}{2}}} - \varepsilon_y(1 - \nu) \left[ 1 - \lambda \left( \frac{1 - \nu_y}{1 - \nu} \right) \right] - \frac{p\bar{a}(1 - \nu)}{2} \frac{1}{(a^2 - \gamma^2)^{\frac{1}{2}}} \frac{1}{\alpha + (a^2 - \gamma^2)^{\frac{1}{2}}}
\]

The above solutions are applicable until the blast pulse ends ($\tau_0 < \tau_3$) or the peak deflection is reached and the outward radial velocity has decreased to zero ($\tau_3 < \tau_0$). This time, $\tau_3$, is given for the latter case by

\[
\tau_3 = \tau_2 + \frac{1}{(\gamma^2 - a^2)^{\frac{1}{2}}} \arctan \left[ \frac{aL + M(\gamma^2 - a^2)^{\frac{1}{2}}}{aM + L(\gamma^2 - a^2)^{\frac{1}{2}}} \right]
\]

\[\alpha < \gamma^2 \quad (22)\]

\[
\tau_3 = \tau_2 + \frac{\left( \frac{du_2}{d\tau} \right)}{a \frac{du_2}{d\tau} + a^2 u_2 + 2\varepsilon_y(1 - \nu) \left[ 1 - \lambda \left( \frac{1 - \nu_y}{1 - \nu} \right) \right]}
\]

\[\alpha = \gamma \quad (23)\]

\[
\tau_3 = \tau_2 + \frac{1}{2(a^2 - \gamma^2)^{\frac{1}{2}}} \ln \left\{ \frac{1}{1 - \frac{\alpha + (\alpha^2 - \gamma^2)^{1/2}}{\sqrt{p} + \alpha + (\alpha^2 - \gamma^2)^{1/2}}} \right\}
\]

\[\alpha > \gamma^2 \quad (24)\]

For the situation in which the blast ends in the initial plastic phase, $\tau_0 < \tau_2$, the plastic response is given by the solutions to eq. (17) with $\bar{p}$ equal to zero. Eqs. (22)-(24) remain unchanged for $\tau_0 < \tau_2$.

The possibility exists that the blast pulse could end during the plastic phase. In this case, the solutions for $\tau_2 < \tau < \tau_0$ are given by eqs. (19)-(21). Following the blast pulse ($\tau_0 < \tau < \tau_3$), the solutions are given by the same equations with $\bar{p}$ equal to zero, all subscripts (2) are replaced by (o), and the "initial conditions", $\frac{du_2}{d\tau}$ and $u_0$, are found for time $\tau = \tau_0$. Eqs. (22)-(24) apply if all subscripts (2) are replaced by (o).

**Elastic Unloading Phase**

The unloading phase can be considered an initial stress problem since outward motion has ceased. The normal force per unit length may be written as

\[
N = N_3 - \int_{-h/2}^{h/2} \left[ \varepsilon_3(C) - \varepsilon(C) \right] \frac{E_1}{(1 - \nu)} \, d\zeta
\]

\[\quad (25)\]
where \( N_3 \) is the normal force at time \( \tau_3 \),

\[
N_3 = h \left[ \sigma_y \left( 1 - \lambda \left( \frac{1 - \nu}{1 - \nu^2} \right) \right) + \frac{E_2 u_3 \beta^2}{2(1 - \nu^2)} \right]
\]

(26)

\( u_3 \) being peak displacement.

The following equation of motion for this elastic phase can be developed from eqs. (1), (25), and (26) for \( \tau_0 > \tau_3 \):

\[
\frac{d^2 u}{d\tau^2} + \beta^2 u + \left[ 1 - \lambda \left( \frac{1 - \nu}{1 - \nu^2} \right) \right] \left[ 2\varepsilon_y (1 - \nu_0) - \beta^2 u_3 \right] = \bar{p}\bar{a}(1 - \nu_0)
\]

(27)

