TRANSIENT RESPONSE OF LINEAR ELASTIC STRUCTURES DETERMINED BY THE MATRIX EXPONENTIAL METHOD*

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ABSTRACT

Determination of the response of components of nuclear power plants to transient inputs is a frequent problem. For many components, response must be limited to linear elastic behavior. The matrix exponential algorithm has been used to determine the response of systems which may be modeled as plane frames in combination with lumped masses and weightless springs. A comparison of the response calculated for simple systems by the matrix exponential method with that determined by modal analysis methods is made.

1. INTRODUCTION

Several areas in structural design confronting the nuclear industry can generally be classified as transient or time varying. Examples of these are aseismic design, emergency action such as blowdown, or accidents involving the shipment of radioactive material. Designers must consider the circumstances and consequences of the situation and take appropriate steps to insure safe operation of the system involved. In doing so, the designer faces several difficulties: the time available to obtain a solution is limited, the problems can generally be classified as complex, and the assumptions made to obtain a model that can be readily analyzed may greatly affect the answers obtained. Fortunately, digital computers have become widely available, and this availability results in some reduction of the difficulties caused by limited time.

Several methods are currently used to develop a model of the physical system and to select a solution technique. Quite often, the structure is modeled as a collection of rigid masses and weightless springs. An alternate choice involves finite element methods to minimize error. When selecting a solution technique, the designer must decide what information is to be obtained as a result of the analysis. This may be a complete time history of displacements or simply estimates of the maximum relative displacements. If only estimates of maximum relative displacement are required, the widely known model superposition methods in combination with a response spectrum may be used. If a complete time history is required,

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some form of integration of the equations of motion will be needed. If an economical, easy to use, and accurate method for performing the direct integration were available, this technique would appear to be the logical choice under all circumstances in that all the data of interest to the designer would be available in the results of the analysis.

Interest in the transient response of linear elastic mechanical systems occurs in many fields. However, the literature surveyed in the course of this investigation was limited primarily to research documents sponsored by the United States Atomic Energy Commission and the National Aeronautics and Space Administration and to standard textbooks.

Most current methods for determining the transient response of multi-degree of freedom systems may be separated into two categories. The first is superposition of modal response patterns, and the second is direct integration. The application of both of these methods is illustrated in a recent review of seismic design analysis methods [1] wherein a linear elastic structural model is formulated by either the lumped parameter or finite element method, and the modal analysis technique is recommended for computing both steady state and transient dynamic responses.

The dynamic equations for linear elastic mechanical structures are characterized by constant coefficients and may be quite readily expressed in matrix form. Since these equations are second order, the solution algorithms generally found in textbooks do not fully exploit the constant coefficient characteristic. The matrix exponential method has been presented by Zadeh and Desoer [2] as a means of solving a set of first-order differential equations that have constant coefficients and are linear. This method has recently received wide attention because of the availability of digital computers. Numerical techniques used in the time domain and in the frequency domain analyses of linear time-invariant systems have been reported by Liou [3,4]. A bound for round-off error involved in digital computation of the transition matrix of a system of linear time-invariant differential equations has been developed and a method of computer selection of the step size and number of series terms in transition matrices has been presented by Mankin and Hung [5,6].

A technique for determining the transient response of structures that is based on a Taylor series expansion for displacement and velocity has been presented by Craggs [7,8]. However, the solution presented was developed only for simple mechanical systems, and the definite relation to the matrix exponential method was not presented.

2. MATHEMATICAL MODEL FOR A COMPLETE STRUCTURE

In order to apply the matrix exponential solution method to the problem of determining transient structural response, the equations of motion for the structure must be written as a coupled set of first-order, linear differential equations. Since only linear elastic structures are considered in this investigation, these equations will have constant coefficients. The equations of motion for the structure are presented in a form compatible with the matrix exponential method in this section.

2.1 Dynamic Equations

The equations of motion for a multi-degree of freedom system may be conveniently written in matrix equation form as

$$M\ddot{x} + C\dot{x} + Kx = f(t) \quad , \tag{1}$$

where

M is the structure mass matrix,

C is the structure damping matrix,

K is the structure stiffness matrix,

x is the structure displacement vector,

$\dot{x}$ is the structure velocity vector,

$\ddot{x}$ is the structure acceleration vector, and

f(t) is the time varying vector of applied loads.

