ANALYSIS OF MODERATELY THICK SHELLS OF REVOLUTION

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ABSTRACT

A method is presented for the analysis of moderately thick shells of revolution under symmetrical loads. Using assumed transverse distributions of stresses and displacements, the governing differential equations are derived by a variational process. These equations are integrated numerically using a procedure developed previously. An illustrative example is included and provides an indication of the accuracy of the theory and the procedure.

1. INTRODUCTION

A method is presented for calculating stresses and deformations in moderately thick shells of revolution subjected to axisymmetric loading such as thermal loads, internal or external pressure, band loads and loadings which vary arbitrarily in the meridional direction. The method has particular application to nuclear reactor primary vessals and may also be used in the analysis of containment and protective shells. The shell of revolution may have any arbitrary shape. The thickness and material properties may vary in any arbitrary manner.

The theory upon which the method is based is an extension of that used by Reissner [1] but incorporates somewhat more realistic stress distributions than have been used by others. Following the "numerical integration method" developed originally by Goldberg and Bogdanoff [2, 3], the governing equations are put in the form of a coupled system of first-order differential equations in the "intrinsic" dependent variables. These equations are integrated with the aid of a computer using any dependable procedure such as the Runge-Kutta fourth-order process. The numerical integration procedure includes a routine previously developed by the senior author [7,8] for precluding poorly conditioned equations at the boundaries. The program, as written, accepts the basic geometry, properties and loads as input; computes the necessary geometrical and mechanical quantities; and calculates deformations and stresses. Stresses are calculated at any desired points within the wall of the shell in two different ways. A comparison of the two sets of stress results provides a basis for appraising the quality of the analysis. The theory and program appear to give results that are considerably better than required for engineering purposes when the radius-to-thickness ratio is as low as four or five.

2. THEORY

Points on the middle surface of the shell are located by the coordinates ϕ,θ where ϕ is the angle between the axis of the shell and the normal, and θ is the polar angle. The third coordinate in the orthogonal system is the normal distance, z, from the middle surface and is positive inward. The principal radii of the middle surface are ρ, R with $\rho = \rho(\phi)$ being the radius of curvature of the meridian. The components of displacement are $\bar{u} = \bar{u}(\phi,z)$ in the meridional direction and $\bar{w} = \bar{w}(\phi,z)$ in the direction of the inner normal.

In view of symmetry, the relevant strains are given by

$$\left.\begin{aligned} (1 - \frac{z}{\rho})\,\varepsilon_\phi &= \frac{\partial\bar{u}}{\rho\partial\phi} - \frac{\bar{w}}{\rho} \\ (1 - \frac{z}{R})\,\varepsilon_\theta &= \frac{\bar{u}\cot\phi}{R} - \frac{\bar{w}}{R} \\ \varepsilon_z &= \frac{\partial\bar{w}}{\partial z} \\ \gamma_{\phi z} &= (\rho-z)\,\frac{\partial}{\partial z}\left(\frac{\bar{u}}{\rho-z}\right) + \frac{1}{\rho-z}\,\frac{\partial\bar{w}}{\partial\phi} \end{aligned}\right\} \qquad (1)$$

Approximations for the stress are taken in the following way:

$$\left.\begin{aligned} \sigma_\phi &= \frac{N_\phi}{h} + \frac{M_\phi}{he\phi}\,\frac{z+(c_\phi-e_\phi)}{\rho-z} \\ \sigma_\phi &= \frac{N_\theta}{h} + \frac{M_\theta}{he\theta}\,\frac{z+(c_\theta-e_\theta)}{R-z} \\ \sigma_z &= \frac{\rho R}{(\rho-z)(R-z)}\left\{-\frac{N_\phi}{\rho}\left(\frac{h}{8R} - \frac{z}{2h} - \frac{z^2}{2hR} + \frac{2z^3}{h^3}\right) - \frac{N_\theta}{R}\left(\frac{h}{8\rho} - \frac{z}{2h} - \frac{z^2}{2h\rho} + \frac{2z^3}{h^3}\right)\right. \\ &\quad + M_\phi\,(E_1 + F_1 z + G_1 z^2 + H_1 z^3) + M_\theta\,(E_2 + F_2 z + G_2 z^2 + H_2 z^3) \\ &\quad \left. - \frac{3}{4}\,p^*\left[\frac{2}{3} - \frac{2z}{h} + \frac{1}{3}\,(\frac{2z}{h})^3\right] - \frac{3}{4}\,q^*\left[\frac{2}{3} + \frac{2z}{h} - \frac{1}{3}\,(\frac{2z}{h})^3\right]\right\} \\ \tau_{\phi z} &= \frac{R}{R-z}\left\{\frac{3N_{\phi z}}{2h}\left[1-(\frac{2z}{h})^2\right](1 + \frac{z}{\rho}) - \frac{p^+}{4}\,(1 - \frac{h}{2R})\left[1- \frac{4z}{h} - 3(\frac{2z}{h})^2\right]\right. \\ &\quad \left. + \frac{p^-}{4}\,(1 + \frac{h}{2R})\left[1 + \frac{4z}{h} - 3\,(\frac{2z}{h})^2\right]\right\} \end{aligned}\right\} \qquad (2)$$

