THE CALCULATION OF TIME-DEPENDENT STRESSES IN PCPVs BY AN APPROXIMATE METHOD

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ABSTRACT

The calculation of time-varying stresses in materials subject to non-homogeneous creep is discussed. Economic solutions are sought using a general theory based on a power variational principle. This permits approximate solutions to the time-varying stresses, in real and complex structures, to be obtained with little effort from a minimum of input data. It is shown that coventional finite element stiffness analyses may be used to provide the necessary input data to the creep problem; these take the form of a set of independent stress distributions throughout the structure. Then, using a flexibility approach these stress distributions are optimised to achieve the best possible fit to the true solutions during the time-dependent phase of the problem. Finally, the theory is applied to two problems and solutions are presented for (a) a thin plate subject to in-plane edge loading and non-uniform temperature distribution, (b) a short cylindrical prestressed concrete reactor pressure vessel, with end caps, subjected to internal pressure and temperature crossfall through the walls and caps.

INTRODUCTION

In the design of engineering structure it is important to have a good knowledge of the working stresses if they are to be kept within permissible limits. No intrinsic difficulties exist if solutions for elastic structures are sought and numerous methods of analysis are currently available which enable even quite complex structures to be analysed in the elastic condition. When the basic structural material is subject to elastic and creep behaviour other methods of stress analysis may need to be employed. When the creep properties of the material are non-homogeneous throughout the structure stresses will undergo redistribution with time and it is then important to be able to calculate not only the initial state of stress at the time of loading but also the time-dependent variations of stress throughout the operating life of the structure.

In this paper a method is described by which the time-dependent stresses in structures subject to sustained loads may be calculated with no more difficulty than is experienced with the initial calculation for the elastic state of stress. A variational principle (1) is used in the form of an optimization procedure to generate an approximate solution for the stresses.

Strains and displacements may then be computed from subsidiary calculations.

In the problems treated the material behaviour relates to that of concrete and the non-homogeneous creep properties are attributed to the non-uniform state of temperature throughout the material. It has been shown elsewhere (2) that the creep behaviour of concrete may, with suitable change of the true variable time, be represented either by a Maxwell body when creep recovery is ignored or by a Burger's body when creep recovery is included. In the former representation the stresses are obtained from the solution of a set of simultaneous first order differential equations and in the latter case from the solution of a set of second order differential equations.

THEORY

For a body undergoing creep deformation which is in equilibrium with its boundary loads and support reactions and which is everywhere compatible, it follows the strain rates $\dot{\varepsilon}$ must be compatible also. Thus, by invoking the principle of Virtual Work during a short time interval Δt, it may be stated that the products of the incremental strains $\dot{\varepsilon}\Delta t$ with a set of stresses in equilibrium $\delta\sigma$, will be zero when integrated over the entire volume of the body*. Thus, the following general statement may be made,

$$\int \dot{\varepsilon}\Delta t \delta\sigma dV = 0 \quad . \quad . \quad . \quad . \qquad (1)$$

or since this is true for all Δt and $\dot{\varepsilon}$, it follows that,

$$\int \dot{\varepsilon}\delta\sigma dV = 0 \quad . \quad . \quad . \quad . \qquad (2)$$

Equation (2) represents a variational principle for bodies resting on rigid supports and may be used to advantage when the internal stresses in a body are specified by a set of different distributions which when proportioned correctly are capable of respresenting the true state of stress closely. It is convenient to choose most of the distributions as self-equilibrating, i.e. the stresses satisfy equilibrium without external loads, and to then satisfy the statical boundary conditions of the problem by one or more distributions of stress.

In what follows it will be assumed that only one distribution of stress will be chosen to satisfy the boundary loading and this is defined as σ_0. The remaining self-equilibrating distributions are, $\sigma_1, \sigma_2, \ldots \sigma_n$. The state of stress, σ, at any time, may then be represented as,

* It is assumed that if there are displacement rates associated with the supports in a particular problem, these may be included in the strain rate term simply by surrounding the given problem by a rigid foundation and thus forcing the support forces and displacement rates to become internal effects in the modified problem.

$$\sigma = \sigma_0 + a_1\sigma_1 + a_2\sigma_2 + \ldots + a_n\sigma_n \qquad . \qquad . \qquad (3)$$

where the weighting of the self-equilibrating distributions is given by the multipliers $a_1, a_2, \ldots a_n$. These quantities are thus functions of time and represent the redundancies in the problem. The stresses, $\sigma_0, \sigma_1, \ldots\sigma_n$, will in general be spatial distributions in three dimensions.

For the continuum, which is infinitely redundant, the stresses calculated from equation (3) will represent only an approximation to the exact values. This is because only a finite number of redundancies, n, have been considered in the solution. For the behaviour of a structure, however, which may be specified completely by a finite number of redundancies, equation (3) will lead to an exact solution.

In order to establish that equation (2) represents the variation of a definable function it is necessary to specify the constitutive laws for the behaviour of the material of the body or structure. It will also be necessary to examine the behaviour of the functional in respect to deviations of the state of stress from the true state and to establish that equilibrium corresponds to a minimum of the functional. These topics are discussed in the next section.