Unless noted otherwise, capital letters are used to denote matrices and lower-case letters are used to denote vectors and scalars. Where necessary to improve clarity of presentation, brackets, [ ], and braces, { }, are also used to denote matrices and vectors.

2.2 Introduction of State Variables

To mathematically simplify the dynamic equations, it is desirable to develop a set of coupled first-order differential equations that is equivalent to the set of second-order differential equations. This may be accomplished by solving explicitly for the acceleration vector in eq. (1) and incorporating an identity relationship involving the velocity vector. Solving eq. (1), the acceleration vector

$$\ddot{x} = -M^{-1}C\dot{x} - M^{-1}Kx + M^{-1}f(t) \quad , \tag{2}$$

where the superscript -1 denotes inversion. The necessary identity is

$$\dot{x} = I\dot{x} \quad , \tag{3}$$

where I is the identity matrix. By combining eqs. (2) and (3), the following set of first-order coupled differential equations is obtained.

$$\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} \Phi & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} + \begin{Bmatrix} \phi \\ M^{-1}f(t) \end{Bmatrix} , \tag{4}$$

where Φ and ϕ denote the null matrix and null vector, respectively.

2.3 Matrix Exponential Solution

For the free vibration case, $f(t) = \phi$, the solution to eq. (4) is as follows. Let

$$A = \begin{bmatrix} \Phi & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} . \tag{5}$$

Integrating from time t to t + τ yields

$$\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_{t+\tau} = \begin{bmatrix} \exp A\tau \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_{t} , \tag{6}$$

where

t is time,

τ is the time increment, and

$[\exp A\tau]$ is the matrix exponential function of A.

The matrix exponential function derives its name from the similarity in definitions to the scalar exponential function. The matrix exponential is defined

$$[e^{A\tau}] = \left[\sum_{k=1}^{\infty} \frac{A^k \tau^k}{k!}\right] . \tag{7}$$

The subscripts t and τ are used to denote the point of evaluation in time. A complete development of this solution has been presented by Zadeh and Desoer [2, Chapter 5].

The exponential matrix, $[\exp A\tau]$, is also called the transition matrix and is the same as that discussed by Craggs [7, page 2] and labeled as T.

For time increments, τ, such that the forcing function may be considered constant within the time step, the solution to the forced vibration problem is derived by Ball and Adams [9] and presented in eq. (8).

$$\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_{t+\tau} = \left[\exp A\tau\right] \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_t + \tau \left[\sum_{k=1}^{\infty} \frac{[A]^{k-1} \tau^{k-1}}{k!}\right] \begin{Bmatrix} \phi \\ M^{-1} f(t) \end{Bmatrix}_t \tag{8}$$

When the forcing function varies linearly within the computation time interval, eq. (9) is the solution.

$$\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_{t+\tau} = \left[e^{A\tau}\right] \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_t + \tau \left[\sum_{k=1}^{\infty} \left(\frac{1}{k!} - \frac{1}{(k+1)!}\right) A^{k-1} \tau^{k-1}\right] \begin{Bmatrix} \phi \\ -M^{-1} f(t) \end{Bmatrix}_t$$

$$+ \tau \left[\sum_{k=1}^{\infty} \frac{A^{k-1} \tau^{k-1}}{(k+1)!}\right] \begin{Bmatrix} \phi \\ -M^{-1} f(t) \end{Bmatrix}_{t+\tau} . \tag{9}$$

2.4 Boundary Conditions

All that remains to be done to develop a complete set of algorithms is to present a method of treating prescribed zero displacement, velocity, and acceleration boundary conditions as are found at restrained node points in structures. In finite element programs for static analysis, it is common practice to accommodate boundary conditions by modifying the stiffness matrix and applied load vectors to incorporate known nodal displacements. All that is required to accommodate a zero displacement is to delete all of the off-diagonal row and column elements of the stiffness matrix, set the diagonal element equal to unity, and set the applied load associated with that particular node equal to zero.

A parallel procedure in dynamic analysis may be used to accommodate zero displacement and velocity boundary conditions. For any degree of freedom of the structure for which the displacement and velocity are zero, the associated off-diagonal row and column elements of the A matrix are deleted, the diagonal element is set equal to unity, and the proper terms in the $M^{-1} f$ vector are deleted.