The terms p^* and q^* are related to the loadings normal to the middle surface associated with normal external (p_o) and internal (p_i) pressures in the following way:

$$\left.\begin{aligned} 4\rho Rp^* &= p_o\,(2\rho + h)\,(2R + h) \\ 4\rho Rq^* &= p_i\,(2\rho - h)\,(2R - h) \end{aligned}\right\} \qquad (3)$$

The terms p+ and p- are the components of the surface loadings at the inner and outer surfaces, respectively and in the direction of the tangent to the meridian at the middle surface; p+ being in the increasing ϕ-direction and p- being opposite. The parameter e_ϕ is the distance between the neutral and centroidal surfaces for bending in the ϕ-direction, and

c_ϕ is the distance between the centroidal and middle surfaces; e_θ and c_θ are analogous parameters associated with the θ-direction. The parameters are evaluated by the following formulas:

$$\left.\begin{aligned} c_\phi &= \frac{h^2}{12R} ; \qquad e_\phi = \frac{m_\phi(\rho + c_\phi)}{1 + m_\phi} \\ m_\phi &= (\rho + c_\phi)\left[\frac{1}{h}\,(1 - \frac{\rho}{R})\log\left(\frac{2\rho+h}{2\rho-h}\right) + \frac{1}{R}\right] - 1 \end{aligned}\right\} \qquad (4)$$

The subscripts are to be interchanged for c_θ and e_θ.

The expression for σ_z involves parameters E., etc. which depend upon the geometry of the shell and are defined as follows:

$$\left.\begin{aligned} E_1\, h\, \rho^2\, e_\phi &= (d_\phi + \rho)\,(R - \rho)\log\left(1 + \frac{h}{2\rho}\right) + \frac{h}{2R}\left(\rho - R + d_\phi - \frac{h}{4}\right) \\ F_1\, h\, \rho^2\, e_\phi &= c_\phi - e_\phi = d_\phi \\ 2G_1\, h\, \rho^3\, R\, e_\phi &= d_\phi\,(R - \rho) + \rho R \\ 3H_1\, h\, \rho^4\, R\, e_\phi &= (d_\phi + \rho)\,(R - \rho) \end{aligned}\right\} \qquad (5)$$

Formulas for E_2, etc. are obtained by interchanging subscripts.

The expressions chosen for σ_ϕ and σ_ϕ were motivated by the engineering theory of curved beams and include the effect of double curvature in a rational manner. This is presumed to be an improvement on the assumptions made previously by others. The relation between σ_z and the tractions N_ϕ, N_θ, M_ϕ, M_θ was obtained by examining transverse equilibrium of the differential element. The relation between σ_z and the surface loads is a modified form [4] of the well-known plane elasticity solution for beams with distributed loading. The relation between $\tau_{\phi z}$ and $N_{\phi z}$ was motivated by Golovin's solution [9] for a curved bar of rectangular section but modified in the present development to account for the double curvature. The distribution of $\tau_{\phi z}$ due to the tangential loadings is a simple equilibrium distribution and was also used by Naghdi [6].

Approximations for the components of displacement which are generally consistant with the approximations for the stress are

$$\bar{u} = u + z\,\beta, \quad \bar{w} = w + z\,w' + z^2\,w''/2 \qquad (6)$$

Substitution of eqs. (b) into eqs. (1) yields

$$\left.\begin{aligned} \varepsilon_\phi\,(\rho - z) &= \frac{du}{d\phi} + z\,\frac{d\beta}{d\phi} - \bar{w} \\ \varepsilon_\theta\,(R - z) &= (u + z\,\beta)\cot\phi - \bar{w} \\ \varepsilon_z &= w' + z\,w'' \\ (\gamma_{\phi z} - \beta)(\rho - z) &= u + z\,\beta - \frac{\partial\bar{w}}{\partial\phi} \end{aligned}\right\} \qquad (7)$$

The governing equations are now derived through the use of a variational equation [1] which, in the φθz system, is

$$\begin{aligned} \delta\Big[&\iiint\{\sigma_\phi\varepsilon_\phi + \sigma_\theta\varepsilon_\theta + \sigma_z\varepsilon_z + \tau_{\phi z}\gamma_{\phi z} - \frac{1}{2E}\,[\sigma_\phi^2 + \sigma_\theta^2 + \sigma_z^2 \\ &-2\nu(\sigma_\phi\sigma_\theta + \sigma_\theta\sigma_z + \sigma_z\sigma_\phi)] - \frac{1}{2G}\,\tau_{\phi z}{}^2\}(1 - \frac{z}{\rho})(1 - \frac{z}{R})r\,\rho\; d_\phi d_\theta d_z \\ &-\iint (p^*\bar{w}_o - q^*\bar{w}_i + b^*\bar{u}_i - a^*\bar{u}_o)\; r r_\phi d_\phi d_\theta \Big] = 0 \end{aligned} \qquad (8)$$

where

r = radius of parallel circle

$$\left.\begin{aligned}
\bar{v}_o &= \bar{v}(\phi, -h/2), \quad \bar{v}_i = \bar{v}(\phi, h/2) \\
\bar{u}_o &= \bar{u}(\phi, -h/2), \quad \bar{u}_i = \bar{u}(\phi, h/2) \\
4\rho R a^* &= p^- (2\rho + h)(2R + h) \\
4\rho R b^* &= p^+ (2\rho - h)(2R - h)
\end{aligned}\right\} \tag{9}$$

After substituting eqs. (2) and (7) into eq. (8) and integrating with respect to z, the Euler equations of (8) become the desired equations; namely, three differential equations of equilibrium, three force-displacement relations and two auxiliary formulas for N_θ and M_θ.