Constitutive laws relating to the Maxwell material are considered in the theory which follows and are utilised by analogy in the worked examples of section which relate to concrete as the material.

In tensorial notation, the material behaviour in three dimensions is represented thus,

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ijE} + \dot{\varepsilon}_{ijV}$$

where $\dot{\varepsilon}_{ijE}$ and $\dot{\varepsilon}_{ijV}$ refer to the strain rates due to the elastic and viscous phases of the Maxwell material.

$$\dot{\varepsilon}_{ijE} = \frac{\dot{\sigma}_{\alpha\alpha}}{9K}\delta_{ij} + \frac{1}{2G}\left[\dot{\sigma}_{ij} - \frac{\dot{\sigma}_{\alpha\alpha}}{3}\delta_{ij}\right] \quad : \quad \dot{\varepsilon}_{ijV} = \frac{\sigma_{\alpha\alpha}}{9K'}\delta_{ij} + \frac{1}{2G'}\left[\sigma_{ij} - \frac{\sigma_{\alpha\alpha}}{3}\delta_{ij}\right] \qquad (4)$$

Here, K and G are the bulk and shear moduli of the elastic phase respectively, and K' and G' are their viscous counterparts.

Rewriting equation (3) in consistent notation with equation (4) it becomes,

$$\sigma_{ij} = \sigma_{o,ij} + a_1\sigma_{1,ij} + a_2\sigma_{2,ij} + \ldots + a_n\sigma_{n,ij} \qquad (5)$$

With the additional knowledge that,

$$\dot{\sigma}_{ij} = \sum_{r=1}^{r=n} \dot{a}_r\sigma_{r,ij} \qquad . \qquad . \qquad (6)$$

$$\text{and} \quad \delta\sigma_{ij} = \frac{\partial\sigma_{ij}}{\partial a_r} = \sigma_{r,ij} \ (\text{for } r = 1, 2, \ldots n) \qquad . \qquad . \qquad (7)$$

equation (2) may be rewritten as a set of 'n' equations after substition of $\dot{\varepsilon}_{ij}$, σ_{ij}, $\dot{\sigma}_{ij}$ and $\delta\sigma_{ij}$ from equations (4) to (7). The 'n' resulting equations may then be written in matrix form thus,

$$[A][\dot{a}] + [B][a] + [C] = 0 \qquad . \qquad . \qquad . \qquad (8)$$

where [A] and [B] are square matrices of order n X n and [c] is a vector of n elements. Typical terms in these matrices are,

$$A_{rs} = \int\left[\left(\frac{1}{9K} - \frac{1}{6G}\right)\sigma_{r,\alpha\alpha}\sigma_{s,\alpha\alpha} + \frac{1}{2G}\sigma_{r,ij}\sigma_{s,ij}\right]dV$$

$$B_{rs} = \int\left[\left(\frac{1}{9K'} - \frac{1}{6G'}\right)\sigma_{r,\alpha\alpha}\sigma_{s,\alpha\alpha} + \frac{1}{2G'}\sigma_{r,ij}\sigma_{s,ij}\right]dV$$

$$C_{r} = \int\left[\left(\frac{1}{9K'} - \frac{1}{6G'}\right)\sigma_{o,\alpha\alpha}\sigma_{r,\alpha\alpha} + \frac{1}{2G'}\sigma_{o,ij}\sigma_{r,ij}\right]dV$$

The set of arbitrary constants which emerge during the solution of equations (8) are obtained from the elastic solution at time t= 0. When $\sigma_{o,ij}$ defines the true elastic solution it follows that [a] = 0, when t = 0. The solution for the time functions is then,

$$[a] = \left[e^{-[A]^{-1}[B]t} - I\right][B]^{-1}[C]$$

The variation of the stresses with time is then obtained from equation (5).

THE FUNCTIONAL

The functional of which equation (2) represents the first variation with respect to the equilibrium state of stress is defined as,

$$\dot{F} = \iint \dot{\varepsilon}_{ij} d\sigma_{ij} dV \qquad . \qquad . \qquad . \qquad . \qquad (9)$$

It is shown easily that the right hand side of equation (9) represents the sum of two quantities which have physical interpretation. When the rate of dissipation of energy due to creep is defined as $\dot{D}$ and the complementary energy of the body (here, of the same value as the strain energy because of the linear stress-strain relationship) is defined as U, the functional, $\dot{F}$, has the form,

$$\dot{F} = \tfrac{1}{2}\dot{D} + \dot{U} \qquad . \qquad . \qquad . \qquad . \qquad (10)$$

$$\dot{D} = \iint \dot{\varepsilon}_{ijV} d\sigma_{ij} dV = \int\left[\frac{\sigma_{\alpha\alpha}{}^2}{9K'}\delta_{ij} + \frac{1}{2G'}\left(\sigma_{ij}{}^2 - \frac{\sigma_{\alpha\alpha}{}^2}{3}\delta_{ij}\right)\right]dV \qquad . \qquad . \qquad (11)$$

$$\dot{U} = \iint \dot{\varepsilon}_{ijE} d\sigma_{ij} dV = \int\left[\frac{\dot{\sigma}_{\alpha\alpha}\sigma_{\alpha\alpha}}{9K}\delta_{ij} + \frac{1}{2G}\left(\dot{\sigma}_{ij}\sigma_{ij} - \frac{\dot{\sigma}_{\alpha\alpha}\sigma_{\alpha\alpha}}{3}\delta_{ij}\right)\right]dV \qquad . \qquad . \qquad (12)$$