The complete solution, with initial conditions \( u(\tau_3) = u_3 \), and \( \frac{du}{d\tau}(\tau_3) = 0 \), is readily found as

\[
u = \left[ u_3 + \left[ \left( 1 - \lambda \left( \frac{1 - \nu}{1 - \nu^2} \right) \right] \left[ 2\varepsilon_y (1 - \nu_0) - \beta^2 u_3 \right] - \frac{\bar{a}}{\bar{p}}(1 - \nu_0) \right] \right] \cos \beta (\tau - \tau_3)
\]

\[
+ \frac{\bar{a}}{\bar{p}}(1 - \nu_0) / \beta^2 \left[ \left[ 1 - \lambda \left( \frac{1 - \nu}{1 - \nu^2} \right) \right] \left[ 2\varepsilon_y (1 - \nu_0) - \beta^2 u_3 \right] / \beta^2 \right]
\]

(28)

\( \tau_3 < \tau < \tau_0 \)

If the blast wave ends before elastic unloading occurs \( (\tau_0 < \tau_3) \), then the right side of eq. (27) is zero, and the solution for \( \tau > \tau_3 \) reduces to

\[
u = \left[ u_3 + \left[ \left( 1 - \lambda \left( \frac{1 - \nu}{1 - \nu^2} \right) \right] \left[ 2\varepsilon_y (1 - \nu_0) - \beta^2 u_3 \right] \right] \cos \beta (\tau - \tau_3)
\]

\[
- \left[ \left( 1 - \lambda \left( \frac{1 - \nu}{1 - \nu^2} \right) \right] \left[ 2\varepsilon_y (1 - \nu_0) - \beta^2 u_3 \right] / \beta^2
\]

(29)

Eqs. (28) and (29) simply indicate that, following the peak deflection, the stress state in the sphere oscillates elastically along a line \( \sigma_1 = \sigma_2 \) in principal stress space. The (undamped) equations remain valid until further plastic yielding or buckling occurs. For an isotropically strain hardening material, in which the yield surface expands isotropically on continued plastic straining, no further plasticity will occur in the absence of buckling. However, for a kinematically hardening material, in which the yield surface translates without distortion, the shell may yield in
compression, so eqs. (28) and (29) are valid only to that time.

For the isotropically hardening material, the final deflections about which the shell oscillates may be found by inspection:

For $\tau < \tau_o$ (Pressure still in effect):

$$u_f = \tilde{a}p (1-\nu_o)/\beta^2 + \left[\lambda \left(\frac{1-\nu_o}{1-\nu}\right) - 1\right] \frac{2\varepsilon_y (1-\nu_o) - \beta^2 u_3}{\beta^2}$$  \hspace{1cm} (30)

For $\tau > \tau_o$

$$u_f = \left[\lambda \left(\frac{1-\nu_o}{1-\nu}\right) - 1\right] \frac{2\varepsilon_y (1-\nu_o) - \beta^2 u_3}{\beta^2}$$  \hspace{1cm} (31)

3. CYLINDRICAL SHELL ANALYSIS

The radial equation of motion of a long, pulse-loaded cylinder, fig. 3, is presented by Stuiver [11]. For small deflections and uniform radial motion, the equation may be written as

$$\frac{d^2w}{dt^2} = \frac{-N}{\rho ab h} + \frac{p(t)}{\rho h}$$  \hspace{1cm} (32)

where

$$N = b \int_{-h/2}^{h/2} \sigma_1 d\zeta$$  \hspace{1cm} (33)

and $w$ is outward radial displacement, $a$ is initial cylinder radius, $b$ denotes cylinder length, and $p(t)$ is the internal pressure pulse. The definitions $\zeta$, $\rho$, and $h$ are unchanged and shell thinning is again neglected. For uniform radial motions, the strain-displacement relation is given by eq. (2). Loading pulse shape and material properties are taken identical to those considered earlier.

The difficulty with the cylinder analysis is that the stress state is no longer one of balanced biaxial stress as it was for the spherical shell. Consider a cylinder undergoing biaxial stress, with circumferential stress denoted by $\sigma_1$ and longitudinal stress denoted by $\sigma_2$. From the von Mises yield condition,

$$\phi(\sigma_1, \sigma_2) = \sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 - \sigma_y^2 = 0$$  \hspace{1cm} (34)
and the condition of plane strain, $\varepsilon_2 = 0$, which dictates that the ratio of elastic principal stresses is

$$\frac{\sigma_2}{\sigma_1} = \nu_o$$

then the circumferential stress at initial contact with the yield surface is given by