In matrix form, the governing equations are as given below.

$$\text{Equilibrium:} \quad \underline{A} = \underline{B}\,\underline{C} \tag{10}$$

$$\text{Force-displacement:} \quad \underline{D} = \underline{F}\,\underline{H} \tag{11}$$

$$\text{Auxiliary } (N_\theta, M_\theta): \quad \underline{J} = \underline{K}^{-1}\,\underline{L}\,\underline{P} \tag{12}$$

$$\text{Auxiliary } (w', w''): \quad \underline{W} = \underline{S}\,\underline{T} \tag{13}$$

$\underline{A}$ and $\underline{D}$ are 3 x 1 column vectors of first derivatives; $\underline{C}$, $\underline{H}$, $\underline{P}$ and I are column vectors made up of dependent variables and loading terms. The detail forms of the matrices are given in the Appendix. The differential equations are to be integrated numerically using, for example, the Runge-Kutta fourth-order process. An efficient computational procedure which has been described elsewhere [7,8] may be conveniently followed.

3. ILLUSTRATIVE EXAMPLE

As an illustrative example, a spherical dome has been analyzed and a portion of the results are listed in Table 1. The shell has a mean radius of 900 inches, a uniform thickness of 180 inches and is clamped at the outer boundary where $\phi = 0.8$ radians. To avoid the singularity at $\phi = 0$, a small hole ($0 \le \phi \le 0.01$) is assumed at the top. Alternate simplifying assumptions, either a small rigid plug or a small elastic plug, are readily accomodated. The modulus of elasticity and Poisson's ratio are $E = 3 \times 10^6$ psi and $\nu = 0.1$. The shell is assumed to be subjected to uniform external pressure of 10 psi over the portion of the shell $0.01 \le \phi \le 0.61$. For computational stability, the path of integration was divided into 50 segments, and each segment was subdivided into 10 intervals. Values of all dependent variables were calculated at all division points.

In addition, the program calculated the stresses wherever desired within the shell by two methods: (1) from the strains by means of Hooke's law, and (2) from the tractions using eqs. (2). A comparison of corresponding values provides the engineer with a basis for appraising the consistency and dependability of the analysis.

Table 1 shows the normal stresses, σ_θ and σ_θ for several locations at the outer surface as calculated (1) from strains, and (2) from tractions. Except where the relative stress level is quite low and of no technical significance, the very good agreement between the two sets of calculated stresses is noteworthy and gratifying.

4. ACKNOWLEDGEMENT

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Table 1. Comparison of Stresses at Outer Surface

ϕ	σ_ϕ (psi)			$\sigma_{\theta\theta}$ (psi)		
	from strains	from tractions	diff. %	from strains	from tractions	diff. %
.026	-71.67	-72.53	1.20	-97.19	-98.18	1.01
.057	-81.64	-82.48	1.05	-86.57	-87.55	1.12
.089	-82.57	-83.44	1.04	-84.30	-85.28	1.15
.136	-81.85	-82.71	1.06	-81.92	-82.90	1.18
.231	-77.34	-78.23	1.14	-75.75	-76.74	1.29
.326	-69.04	-69.95	1.30	-66.67	-67.67	1.47
.421	-55.84	-56.77	1.63	-54.54	-55.53	1.79
.500	-39.98	-40.91	2.41	-42.26	-43.23	2.25
*.610	- 8.35	- 9.24	9.64	-22.58	-23.50	3.90
.674	8.75	9.11	4.05	- 9.10	- 8.74	4.08
.721	18.23	18.62	2.08	- 3.13	- 2.76	13.71
.768	24.71	25.11	1.60	1.07	1.46	-
.800	27.55	27.96	1.47	2.82	3.22	12.48

*Outer edge of loaded region.

APPENDIX

Column vectors $\underline{A}$, $\underline{D}$, $\underline{J}$, and $\underline{W}$ in eqs. (10)-(13) are

$$A = \begin{Bmatrix} \dfrac{d(rN_\phi)}{ds} \\ \dfrac{d(rN_{\phi z})}{ds} \\ \dfrac{d(rM_\phi^*)}{ds} \end{Bmatrix}, \quad \underline{D} = \begin{Bmatrix} \dfrac{du}{ds} \\ \dfrac{dw}{ds} \\ \dfrac{d\beta}{ds} \end{Bmatrix}, \quad \underline{J} = \begin{Bmatrix} N_\theta \\ M_\theta \end{Bmatrix}, \quad \underline{W} = \begin{Bmatrix} w' \\ w'' \end{Bmatrix} \tag{14}$$

where $M_\phi^* = M_\phi - c_\phi N_\phi$ and $ds = \rho d\phi$. (15)