2.5 Structural Damping

Because of the general lack of knowledge about the exact velocity dependence of energy dissipative processes in structures, it is common practice to assume that the damping in the structure is a linear function of node point velocities. This may be readily incorporated into the mathematical model of the structure when modal analysis procedures are used. The same procedure used in modal analysis could be used with the matrix exponential solution, but that course was not followed in this investigation. An approximate representation of damping may be incorporated into the structure by considering two sets of dampers: one associated with the node point inertial characteristics and the other associated with the node point stiffness characteristics, as suggested by Biggs [10, pages 140-147]. The magnitude of the inertial associated damping coefficient matrix, C_r, is

$$C_r = c_r M , \tag{10}$$

where c_r is a scalar constant defined explicitly later. The magnitude of the stiffness associated damping coefficient matrix, C_g, is

$$C_g = c_g K , \tag{11}$$

where c_g is a scalar constant defined explicitly later. Biggs [10, pages 140-147] presents a method for determining these two sets of coefficients by substitution into the following equation.

$$c_g \omega^2 + c_r = \eta 2\omega , \tag{12}$$

where η is the ratio of actual to critical damping at the circular frequency ω. Thus, the damping ratio, η, may be set at any desired level at two separated frequencies. This determines the damping ratio at all other frequencies. The total structure damping matrix is therefore determined by

$$\begin{aligned} C &= C_r + C_g \\ &= c_r M + c_g K . \end{aligned} \tag{13}$$

3. COMPUTATIONAL ASPECTS

The evaluation of the matrix functions in eq. (8) may most readily be accomplished by use of either Sylvester's theorem or truncated series expansion. Computer codes based on both techniques [9,11] have been developed at Oak Ridge National Laboratory. The truncated series expansion approach was used by Stoddart [12] to develop a code for the analysis of plane motion problems. The systems which may be analyzed are characterized by distributed weight beams, weightless springs, and lumped masses. The beam element used is that recommended by McCalley [13], neglecting the effect of shear deformation.

The truncated series approach to determination of the matrix function allows control of error in the integration process through the setting of step size and number of terms used in the series [3,4,5,6]. A further precaution should be that the minimum step size and maximum number of terms are set through criteria related to the smallest number that may be represented on the computer. An emperical rule which has proved useful in the choice of k and τ is

$$\xi \leq k \ln\tau - \ln(K!) , \tag{14}$$

where ξ is the exponent associated with the smallest floating point that may be represented within the computer.

4. TRANSIENT RESPONSE OF SIMPLE STRUCTURES

To demonstrate the use of the computer program developed in this investigation, two example problems are presented and compared with known solutions.

4.1 First Example Problem

The first example problem involves the determination of the time history of displacements for the three-degrees-of-freedom problem illustrated in Fig. 1. The displacements indicated in Fig. 1 are measured from the static equilibrium position of the node points indicated as circled numerals. The time relationship and magnitudes of the applied loads $f_2(t)$, $f_3(t)$, and $f_4(t)$ are indicated in Fig. 2. The time increment used in the solution of the problem was 0.005 second, and the number of terms in the series approximation of the matrix exponential was ten. The displacement of node point 3 as determined with the computer program is plotted in Fig. 3 and may be compared with the solution developed through the use of modal methods reported by Biggs [10, pages 121-123]. The continuous line in Fig. 3 represents the theoretical solution and the symbols "X" represent the approximate solution as output from the computer program.

4.2 Second Example Problem

In this problem, the response of the point of dynamic load application for a simply supported beam, as illustrated in Fig. 4, is to be determined. The beam is a wide-flange steel section 14 in. deep that weighs 142 lb per lineal foot. The dynamic load, f(t), is initially 50,000 lb, decreases linearly to zero at 0.01 second, and remains zero for all later time.

The response of this beam was determined by using two combinations of beam elements connected in series. The time increment used in the solution of the problem was 0.0001 second, and eight terms were used in the series approximation. A comparison of the predicted response and that determined through modal analysis using the first three modes is illustrated in Fig. 5. The continuous line represents the theoretical solution obtained by modal superposition. The computer solutions for two- and four-beam elements are plotted with the symbols X and Δ, respectively.

5. CONCLUSIONS

It has been shown that the dynamic equations for a linear elastic structure may be written as a set of coupled first-order differential equations with constant coefficients. The matrix exponential solution method was presented to show the close similarity between it and the solution of a single first-order constant coefficient differential equation.