An alternative representation of $\dot{F}$ is obtained from the knowledge that the rate of change of strain energy plus the rate at which energy is dissipated from the body due to creep, is equal to the total rate of working of the external loads on the surface of the body, $\dot{W}$.

i.e. $$\dot{W} = \dot{D} + \dot{U} \qquad . \qquad . \qquad . \qquad . \qquad (13)$$

Hence, the functional has also the following form,

$$\dot{F} = \dot{W} - \tfrac{1}{2}\dot{D} \qquad . \qquad . \qquad . \qquad . \qquad (14)$$

Equation (2) states that the functional, $\mathfrak{F}$, has a stationary value with respect to variations of stress from the true state of equilibrium provided that the new state of stress also satisfies equilibrium. The use of this equation thus enables the time functions a_r, of equations (5) and (6) to be evaluated as described previously. With the value of a_r known the functional can be evaluated.

The value of the functional is now to be examined as the state of stress is allowed to vary from the true state which satisfies both equilibrium and the continuity of the material of the body or structure. Let it be assumed that the true state of stress for a body in equilibrium is described by a finite number of redundancies, n, of the form, a_1, a_2 . . a_n. It is then necessary to examine the value of the functional when any or all of these redundancies are allowed to vary. If the true state of stress and equilibrium is characterised by the values of a_r, r = 1 to n, calculated from the use of equation (2) it is necessary to show that these values of a_r give rise to a minimum value of $\mathfrak{F}$.

For simplicity, the arguments which follow are presented for the uniaxial state of stress only. For this case,

$$\mathfrak{F} = \dot{U} + \tfrac{1}{2}\dot{D}$$

and

$$\dot{U} = \int \frac{\sigma\dot{\sigma}}{E} dV$$

$$\dot{D} = \int \frac{\sigma^2}{\eta} dV$$

where E and η refer to the elastic modulus and viscosity of the material.

Thus

$$\mathfrak{F} = \int \left[\frac{\sigma\dot{\sigma}}{E} + \frac{\sigma^2}{2\eta}\right] dV \quad . \quad . \quad . \quad . \qquad (15)$$

Also

$$\sigma = \sigma_0 + \sum_{r=1}^{r=n} a_r \sigma_r \quad . \quad . \quad . \quad . \qquad (16)$$

If now the pth stress distribution of equation (16) undergoes a change to $(1 + e)a_p \sigma_p$ then,

$$\mathfrak{F} + \Delta\mathfrak{F} = \int \left(\sigma_0 + \sum_{r=1}^{r=n} a_r\sigma_r + ea_p\sigma_p\right)\left[\frac{\dot{\sigma}}{E} + \left(\sigma_0 + \sum_{r=1}^{r=n} a_r\sigma_r + ea_p\sigma_p\right)\frac{1}{2\eta}\right] dV$$

$$= \int \left(\sigma_0 + \sum_{r=1}^{r=n} a_r\sigma_r\right)\left[\frac{\dot{\sigma}}{E} + \left(\sigma_0 + \sum_{r=1}^{r=n} a_r\sigma_r\right)\frac{1}{2\eta}\right] dV + \int ea_p\sigma_p\left[\frac{\dot{\sigma}}{E} + \left(\sigma_0 + \sum_{r=1}^{r=n} a_r\sigma_r\right)\frac{1}{2\eta}\right] dV$$

$$+ \int ea_p\sigma_p\left(\frac{ea_p\sigma_p}{2\eta}\right) dV + \int \frac{ea_p\sigma_p}{2\eta}\left(\sigma_0 + \sum_{r=1}^{r=n} a_r\sigma_r\right) dV \quad . \quad . \quad . \qquad (17)$$

The first term on the right hand side of equation (17) is the functional, $\mathfrak{F}$, the second and fourth terms when added together give rise to the integral $\int \dot{\varepsilon}\sigma_p dV$ which is zero by consequence of equation (2).

It therefore follows that,

$$\Delta \mathfrak{F} = \int \frac{(ea_p \sigma_p)^2}{2\eta} dV \quad . \quad . \quad . \quad (18)$$

i.e. $\Delta \mathfrak{F} > 0$ for all e, since $\eta > 0$ always.

Since 'p' may take any or all of the values in the range, 1 to n, it follows that $\mathfrak{F}$ is a minimum when the structure is in a true state of equilibrium. Also, since $a_p = a_p(t)$, it follows that $\mathfrak{F}$ has a minimum value, with respect to variations of the stresses, while the time-varying stresses of the body conform continuously to true states of equilibrium.