$$\sigma_{1y} = \frac{\sigma_y^D}{\left[1 - \nu_o + \nu_o^2\right]^{1/2}}$$

Unfortunately, following contact with the yield surface, the stress state moves along the yield surface in a nonlinear manner during subsequent plasticity, approaching the principal stress ratio, $\sigma_2/\sigma_1 = 1/2$. As a result, the principal stresses no longer remain proportional in elastic and plastic ranges, and it is necessary to take this motion of the stress state into account. For the special case of Poisson's ratio equal to one-half, however, the stress state remains proportional during elastic and plastic phases, so that the approach utilized previously for the spherical shell can be used here with the above restriction. The solution for the cylinder ($\nu_o = 1/2$ only) is thus found beginning with eqs. (32) and (33) by partitioning the transient response into an initial elastic phase, a plastic phase, and an elastic unloading phase.

The resulting cylindrical shell response equations are quite closely related to those of the spherical shell for the case of uniform radial motion. The relations between the two sets of equations are simplified if the spherical shell equations are written in terms of the following additional parameters:

$$T = c_p t/\alpha, \quad c_p = \left[\frac{E_1}{\rho (1-\nu_o^2)}\right]^{1/2},$$

$$\eta^2 = \alpha \ln \left[\frac{2a/h+1}{2a/h-1}\right],$$

and where a strain rate sensitivity parameter, $k_p$ is given by

$$\frac{\sigma_y^D}{\sigma_y} = 1 + k_p \frac{dc}{dT}.$$

The resulting response equations of the previous section, eqs. (4)-(6), (10)-(14), (18)-(24), and (28)-(31) then apply directly to the cylindrical shell for $\nu_o = 1/2$ providing the following substitutions are made:

Replace $\tau$ by $T$, $\beta$ by $\eta$, the term $\ddot{\alpha}p(1-\nu_o)$ by $\ddot{\alpha}p(1-\nu_o^2)$, the term
\[ \beta \varepsilon_y k_y (1-\nu_o) \text{ by } \eta \varepsilon_y k_p \frac{(1-\nu_o^2)}{(1-\nu_o+\nu_o^2)^{1/2}}, \text{ the term } \beta^2 \varepsilon_y k_y (1-\nu_o) \text{ by} \]

\[ \eta^2 \varepsilon_y k_p \frac{(1-\nu_o^2)}{(1-\nu_o+\nu_o^2)^{1/2}}, \text{ the term } 2 \varepsilon_y (1-\nu_o) \text{ by } \frac{\varepsilon_y (1-\nu_o^2)}{(1-\nu_o+\nu_o^2)^{1/2}}, \text{ and } \frac{\varepsilon_y (1-\nu_o)}{\delta (1-\nu_o+\nu_o^2)^{1/2}}. \]

It should be cautioned that the above substitutions only apply to the equations indicated.

4. DISCUSSION

An example of transient sphere radial displacement showing the various phases of the response is presented in fig. 4 for the parameters indicated on that figure. A moderate degree of strain hardening and strain rate sensitivity is included. In this example, the blast loading ends in the plastic phase (\( \tau_o = 1 \)). The response in the elastic phase is determined from eq. (4). Yielding time is calculated from eq. (10). Plastic response prior to \( \tau_o \) is given by eq. (19) for this example, and following \( \tau_o \) by eq. (19) modified as discussed after eq. (24). Time of peak sphere deflection is found from eq. (22). Elastic unloading displacement-time behavior is given by eq. (29).

The influence of pulse duration on peak deflection of the sphere is shown in fig. 5 for two values of nondimensional total impulse. The value, \( I_o \), represents the nondimensional impulse, \( \bar{\tau} \tau_o \), to just cause yielding for a sphere subjected to an ideal impulse and constructed of a rate-insensitive material. Both rate-sensitive and rate-insensitive curves of peak deflection as a function of pulse duration are plotted for each value of impulse considered. It is seen that, consistent with other investigations by Florence [3] and Jones [4], peak deflection increases as pulse duration decreases. Comparisons of corresponding rate-sensitive and rate-insensitive curves for the same total impulse level indicate that the rate-sensitive peak deflections decrease less rapidly with increased pulse duration than corresponding rate-insensitive deflections, causing the two curves to approach each other for longer duration pressure pulses. Thus, for a given total impulse, the influence of strain rate sensitivity on peak deflections continually decreases as the pulse duration increases. Consequently, the influence of
strain rate sensitivity is most severe for a given impulse level if that loading is applied as a pure impulse.