A computer program based on the equations presented was developed, and the transient response of two simple structures was determined through the use of this program. The transient responses determined in this manner were compared with previously reported analytical and experimental data.

The objective of this investigation was to develop a numerical solution for the transient response of linear elastic mechanical systems by using the matrix exponential method. With regard to this objective, the following conclusions may be drawn.

1. The matrix exponential solution method was applied successfully to determine the structural response of linear elastic mechanical systems.

2. The computer program developed in this investigation provided accurate solutions to the response of mechanical systems.

3. From limited comparisons with a modal analysis computer program, the time of formation of the matrix functions in eq. (8) is approximately the same as required to compute eigenvalues and eigenvectors.

4. The computer time per integration step is approximately the same for the matrix exponential solution using full matrices and the modal superposition of nearly all modes. The computer time per integration for the matrix exponential solution could be significantly reduced by using sparse matrix-vector multiplication methods.

In summary, the matrix exponential solution implemented in full matrix form is competitive with the more commonly used modal analysis solution in terms of computer time required and accuracy achieved, and it requires significantly greater core storage for in-core solution. If the characteristics of sparseness are recognized for the matrix functions, the storage allocations would be more nearly equal.

6. ACKNOWLEDGEMENT

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LIST OF REFERENCES

1. United Nuclear Corporation, "An Interpretive Review of Seismic Design Methods," USAEC Report ORNL TM-2900, Oak Ridge National Laboratory, May 1970.

2. L. A. Zadeh and C. A. Desoer, Linear System Theory: The State Space Approach, McGraw-Hill Book Company, New York, 1963.

3. M. L. Liou, "Time and Frequency Domain Analysis of Linear Time-Invariant Systems," System Analysis By Digital Computer, pp. 99-129, John Wiley and Sons, Inc., New York, 1966.

4. M. L. Liou, "A Novel Method of Evaluating Transient Response," Proceedings of the IEEE, Vol. 54, No. 1, pp. 20-23, January 1966.

5. J. B. Mankin, Jr., and J. C. Hung, "On Round-Off Errors in Computation of Transition Matrices," USAEC Report CTC-INF-597, Oak Ridge Gaseous Diffusion Plant, Computing Technology Center, June 27, 1969.

6. J. B. Mankin, Jr., and J. C. Hung, "Selection of Computational Parameters for Transition Matrices," USAEC Report CTC-INF-969, Oak Ridge Gaseous Diffusion Plant, Computing Technology Center, September 17, 1969.

7. A. Craggs, "Transient Vibration Analysis of Linear Systems Using Transition Matrices," NASA Contractor Report 1237, National Aeronautics and Space Administration, Washington, D. C., 1968.

8. A. Craggs, "The Transient Response of Coupled Acousto-Mechanical Systems," NASA Contractor Report 1421, National Aeronautics and Space Administration, Washington, D. C., August 1969.

9. S. J. Ball and R. K. Adams, "MATEXP, A General Purpose Digital Computer Program for Solving Ordinary Differential Equations by the Matrix Exponential Method," USAEC Report ORNL TM-1933, Oak Ridge National Laboratory, August 30, 1967.

10. J. M. Biggs, Introduction to Structural Dynamics, McGraw-Hill Book Company, New York, 1964.

11. J. V. Wilson, "IDEA - A Program for Integrating Differential Equations Approximately," USAEC Report ORNL CF-68-9-57, Oak Ridge National Laboratory, September 26, 1968.

12. W. C. Stoddart, "Transient Response of Linear Elastic Structures Determined by The Matrix Exponential Method," USAEC Report ORNL TM-3248, Oak Ridge National Laboratory, January 1971.

13. R. B. McCalley, Jr., "Mass Matrix for a Prismatic Beam Segment," USAEC Report KAPL-M-6913, Knolls Atomic Power Laboratory, March 1968.

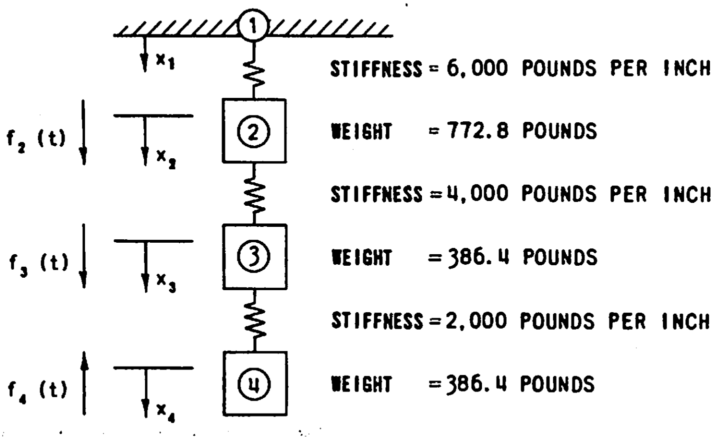


Figure 1. Three-Degrees-of-Freedom Model With Weightless Springs and Lumped Masses for First Example Problem.