The arguments outlined may be extended readily into three dimensions and for any number of redundancies, n. For the continuum solution, a logical extension to the case of $n = \infty$ is required. In practice this is not possible and an approximate solution is obtained by adopting a finite value for 'n'. In this case the true minimum of the functional is never achieved but some knowledge of the accuracy of the solution is obtained by observing the reduction of $\mathfrak{F}$ with increasing 'n'. $\mathfrak{F}$ varies with time and in the long term becomes equal to one half of the steady state dissipation rate, $\dot{D}_{ss}$, and although it is readily shown that $\dot{D}_{ss}$ is the minimum dissipation rate at any time, there appears to be little significance associated with the variation of $\mathfrak{F}$ with t.

Figure 1 illustrates the variation in the value of the functional for a problem containing two stress parameters only, a_1 and a_2. The functional is plotted at t = 0, i.e. a time corresponding to the elastic solution, and at two other times during the creep phase of the problem.

In the next section the variational principle is used to determine the time-varying stresses in a flat plate subjected to sustained in-plane edge loading and non-uniform temperature distribution. Two methods of solution are used and the results are compared to a step-by-step creep calcuation.

NUMERICAL EXAMPLES

Of the three solutions compared two are generated from the variational principle of equation (2) and of these the first uses stress functions and is termed, analytical, while the second relies on the existance of distributions of self-equilibrating stresses throughout the plate, and is achieved by using finite element techniques. The third solution is form a numerical step-by-step analysis.

Details of the problem solved are shown in figure 2.

Analytical solution

When stress functions are used in conjunction with the variational principle method of solution, equation (3) is replaced by a similar expression for the stress functions, thus,

$$\phi = \phi_0 + a_1\phi_1 + \ldots + a_n\phi_n \quad . \quad . \quad (19)$$

where ϕ_o is a stress function which satisfies the boundary conditions of the plate element and ϕ_1, ϕ_2, . . . ϕ_n, are stress functions which allow freedom of stress variation within the plate only, i.e. they are consistent with zero loads on the boundary. The following expressions for ϕ was chosen for the numerical calculations,

$$\phi = \phi_o + (x^2 - a^2)^2(y^2 - b^2)^2(a_1 + a_2x^2 + a_3y^2 + a_4x^4 + a_5y^3 + a_6x^6 + a_7y^4 + \ . \ . \ .) \quad . \ (20)$$

or,

$$\phi = \phi_0 + (x^2 - a^2)^2 (y^2 - b^2)^2 \left[a_1 + \sum_{n=1}^{n=N}\left(a_{2n}x^{2n} + a_{2n+1}y^{n+1}\right)\right] \quad . \quad (21)$$

where N is a number to be chosen by the investigator.

In this example ϕ_o satisfies the boundary loading on the plate but does not correspond to the elastic solution. The values of a_1, a_2, . . . were therefore non-zero at time t = 0 and were thus influential in both the elastic and creep solutions. An initial check on the solution was possible for the elastic solutions which refer to aspect ratios of $\frac{a}{b} = 1$ and 2(3). Timoshenko's solutions are tabulated along with calculated values in Table 1.

Then, adopting an aspect ratio of $\frac{a}{b} = 3$, a complete creep solution was obtained using ϕ of equation (21), with N = 4. Various problems and values of N were investigated and N = 4 was found to be adequate in most cases. The results of this analysis are shown in figures 3 to 6 for various values of the creep parameter, c. This parameter is the analogous quantity to time in the Maxwell equation and represents a normalised creep property of concrete with respect to stress and temperature. Experiments have indicated that the strain behaviour of concrete may be approximated to equation (4) when the derivatives, $\dot{\epsilon}$ and $\dot{\sigma}$, relate to differentiations with respect to 'c', and when K' and G' take the following values.

$$K' = \frac{1}{3\phi(T)(1 - 2\nu_c)}$$

$$G' = \frac{1}{2\phi(T)(1 + \nu_c)} \quad . \quad . \quad . \quad (22)$$

ν_c and $\phi(T)$ are respectively the creep Poisson's ratio and the temperature function which results from the normalisation of the creep data.

Thus, after substitution of K' and G' into equation (4), the total strain rates for the two-dimensional plate problem become,

$$\dot{\epsilon}_x = \frac{1}{E}\left(\dot{\sigma}_x - \nu\dot{\sigma}_y\right) + \phi(T)\left(\sigma_x - \nu_c\sigma_y\right)$$
$$\dot{\epsilon}_y = \frac{1}{E}\left(\dot{\sigma}_y - \nu\dot{\sigma}_x\right) + \phi(T)\left(\sigma_y - \nu_c\sigma_x\right)$$
$$\dot{\epsilon}_{xy} = \frac{2(1 + \nu)}{E}\dot{\sigma}_{xy} + 2\phi(T)(1 + \nu_c)\sigma_{xy} \quad . \quad . \quad (23)$$

From equation (21),

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} \ ; \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2} \ ; \text{ and } \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \ . \quad (24)$$

Also,

$$(\delta\sigma_x)_p = \frac{\partial}{\partial a_p}\left[\frac{\partial^2\phi}{\partial y^2}\right] ; \qquad (\delta\sigma_y)_p = \frac{\partial}{\partial a_p}\left[\frac{\partial^2\phi}{\partial x^2}\right]; \qquad (\delta\sigma_{xy})_p = - \frac{\partial}{\partial a_p}\left[\frac{\partial^2\phi}{\partial x \partial y}\right] \quad (25)$$

where p ranges from 1 to N

The variational principle equation is then, for the pth stress variation.