Observations regarding influence of pulse duration on peak cylinder response, as shown in fig. 6, are similar to those made for the spherical shell. The value of nondimensional impulse, \( I_0 = \frac{\Delta T}{T_0} \), represents the value of ideal impulse to just cause plastic yielding for a cylindrical shell constructed of a rate-insensitive material.

The transient response of a spherical shell for several values of pulse duration, \( \tau_0 \), is shown in fig. 7 for the parameters indicated. The total impulse is the same for each curve shown.

5. CONCLUSIONS

The following observations are made for both the cylindrical and spherical shells.

(1) For a given pulse duration, peak deflections are more highly influenced by material strain rate sensitivity as the impulse level is increased.

(2) For a given total impulse, the influence of strain rate sensitivity on peak deflections continually decreases as the pulse duration increases. Consequently, the influence of strain rate sensitivity is most severe for a given impulse level if that loading is applied as a pure impulse. This observation suggests that, for design purposes, a structure subjected to a finite duration blast pulse could be analyzed using a strain rate-insensitive theory for the finite pressure pulse loading, and with a rate-sensitive theory for the more tractible ideal impulse loading. The latter would be useful in bracketing rate effects in the finite pressure pulse solution.

It is believed that the solutions presented here could be extended to other blast pulse shapes, such as the more realistic exponential decay. However, shell response due to other pulse shapes can be related to the rectangular pulse by Youngdahl [12], so extension of this work to other pulses may provide little new information.

6. REFERENCES


Figure 1. Segment of Blast-Loaded Spherical Shell

Figure 2. Uniaxial Elastic-Plastic Strain Rate Sensitive Material Behavior
Figure 3. Undeformed State of Pulse-Loaded Cylinder

Figure 4. Example Transient Sphere Displacement Showing Various Phases of Response

\[ \lambda = 0.1 \]
\[ \mu_0 = 0.3 \]
\[ \gamma_0 = 1.0 \]
\[ \bar{a} = 50 \]
\[ k_\mu = 119.5 \]
\[ \bar{P} = 1.705 \times 10^{-4} \]

\[ (I/I_0 = 5) \]
\[ \epsilon_\nu = 0.001206 \]
Figure 6. Influence of Pulse Duration on Peak Cylinder Figure 5 Response
Figure 7.  Influence of Pulse Duration on a Spherical Shell For a Constant Total Impulse
DISCUSSION

Q  J. BOWEN, U. K.
Explanation of \( K_V = 104 \) ?

A  T. A. DUFFEY, U. S. A.
\( K_V = 104 \) corresponds (for the cylinder) to a material whose dynamic yield stress doubles (compared to the static value) at a strain rate of 1000 in. /in/sec, and for a 2-inch radius. The problem with \( K_V \) is that it contains a size effect similar to that brought out in the previous paper. Thus the value is a function of both material and geometry.

Q  R. A. VALENTIN, U. S. A.
What pulse shapes were used in the analysis ?
Have you made any attempt to check whether C. K. Youngdahl's correlation parameters for ideal dynamic plasticity will work in the case of strain rate sensitive deformation ?

A  T. A. DUFFEY, U. S. A.
Only the rectangular pulse shape was considered. Youngdahl has a method of "rapidly relating the response of various shaped pulses, so it seemed of limited value to repeat the analysis for other shapes. I have not checked Youngdahl's correlation parameters for the case of rate-sensitive deformation.

Q  R. A. Müller, Germany
Is it intended to extend the work to other geometries, e.g. tubes with a hexagonal cross section, or do you know some work going on in this field ?
Tubes with a hexagonal cross section are used as wrapper tubes for the fuel elements of sodium cooled fast reactors. Under accident conditions, blast loadings can exist.

A  T. A. DUFFEY, U. S. A.
Extending the present type of analysis to more complex geometries does not appear feasible. More approximate rigid-plastic analyses or structural computer programs would seem appropriate. I am not familiar with dynamic analyses of tubes with hexagonal cross section.