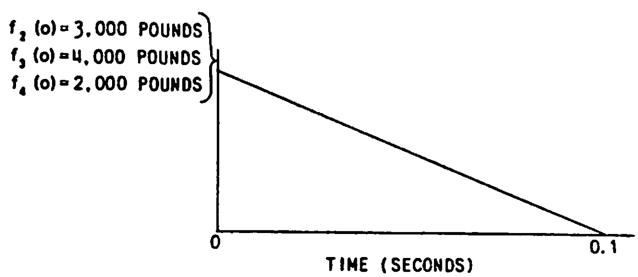


Figure 2. Applied Loads for the Three-Degrees-of-Freedom Model of the First Example Problem.

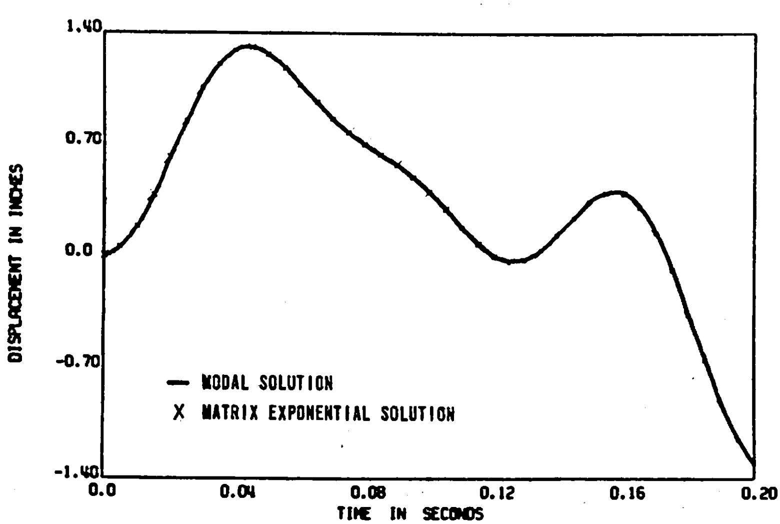


Figure 3. Example 1, Response of Node 3.

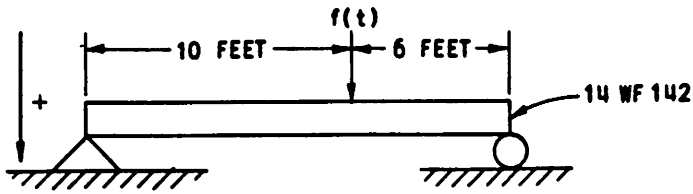


Figure 4. Simply Supported Beam of Second Example Problem.

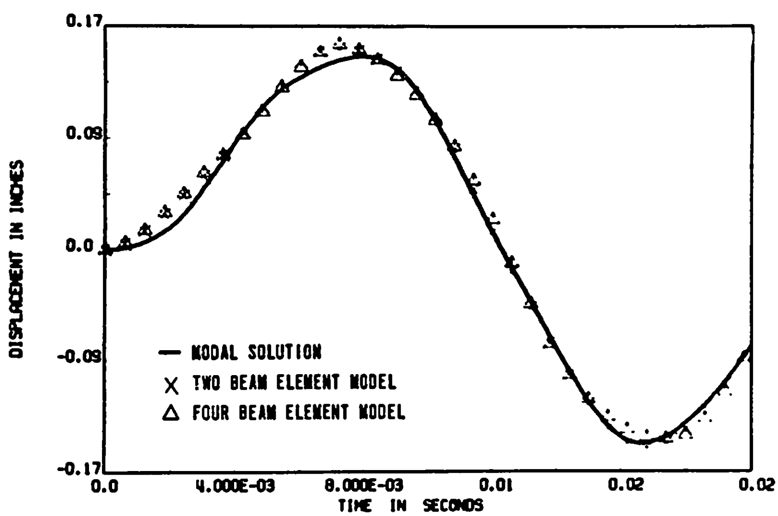


Figure 5. Example 2, Response at Point of Loading.