The elements of column vectors $\underline{C}$, $\underline{H}$, $\underline{P}$, and $\underline{T}$, written horizontally, are

$$\underline{C} = (N_\phi,\ M_\phi,\ N_{\phi z},\ M_\theta^*,\ a^*,\ b^*,\ p^*,\ q^*) \tag{16}$$

$$\underline{H} = (u,\ v,\ \beta,\ w',\ w'',\ \frac{dw'}{ds},\ \frac{dw''}{ds},\ N_\phi,\ N_\theta,\ N_{\phi z},\ M_\phi,\ M_\theta,\ p^*,\ q^*,\ p^+,\ p^-) \tag{17}$$

$$\overline{P} = (u,\ w,\ w',\ w'',\ N_\phi,\ M_\phi,\ p^*,\ q^*) \tag{18}$$

$$\overline{T} = (N_\phi,\ N_\theta,\ M_\phi,\ M_\theta,\ p_o,\ p_1) \tag{19}$$

For use in eqs. (10) and (13)

$$\underline{B} = r \begin{bmatrix} 0 & \cot\phi/R & 1/\rho & 0 & 1 & -1 & 0 & 0 \\ -1/p & -1/R & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & \cot\phi/R & -h/2 & -h/2 & 0 & 0 \end{bmatrix} \tag{20}$$

$$\underline{S} = \frac{-1}{Eh} \begin{bmatrix} \nu & \nu & s_\phi & s_\theta & h/2 & h/2 \\ 0 & 0 & v_\phi & v_\theta & -1 & 1 \end{bmatrix} \tag{21}$$

$$s_\phi = \nu(4d_\phi\rho - h^2)/e_\phi(4\rho^2 - h^2), \quad v_\phi = 4\nu(d_\phi + \rho)/e_\phi(4\rho^2 - h^2) \tag{22}$$

Matrices $\underline{K}$ and $\underline{L}$, in eq. (12), are

$$\underline{K} = [k_{mn}] \quad \text{and} \quad \underline{L} = [\ell_{mn}] \tag{23}$$

Matrix $\underline{F}$ may be written

$$\underline{F} = [f_{m,n}] \tag{24}$$

in which the non-zero elements are

$$f_{1,2} = \frac{1}{r_\phi},\ f_{1,4} = c_\phi n_{1\phi},\ f_{1,5} = c_\phi \ \frac{3R}{10\rho} - n_{1\theta}^{\phi},\ f_{1,8} = \frac{1}{Eh} - n_{14} + (A_{19} + c_\phi A_1)$$

$$- x_\phi y_{1\phi}^{\theta},\ f_{1,9} = -\frac{\nu}{Eh} - n_{14} + (A_{20} + c_\phi A_2) - x_\theta n_{13},\ f_{1,11} = \frac{1}{E}(t_\phi - \nu v_1)$$

$$+ (A_{21} + c_\phi A_3) + 6x_\phi[s_{11}n_9 n_{23} + 2s_{13}(n_{24} - n_{30}x_{1\phi}) + 2s_{15}(n_{25} - r_\phi^{\ 2} n_{30} v_{1\phi})]$$

$$f_{1,12} = \frac{\nu}{E}(t_\theta + v_2) + (A_{22} + c_\phi A_4) + \frac{hy_\theta}{2\rho R}[s_{21}n_{26}n_{22} + s_{23}(n_{27} - n_{31}x_{1\theta})$$

$$+ s_{25}(n_{28} - R^2 n_{31} v_{1\theta})] - \frac{h^3}{6} n_{15} t_{12}^{\theta} - c_\phi n_{15} n_{29},\ f_{1,13} = \frac{\nu}{2E} + (A_{23} + c_\phi A_5)$$

$$+ \frac{h^3}{12} n_{15}[s_{11}n_9 + s_{13}(n_{10} + n_{11}) + s_{15}n_{12}],\ f_{1,14} = \frac{\nu}{2E} + (A_{24} + c_\phi A_6)$$

$$+ \frac{h^3}{12} n_{15}[s_{11}n_9 + s_{13}(n_{10} - n_{11}) - s_{15}n_{12}]$$

$$f_{2,6} = -\frac{3}{5} c_\theta,\ f_{2,7} = -\frac{h}{40},\ f_{2,10} = \frac{9n_{21}}{4h^2G},\ f_{2,15} = \frac{3}{8hG}(1 - \frac{h}{2R})[s_{21}n_2 + s_{23}(n_3 + n_4)$$

$$+ s_{25}(n_5 + n_6) + s_{27}(n_7 + n_8)],\ f_{2,16} = -\frac{3}{8hG}(1 + \frac{h}{2R})[s_{21}n_2 + s_{23}(n_4 - n_3)$$

$$+ s_{25}(n_6 - n_5) + s_{27}(n_8 - n_7)]$$

$$f_{3,4} = n_{1\phi},\ f_{3,5} = -n_{1\theta}^{\phi} + \frac{c_\theta}{2e_\phi},\ f_{3,8} = \frac{1}{E}(t_\phi - \nu v_1) + A_1 - \frac{y_\phi y_{1\phi}^{\theta}}{2\rho h},\ f_{3,9} = -\frac{\nu}{E}(t_\phi + v_1)$$

$$+ A_2 - n_{13}n_{15},\ f_{3,11} = A_3 + y_\phi t_{11}^{\phi},\ f_{3,12} = A_4 - y_\theta(s_{21}z_{1\theta}^{\phi} + 2\rho^2 s_{23}x_{1\phi}$$

$$+ 2R^4 s_{25}v_{1\theta}) - y_\phi t_{12}^{\phi} - n_{15}n_{29},\ f_{3,13} = A_5 + \frac{y_\phi}{2}[s_{11}n_9 + s_{13}(n_{10} + n_{11}) + s_{15}n_{12}],$$

$$f_{3,14} = A_6 + \frac{y_\phi}{2}[s_{11}n_9 + s_{13}(n_{10} - n_{11}) - s_{15}n_{12}]$$

$$E^* = 2E(R - \rho),\ s_{ji} = \sum_{n=0}^{\infty} \frac{1}{i + 2n}\left(\frac{h}{2r_j}\right)^{i+2n},\ i = 1, 3, \ldots \text{ and } r_1 = \rho,\ r_2 = R$$