$$\int \dot{\varepsilon}_{ij}\delta\sigma_{ij}dV = \int\left[\dot{\varepsilon}_x(\delta\sigma_x)_p + \dot{\varepsilon}_y(\delta\sigma_y)_p + \dot{\varepsilon}_{xy}(\delta\sigma_{xy})_p\right]dV = 0 \quad (26)$$

After inclusion of the strain rates from equation (23) and the simplifying assumption that the elastic and creep Poisson's ratios may be set equal, and to ν , equation (26) becomes,

$$0 = \int\left(\frac{D}{E} + \phi(T)\right)\left[(\sigma_x - \nu\sigma_y)(\delta\sigma_x)_p - (\sigma_y - \nu\sigma_x)(\delta\sigma_y)_p + 2(1 + \nu)\sigma_{xy}(\delta\sigma_{xy})_p\right]dV \quad (27)$$

In this equation the differential operator, $D = \frac{d(\;)}{dc}$

Substitution of σ, $\dot{\sigma}$, $\delta\sigma$ in equation (27) leads to a first order matrix differential equation of the type of equation (8). The results from the solution of this equation are shown in figures 3 to 6 , and are compared with values from two other forms of analysis, in figures 7 to 11.

Finite element solution

For the first comparison a finite element analysis was performed and this used a mesh of 25X9 rectangular elements. A conventional stiffness analysis was carried out to determine the intial elastic and thermo-elastic stresses at time t = 0.

A further stiffness analysis revealed the long term steady-state stress distribution for the creep problem. This calculation is of a pseudo-elastic nature and may be carried out in isolation from the transient creep analysis (4).

The results of figures 7 to 10 refer to the inclusion of only one self-equilibrating stress distribution and this was derived from the difference between the intial stresses due to mechanical loads and temperature, σ_o, and the steady-state stresses, σ_s. As a result of this single self-equilibrating distribution, only one time function, a_1, was sought in the subsequent creep analysis. The time-varying stresses thus have the representation,

$$\sigma_{ij} = \sigma_{o,ij} + a_1(\sigma_{s,ij} - \sigma_{o,ij})$$

Although in the solution of the plate problem it was not considered necessary to introduce more distributions of stress, in other problems one distribution only may be insufficient. This topic is discussed in detail elsewhere (5). When more self-equilibrating distributions are required they may be obtained by repeated use of the stiffness analysis. Suitable distributions will result from the difference of any two different non-homogeneous elastic solutions for the same boundary loads or more simply from a single calculation of the thermal stress type. A solution from the latter type of calculation will automatically be self-equilibrating, and in many cases may be obtained without further inversion of the stiffness matrix if this has been stored previously. In the former type of solution, however, a new stiffness matrix and inversion is required for each calculation.

Step-by-step solution

Although this method of solution has been described elsewhere (6) a brief description is given here.

Firstly, a finite element solution for the elastic problem is obtained at t = 0, then with knowledge of the creep laws of the material the free creep strains which would occur during a short interval of time, Δt, are computed on the assumption that the stresses in the elements remain constant and that compatibility is not enforced. At the end of the interval compatibility is enforced and the resulting elastic deformation and associated stresses are added to those corresponding to the start of the interval. In this way the state of stress and deformation at the end of the interval is obtained, and this is used to define the initial state for the next time interval. The calculation thus proceeds in a step-by-step manner with the information of any step being dependent upon the knowledge of the stresses and strains of previous time intervals. By necessity the time intervals must be kept small and in consequence many steps in the calculation are then required.

Figures 11a to 11c show the results from a calculation of this type which was generated from a finite element mesh of 10 X 15 triangular elements. The comparison of stresses is made for a large value of the creep parameter, c, and thus approximates to the steady-state stress condition. It serves to check the accuracy of the step-by-step calculation because the steady-state stress solution of the variational principle approach is deduced from an essentially 'elastic' calculation. With the knowledge that the step-by-step solution is satisfactory it may then be used as the basic for checking against other solutions during the transient creep period of the analysis. Other comparisons made during this period indicate similar agreement with the finite element variational principle solution, while the analytical solutions show some deviation from both the finite element methods, in regions close to the edge of the plate. It is believed that the analytical solution could be improved by selecting additional stress functions which allowed more freedom of stress variation in these regions.

ADDITIONAL PROBLEM

Although the plate problem of the previous section is of a somewhat academic nature when concrete properties are assigned to the material, the example has demonstrated the use of the variational principle and its ability to generate approximate stress solutions of almost any desired degree of accuracy, by the selection of suitable stress distributions, σ_{rij}, of equation (5), or stress functions from equation (19).