DISCUSSION

Q H. RIEKERT, Germany

I should like to know what method is used to evaluate $e^{A\tau}$ and if there are any numerical difficulties to expect in cases which deal with more complex structures than those shown at the session, especially when the stiffnesses of springs are different.

A W. C. STODDART, U. S. A.

Series expansion is used to determine all matrix functions in the program. This technique is preferred to alternates due to simplicity of coding and avoidance of problems with duplicate eigenvalues when techniques such as Sylvesteis formula are used. As with all numerical methods using finite arithmetics, difficulties may be expected when stiffness, and mass vary greatly within a structure. Even model analysis techniques run into problems unless condensation techniques are used wisely. Limited experience has shown that for structures modelled with care acceptable results are obtained with the series expansion method of obtaining $e^{A\tau}$. An excellent discussion of problems associated with round-off errors in this context was presented by J. B. Mankin and J. C. Hung at the Joint Automatic Control Conference, Boulder, Col. USA, Aug. 3-8, 1969, "On Round-Off Error in Computation of Transition Matrices".

Q T. H. LEE, U. S. A.

In the customary modal solution, we gain the advantage of obtaining equations in which the number of equations is reduced.
In your approach, the number of equations is actually increased (doubled). What is the advantage of using your approach which increases the number of equations ?

A W. C. STODDART, U. S. A.

In the modal solution method eigen-values and eigen-vectors are computed for all degrees of freedom which may be of interest. The response may be dominated by lower modes but still significant effort is required. The solution method proposed offers a trade between calculation of eigen-vectors and eigen-values with associated transformations to simplify arithmetic for a simple series expansion function of a 2N x 2N matrix. The additional gain is that "exact" structure velocities are obtained. The choice is based on convenience of formulation at the expense of increase in number of variables. The result is a slight decrease in running time with a precise method of evaluating errors in round-off (see comments to Mr. Riekert).

Q T. SEKIYA, Japan

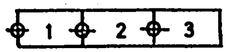
In the second example, where is the mass of each beam element concentrated ?

Is it at the nodal point or at the center of gravity of each element ? By our experience in the case of the finite difference method, it is preferable to concentrate the mass of each element at the center of gravity rather than at the nodal point for dynamic problems.

Example : Natural frequency of beam with clamped-free ends.

Fig. 1 - Mass at center of gravity

Fig. 2 - Mass at nodal point



The results of natural frequency using finite difference method for Fig. 1 and 2 are compared in Table 1 with the exact values where

$$\xi_i = \frac{\mu \omega_i^2 \ell^4}{\alpha}$$

α = bending stiffness of beam
μ = mass per unit length
ω_i = i - th natural circular frequency
ℓ = length of beam

	for Fig. 1 (1)	for Fig. 2 (2)	Exact value
ξ_1	8. 58	13. 9	12. 4
ξ_2	119. 4	472. 1	485. 5

References:

(1) Y. Sugiyama, T. Katayama and T. Sekiya : "Studies on Non-Conservative Problems of Instability of Columns by Difference Method", Proc. 19th Jap. Natl. Congr. Appl. Mech., 1969, p. 23.

(2) H. Leipholz, Z. A. M. P., 13, 581 (1962).

A W. C. STODDART, U. S. A.

The computer code developed uses a consistent mass matrix for distributed mass elements such as beams. The mass matrix used is similar to that in Refs. (1) and (2) below. In Ref. (3) a comparison of the eigenvalues for several formulations for stiffness and mass matrices is made and detailed discussion is given. In general it has been found that the number of elements, subdivisions in length of a beam, should be two times the number of the greatest eigenvalue.

References:

(1) J. S. Archer : "Consistent Mass Matrices for Distributed Mass Systems", Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, pp. 161-178, August 1963.

(2) R. B. McCalley, Jr. : "Mass Matrix for a Prismatic Beam Segment", USAEC Report KAPL-M-6913, Knolls Atomic Power Laboratory, March 1968.

(3) K. K. Kapur : "Vibrations of a Timoshenko Beam, Using Finite Element Approach", The Journal of the Acoustical Society of America, pp. 1058-1063, Vol. 40, No. 5, 1966.