Matrix $\underline{K}$ may be written as

$$\underline{K} = [k_{mn}] \tag{25}$$

where

$$k_{11} = \frac{1}{Eh}[\frac{c_\phi}{\rho}(1 + 2\nu) + 1] + A_8,\quad k_{12} = k_{21} = \frac{1}{E}(t_\theta - \nu v_2) + A_{10} - \frac{y_\theta y_{2\theta}^{\phi}}{2hR},$$

$$k_{22} = A_{16} + \frac{y_\theta n_{20}}{\nu Rh} - 2y_\theta t_{22}^{\theta}$$

Matrix $\underline{L}$ may be written as

$$\underline{L} = [\ell_{mn}]$$

in which the non-zero elements are

$$\ell_{11} = \ell_{22} = \frac{\cot\phi}{R},\ \ell_{12} = -\frac{c_\phi \cot\phi}{\rho},\ \ell_{13} = -\frac{1}{R},\ \ell_{15} = -\frac{3}{10}c_\phi,\ \ell_{16} = \frac{\nu}{Eh}(1 - \frac{c_\phi}{\rho}) - A_7,$$

$$\ell_{17} = \frac{\nu}{E}(t_\phi + v_1) - A_9 + n_{13}n_{15},\ \ell_{18} = -\frac{\nu}{2E} - A_{11},\ \ell_{19} = -\frac{\nu}{2E} - A_{12},\ \ell_{24} = -n_{2\theta},$$

$$\ell_{25} = n_{2\phi}^{\theta},\ \ell_{26} = \frac{\nu t_\theta}{E},\ \ell_{27} = y_\theta(s_{21}z_{1\theta}^{\theta} + s_{23}z_{2\phi}^{\phi} + 2R^4 s_{25}v_{1\theta}) + y_\phi t_{12}^{\phi} - A_{15},$$

$$\ell_{28} = \frac{y_\theta}{2h}[s_{21}n_{16} + s_{23}(n_{17} + n_{18}) + s_{25}n_{19}] - A_{17},\ \ell_{29} = \frac{y_\theta}{2h}[s_{21}n_{16} + s_{23}(n_{18} - n_{17})$$

$$- s_{25}n_{19}] - A_{18}$$

The symbols A_i have the following definitions:

$$\begin{aligned}A_1 = \frac{1}{E^*}\Big[& s_{13}\left(\frac{2E_1\rho R}{h} + \frac{2E_1\rho^2}{h} - \frac{F_1h\rho}{2} - \frac{G_1h\rho^2}{2}\right) - s_{15}\left(\frac{8E_1\rho^3}{h^3}\right. \\ & \left. - \frac{2F_1\rho^3}{h} + \frac{8F_1\rho^4R}{h^3} - \frac{2G_1\rho^3R}{h} - \frac{2G_1\rho^4}{h} + \frac{H_1h\rho^3}{2} - \frac{2H_1\rho^4R}{h}\right) \\ & - s_{17}\left(\frac{8G_1\rho^5R}{h^3} - \frac{2H_1\rho^5}{h} + \frac{8H_1\rho^6R}{h^3}\right) + (s_{21} - s_{11})\frac{E_1h}{2} \\ & - s_{23}\left(\frac{4E_1R^2}{h} - \frac{F_1hR}{2} + \frac{2F_1R^3}{h} - \frac{G_1hR^2}{2}\right) + s_{25}\left(\frac{8E_1R^4}{h^3} - \frac{2F_1R^3}{h} + \frac{8F_1R^5}{h^3}\right. \\ & \left. - \frac{4G_1R^4}{h} + \frac{H_1hR^3}{2} - \frac{2H_1R^5}{h}\right) + s_{27}\left(\frac{8G_1R^6}{h^3} - \frac{2H_1R^5}{h} + \frac{8H_1R^7}{h^3}\right)\Big]\end{aligned}$$

$$\begin{aligned}A_2 = \frac{1}{E^*}\Big[& (s_{21} - s_{11})\frac{E_1h}{2} + s_{13}\left(\frac{4E_1\rho^2}{h} - \frac{F_1h\rho}{2} + \frac{2F_1\rho^3}{h}\right) - s_{15}\left(\frac{8E_1\rho^4}{h^3}\right. \\ & \left. - \frac{2F_1\rho^3}{h} + \frac{8F_1\rho^5}{h^3} - \frac{4G_1\rho^4}{h} + \frac{H_1h\rho^3}{h}\right) - s_{17}\left(\frac{8G_1\rho^6}{h^3}\right. \\ & \left. - \frac{2H_1\rho^5}{h} + \frac{8H_1\rho^7}{h^3}\right) - s_{23}\left(\frac{2E_1\rho R}{h} + \frac{2E_1R^2}{h} - \frac{F_1hR}{2} + \frac{2F_1\rho R^2}{h} - \frac{G_1hR^2}{2}\right) \\ & + s_{25}\left(\frac{8E_1\rho R^3}{h^3} - \frac{2F_1R^3}{h} + \frac{8F_1\rho R^4}{h^3} - \frac{2G_1\rho R^3}{h} - \frac{2G_1R^4}{h} + \frac{H_1hR^3}{2} - \frac{2H_1\rho R^4}{h}\right) \\ & + s_{27}\left(\frac{8G_1\rho R^5}{h^3} - \frac{2H_1R^5}{h} + \frac{8H_1\rho R^6}{h^3}\right)\Big]\end{aligned}$$