A more realistic problem in civil engineering terms is that depicted by figure 12. This figure shows, in idealised form, a short cylindrical prestressed concrete pressure vessel of the type used in nuclear reactor generating stations. This structure has been analysed for creep by the combined use of finite elements and the variational principle. It has been assumed that the external prestress corresponds to a radially applied stress of 1000 lb/in^2, the effect which would be produced by wire winding techniques. The longitudinal stress in the walls of the cylinder corresponds to the effect of a uniform pressure of 1000 lb/in^2 applied over the top cap, and the internal pressure of the vessel is 500 lb/in^2. The temperature crossfall through the wall is assumed to be that relating to steady-state heat flow at the mid plane, and in the cap it has been assumed that the crossfall is linear and varies from 80°C at the inner face to 30°C at the outer face of the vessel. No liner is included in the analysis.

Although in the analysis of a real structure a subsidiary calculation would be required to yield the actual temperature distribution and thus provide the correct data for the creep analysis, the simplified data used in the analysis quoted here in no way alters the simplcity with which a solution is obtained. Figure 13 shows the circumferential stress contours for the prestressed cylinder at three times; initially, for the elastic solution (inclusive of temperature effects), a transient solution corresponding to approximately 2 to 3 months after heating, and the long term steady-state solution. The effect of the number of self-equilibrating distributions of stress incorporated in the analysis may be seen in part b of the figure.

The significant effect of a temperature shut-down may be assessed by the superposition of the stresses caused by the temperature crossfall alone on to the steady-state distribution. This operation reveals that cracking is likely to be experienced on the inside face if the temperature crossfall is removed completely late in the life of a pressure vessel, after creep has caused major stress redistribution under operating conditions.

DISCUSSION

The numerical examples have illustrated the effectiveness of the variational principle approach to the solution of stresses in structures subjected to sustained loads, when the material behaviour relates to that of the Maxwell material. When it is desired to include creep recovery in the analysis, more sophisticated material representation is required.

The arguments leading to the statement of equation (2) may be extended to the following statement also,

$$\int \ddot{\varepsilon}\,\delta\sigma\,dV = 0 \qquad . \qquad . \qquad . \qquad . \qquad (28)$$

An interesting feature emerges when equations (2) and (28) are applied to the analysis of a structure in which the material behaviour is represented by a Burger's body, figure 14. For this body the constitutive equation is of the form,

$$p_1\dot{\varepsilon} + p_2\ddot{\varepsilon} = q_0\sigma + q_1\dot{\sigma} + q_2\ddot{\sigma} \qquad . \qquad . \qquad (29)$$

where the quantities, p and q, contain the elastic and viscous parameters of the individual elements of the model.

To achieve a zero on the left hand side of equation (29) requires the statement that,

$$\int (p_1\dot{\varepsilon} + p_2\ddot{\varepsilon})\delta\sigma dV = 0 \qquad . \qquad . \qquad . \qquad (30)$$

In utilizing the separate equations, (2) and (28), if the statement of equation (31) is true, then it follows that equation (30) represents a true statement.

$$\int p_1\dot{\varepsilon}\delta\sigma dV + \int p_2\ddot{\varepsilon}\delta\sigma dV = 0 \qquad . \qquad . \qquad . \qquad (31)$$

For this statement to be true, however, it is necessary in consequence of equations (2) and (28) that the two terms may be written also in the following form,

$$\int \dot{\varepsilon}\delta\sigma dV + \frac{p_2}{p_1}\int \ddot{\varepsilon}\delta\sigma dV = 0 \qquad . \qquad . \qquad . \qquad (32)$$

Equation (32) implies that either the value of the ratio $\frac{p_2}{p_1}$ or the individual values of both, p_1 and p_2, must be constant throughout the volume of the structure. In other words a restiction is placed on the spatial variation of the material properties associated with p_1 and p_2. These properties relate to the recovery strains only, in the Burger's model representation.

It is thus possible to utilize the statements of equations (2) and (28) simultaneously in the analysis of structures for which the material may be considered as non-homogeneous with respect to elastic strains and irreversible creep but homogeneous with respect to reversible creep strains. Solutions to problems with material behaviour defined in this way are presented in the work of reference (5).