$$\begin{aligned}A_3 = \frac{1}{E^*}\Big[& 4E_1^2\rho R(s_{11} - s_{21}) + s_{13}(8E_1F_1\rho^2R + 8E_1G_1\rho^3R + 4F_1^2\rho^3R) \\ & + s_{15}(8E_1H_1\rho^4R + 8F_1G_1\rho^4R + 8F_1H_1\rho^5R + 4G_1^2\rho^5R) \\ & + s_{17}(8G_1H_1\rho^6R + 4H_1^2\rho^7R) - s_{23}(8E_1F_1\rho R^2 + 8E_1G_1\rho R^3 + 4F_1^2\rho R^3) \\ & - s_{25}(8E_1H_1\rho R^4 + 8F_1G_1\rho R^4 + 8F_1H\rho R^5 + 4G_1^2\rho R^4) - s_{27}(8G_1H_1\rho R^6 + 4H_1^2\rho R^7)\Big]\end{aligned}$$

$$\begin{aligned}A_4 = \frac{1}{E^*}\Big[& 4E_1E_2\rho R(s_{11} - s_{21}) + 4s_{13}(e_1F_2\rho^2R + E_1G_2\rho^3R + F_1E_2\rho^2R \\ & + F_1F_2\rho^3R + G_1E_2\rho^3R) + 4s_{15}(E_1H_2\rho^4R + F_1G_2\rho^4R + F_1H_2\rho^5R \\ & + G_1F_2\rho^4R + G_1G_2\rho^5R + H_1E_2\rho^4R + H_1F_2\rho^5R) \\ & + 4s_{17}(G_1H_2\rho^6R + H_1G_2\rho^6R + H_1H_2\rho^7R) - 4s_{23}(E_1F_2\rho R^2 \\ & + E_1G_2\rho R^3 + F_1E_2\rho R^2 + F_1F_2\rho R^3 + G_1E_2\rho R^3) - 4s_{25}(E_1H_2\rho R^4 \\ & + F_1G_2\rho R^4 + F_1H_2\rho R^5 + G_1F_2\rho R^4 + G_1G_2\rho R^5 + H_1E_2\rho R^4 + H_1F_2\rho R^5) \\ & - 4s_{27}(G_1H_2\rho R^6 + H_1G_2\rho R^6 + H_1H_2\rho R^7)\end{aligned}$$

$$A_5 = \frac{1}{E^*}\Big[(s_{11} - s_{21})h_1 + s_{13}(z_\phi + n_\phi) + s_{15}(\lambda_\phi - \xi_\phi) + s_{17}\Lambda_\phi + s_{23}(z_\theta + n_\theta)$$

$$+ s_{25}(\xi_\theta - \lambda_\theta) + s_{27}\Lambda_\theta \Big]$$

$$A_6 = \frac{1}{E^*}\Big[(s_{11} - s_{21})h_1 + s_{13}(z_\phi - n_\phi) - s_{15}(\xi_\phi - \lambda_\phi) - s_{17}\Lambda_\phi + s_{23}(z_\theta - n_\theta)$$

$$+ s_{25}(\xi_\theta + \lambda_\theta) - s_{27}\Lambda_\theta \Big]$$

$$A_7 = \frac{1}{E^*}\Big[(s_{11} - s_{21}) \frac{h^2}{16\rho R} - s_{13}\left(\frac{3\rho}{4R} - \frac{\rho^2}{h^2} + \frac{1}{4}\right) + s_{15}\left(\frac{3\rho^3}{h^2 R} + \frac{2\rho^2}{h^2} - \frac{8\rho^4}{h^4}\right)$$

$$- s_{17}\left(\frac{4\rho^5}{h^4 R} + \frac{4\rho^4}{h^4} - \frac{16\rho^6}{h^6}\right) + s_{23}\left(\frac{1}{4} - \frac{3R}{4\rho} - \frac{R^2}{h^2}\right)$$

$$- s_{25}\left(\frac{2R^2}{h^2} + \frac{3R^3}{h^2\rho} - \frac{8R^4}{h^4}\right) + s_{27}\left(\frac{4R^4}{h^4} + \frac{4R^5}{h^4\rho} - \frac{16R^6}{h^6}\right)\Big]$$

$$A_8 = \frac{1}{E^*}\Big[(s_{11} + s_{21}) \frac{h^2}{16\rho R} - s_{13}\frac{\rho}{R} + s_{15}\left(\frac{5\rho^3}{h^2 R} - \frac{8\rho^5}{h^4 R}\right) - s_{17}\left(\frac{8\rho^5}{h^4 R} - \frac{16\rho^7}{h^6 R}\right)$$