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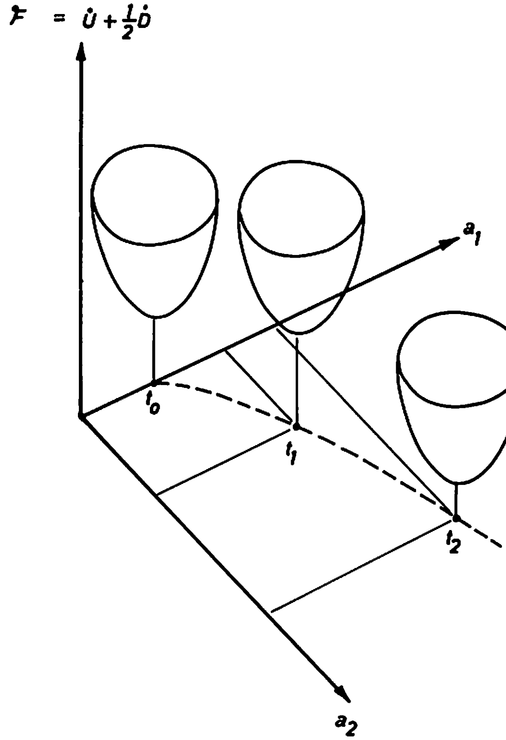
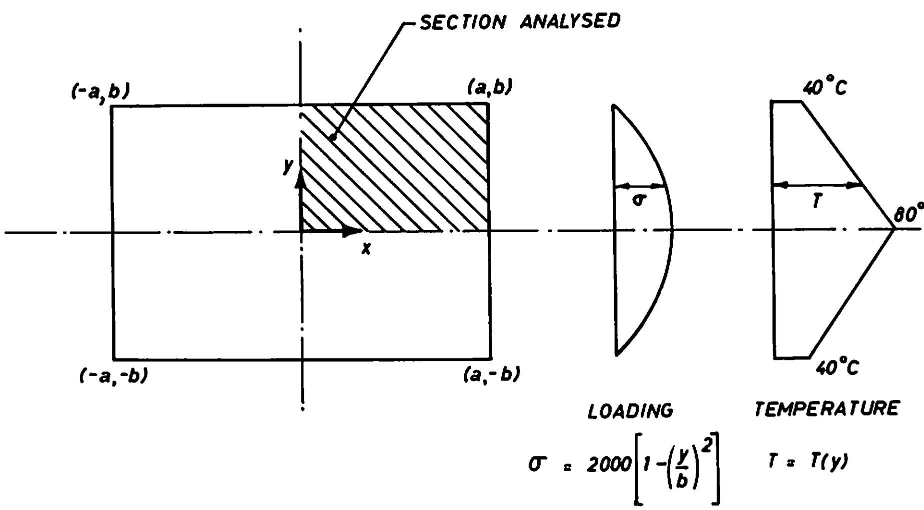
6. England, G.L. "Numerical Creep Analyses applied to Concrete Structures". Proc. of American Concrete Institute, Vol. 64, pp 301-311, 1967.

TABLE 1

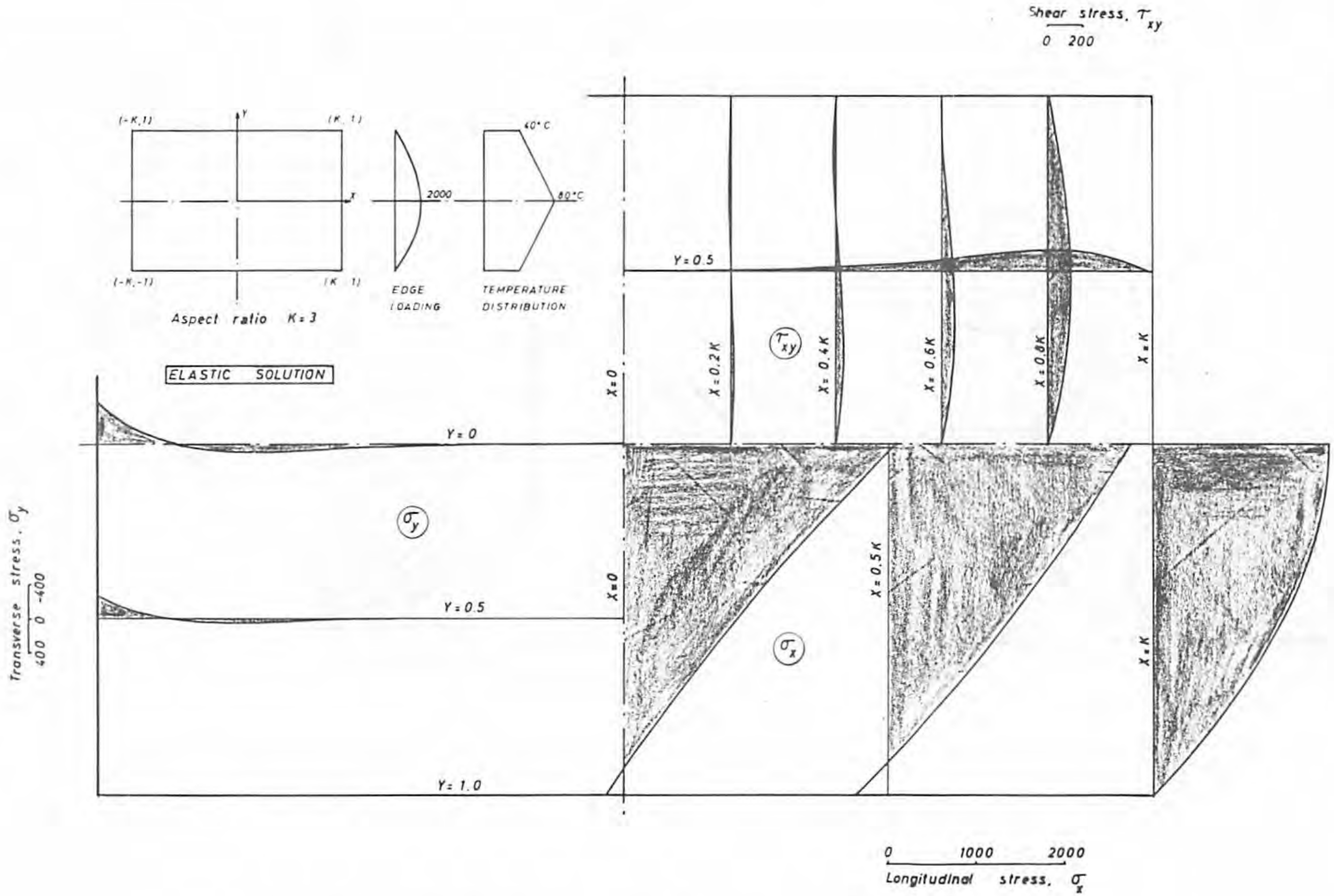
Plate aspect ratio	No. of Coeffs	Timoshenko[3]	Computed [5]
a/b = 1	1	$a_1 = 0.04253$	$a_1 = 0.04253$
a/b = 1	3	$a_1 = 0.04040$ $a_2 = 0.01174$ $a_3 = 0.01174$	$a_1 = 0.04040$ $a_2 = 0.01172$ $a_3 = 0.01172$
a/b = 2	3	$a_1 = 0.07983$ $a_2 = 0.1250$ $a_3 = 0.01826$	$a_1 = 0.07982$ $a_2 = 0.1250$ $a_3 = 0.01846$