$$+ s_{23}\left(\frac{R}{2\rho} + \frac{1}{2}\right) - s_{25}\left(\frac{4R^2}{h^2} - \frac{8\rho R^3}{h^4} + \frac{R^3}{h^2\rho}\right) + s_{27}\left(\frac{8R^4}{h^4} - \frac{16\rho R^5}{h^6}\right)$$

$$A_9 = A_2$$

A_{10}, A_{16}, A_{17}, A_{18} = A_2, A_3, A_5, A_6 with E_1, F_1, G_1, H_1 replaced by E_2, F_2, G_2, H_2

$$A_{11} = \frac{1}{E^*}\Big[(s_{11} - s_{21})h_2 - s_{13}(\zeta_{\phi\phi} + z_{\phi\phi}) + s_{15}(y_{\phi\phi} + h_{\phi\phi}) - s_{17}\frac{\rho^2 h_{\phi\phi}}{h^2}$$

$$+ s_{23}(\zeta_{\theta\phi} + z_{\theta\phi}) - s_{25}(y_{\theta\phi} + h_{\theta\phi} + s_{27}\frac{R^2 h_{\theta\phi}}{h^2}\Big]$$

$$A_{14} = A_{10}$$

$$A_{15} = A_4$$

$$A_{19} = \frac{1}{E^*}\Big[(s_{11} - s_{21}) \frac{h^2}{16\rho R} - s_{13}\left(\frac{\rho}{2R} - \frac{\rho R}{h^2} + \frac{1}{2}\right) + s_{15}\left(\frac{4\rho^2}{h^2} - \frac{8\rho^3 R}{h^4} + \frac{\rho^3}{h^2 R}\right)$$

$$- s_{17}\left(\frac{8\rho^4}{h^4} - \frac{16\rho^5 R}{h^6}\right) + s_{23}\left(\frac{R}{\rho} - \frac{R^3}{h^2\rho}\right) - s_{25}\left(\frac{5R^3}{h^2\rho} - \frac{8R^5}{h^4\rho}\right)$$

$$+ s_{27}\left(\frac{8R^5}{h^4\rho} - \frac{16R^7}{h^6\rho}\right)\Big]$$

$$A_{20} = A_7,\ A_{21} = A_1,\ A_{22} = A_{13}$$

$$A_{23} = \frac{1}{E^*}\Big[(s_{11} - s_{21})h_2 - s_{13}(\zeta_{\phi\theta} + z_{\phi\theta}) + s_{15}(y_{\phi\theta} + h_{\phi\theta}) - s_{17}\frac{\rho^2 h_{\phi\theta}}{h^2}$$

$$+ s_{23}(\zeta_{\theta\theta} + z_{\theta\theta}) - s_{25}(y_{\theta\theta} + h_{\theta\theta}) + s_{27}\frac{R^2 h_{\theta\theta}}{h^2}\Big]$$

$$A_{24} = \frac{1}{E^*}\Big[(s_{11} - s_{21})h_2 - s_{13}(\zeta_{\phi\theta} - z_{\phi\theta}) + s_{15}(y_{\phi\theta} - h_{\phi\theta}) + s_{17}\frac{\rho^2 h_{\phi\theta}}{h^2}$$

$$+ s_{23}(\zeta_{\theta\theta} - z_{\theta\theta}) - s_{25}(y_{\theta\theta} - h_{\theta\theta}) - s_{27}\frac{R^2 h_{\theta\theta}}{h^2}.$$

Symbols n_2, ..., n_{31} are defined in the following way:

$$n_2 = 2R,\ n_3 = -\frac{8R^2}{h},\ n_4 = -\frac{2R^3}{\rho^2} - \frac{32R^3}{h^2},\ n_5 = \frac{8R^4}{h\rho^2} + \frac{32R^4}{h^3},$$

$$n_6 = \frac{32R^5}{h^2\rho^2} + \frac{96R^5}{h^4},\ n_7 = -\frac{32R^6}{h^3\rho^2},\ n_8 = -\frac{96R^7}{h^4\rho^2},\ n_9 = 2d_\phi R,$$

$$n_{13} = s_{11}d_\phi \frac{h^2}{2} + s_{13}\left(2\rho^3 + 4d_\phi\rho^2 - \frac{\rho h^2}{2}\right) + s_{15}\left(2\rho^3 - \frac{8\rho^5}{h^2} - \frac{8d_\phi\rho^5}{h^2}\right),$$

$$n_{14} = \frac{\nu h}{12E\rho R}\left(E_1\rho h + \frac{G_1\rho h^3}{12}\right),\ n_{15} = \frac{\nu}{2E\rho Re_\phi h^2},\ n_{16} = -2Rd_\theta h,$$

$$n_{17} = 6R^2(R + d_\theta),\ n_{18} = -2R^2h,\ n_{19} = -\frac{8R^4}{h^2}(R + d_\theta),$$

$$n_{20} = 2s_{21}Rd_\theta^{\,2} + 2s_{23}\frac{R^2}{\rho}(R\rho + 2d_\theta\rho - 2d_\theta R - d_\theta^{\,2}) - 2s_{25}\frac{R^4}{\rho},$$

$$n_{21} = 2s_{21}R + 2s_{23}\left(\frac{R^2}{\rho} - \frac{R^3}{\rho^2} - \frac{8R^3}{h^2}\right) - s_{25}\left(\frac{2R^4}{\rho^3} + \frac{16R^4}{h^2\rho} - \frac{16R^5}{h^2\rho^2} - \frac{32R^5}{h^4}\right)$$

$$+ \frac{16R^6}{h^2\rho}s_{27}\left(\frac{1}{\rho^2} + \frac{2}{h^2} - \frac{2R}{h^4\rho}\right) - \frac{32R^8}{h^4\rho^3}s_{29},\ n_{22} = \frac{1}{2} - \frac{h\rho}{3}E_1,$$

$$n_{23} = \frac{1}{2} + \frac{d_\phi}{3\nu e_\phi} - \frac{2\rho E_1 h}{3},\ n_{24} = \frac{\rho^2}{2}\left(1 - \frac{4\rho R}{h^2}\right) - \frac{2\rho^2}{h}d_\phi\left(1 + \frac{R}{h}\right)$$

$$+ \frac{\rho^2}{3\nu e_\phi}\left(\rho + 2d_\phi - 2d_\phi\frac{\rho}{R} - \frac{d_\phi^{\,2}}{R}\right),\ n_{25} = \frac{\rho^4}{h^2}\left(\frac{8\rho R}{h^2} + \frac{8d_\phi R}{h^2} - \frac{2h^2}{6\nu e_\phi R} - 2\right),$$

$$n_{26} = d_\theta R,\ n_{27} = \frac{R^2}{2}\left(1 - \frac{4R^2}{h^2} - \frac{8Rd_\theta}{h^2}\right),\ n_{28} = \frac{8R^4}{h^4}\left(R^2 + d_\theta R - \frac{h^2}{4}\right),$$

$$n_{29} = \frac{h}{e_\theta}\left(\frac{h^2}{6} + 2d_\phi d_\theta\right),\ n_{30} = \frac{2h\rho^3}{3},\ n_{31} = \frac{h\rho R^2}{3}$$

Symbols h_1 and h_2 have the following meanings:

$$h_1 = -2E_1\rho R,\ h_2 = \frac{h}{4}$$

Symbols t_i, x_i, y_i, z_i, n_i, ξ_i, λ_i, Λ_i are defined as follows:

$$t_i = \frac{d_i}{r_i e_i h} - \frac{h}{12\rho Re_i},\ x_i = \frac{\nu}{24E\rho Re_\phi r_i},\ y_i = \frac{\nu}{Er_i e_i h},\ z_i = -2\rho Rr_i(F_1 + G_1 r_i),$$

$$n_i = \frac{6\rho R}{h}r_i(E_1 + F_1 r_i),\ \xi_i = 2H_1\rho Rr_i^{\,3},\ \lambda_i = \frac{R\rho}{h}r_i^{\,3}\left(6G_1 + 6H_1 r_i - \frac{8E_1}{h^2}r_i\right),$$

$$\Lambda_i = -\frac{8\rho R}{h^3}r_i^{\,5}(G_1 + H_1 r_i),\ i = \phi, \theta,\ r_\phi = \rho,\ r_\theta = R$$

Symbol v_i has the following meaning:

$$v_i = E_i + \frac{G_i h^2}{12},\ i = 1, 2$$

Symbols n_{ij}, v_{ij}, x_{ij}, h_{ij}, y_{ij}, z_{ij}, ζ_{ij} have the following definition:

$$n_{ij} = \frac{1}{r_j} - E_i h - \frac{G_i h^3}{12}, \; v_{ij} = G_i + H_i r_j + H_i d_j, \; x_{ij} = E_i + F_i r_j + F_i d_j + G_i d_j r_j$$

$(i = 1, 2;\ j = \phi, \theta, r_\phi = \rho, r_\theta = R)$;

$$h_{ij} = \frac{4}{h^4}(h^2 r_i{}^3 - 16 r_i{}^4 r_j), \; y_{ij} - \frac{4}{h^3} r_i{}^3 r_j, \; z_{ij} = 3r_i\left(\frac{1}{4} - \frac{r_i r_j}{h^2}\right)$$

$$\zeta_{ij} = \frac{r_i}{h}(r_i + r_j) \text{ with } (i, j = \phi, \theta;\ r_\phi = \rho, r_\theta = R)$$

Symbols t_{ij}^k, y_{ij}^k, z_{ij}^k, η_{ij}^k are defined as follows:

$$y_{ij}^k = -s_{11}\frac{r_j d_j h^2}{2r_k} + s_{13}\left(2r_j{}^3 + 2d_j r_j{}^2 + \frac{2d_j r_j{}^3}{r_k} - \frac{r_j{}^2 h^2}{2r_k}\right) + s_{15}\left(\frac{2r_j{}^4}{r_k} - \frac{8r_j{}^5}{h^2} - \frac{8d_j r_j{}^4}{h^2}\right),$$

$$z_{ij}^k = 2E_i d_j r_k, \; \eta_{ij}^k = F_i\frac{h^3}{12} + H_i\frac{h^5}{80} + \frac{c_j}{2e_k} - \frac{d_k c_j}{2r_j e_k} - \frac{1}{2}, \; (i = 1, 2;\ j, k = \phi, \theta)$$

$$t_{ij}^k = 2s_{11}E_k r_j d_j + 2s_{13} r_j{}^2(E_k + F_k r_j + F_k d_j + G_k d_j r_j)$$

$$+ 2_{15} r_j{}^4(G_k + H_k r_j + H_k d_j), \; (i, k = 1, 2;\ j = \phi, \theta)$$