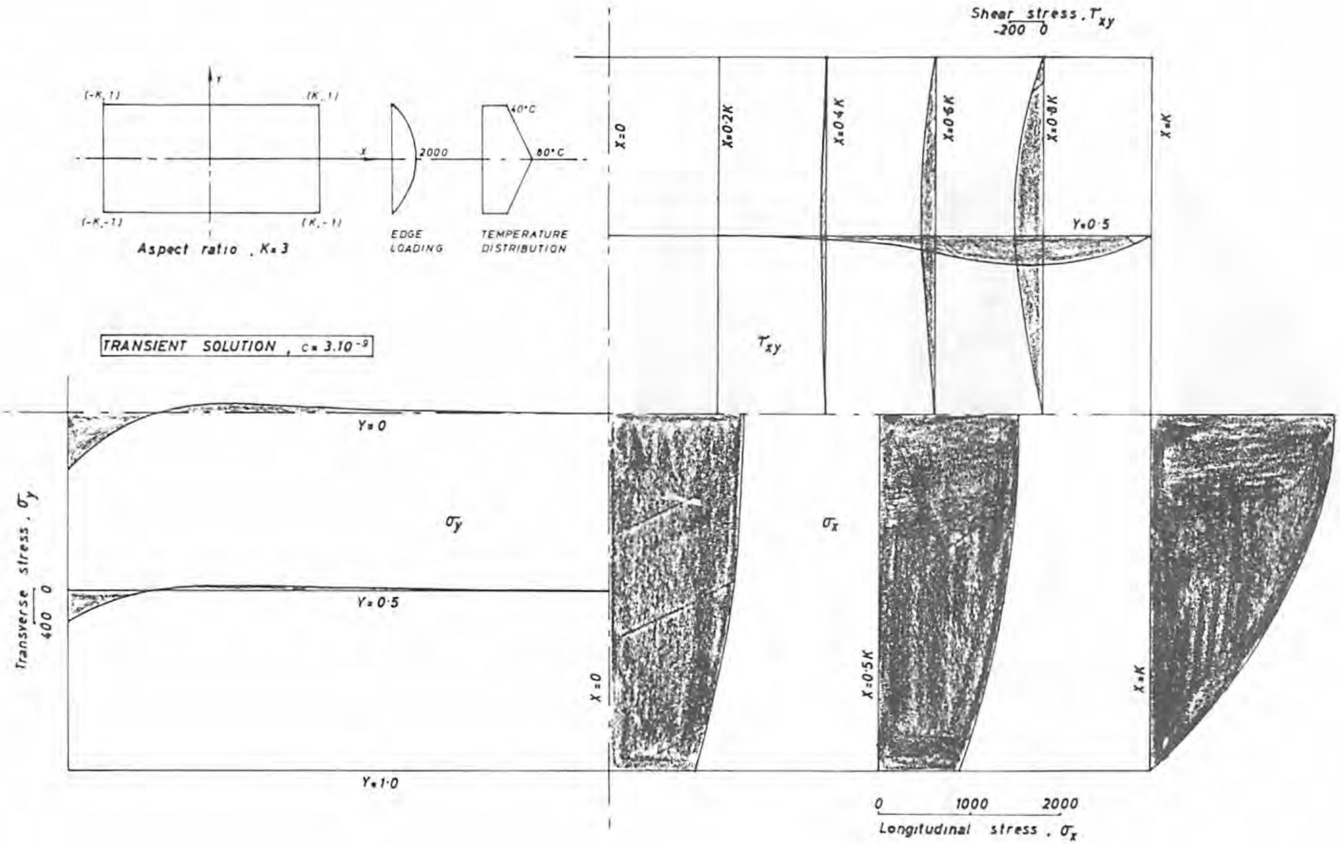
1. Variation of functional, $\mathcal{F}$, for deviations of stress from the true state of equilibrium at times, $t = t_o$, (elastic solution), and $t = t_1$ and t_2, (creep solutions).



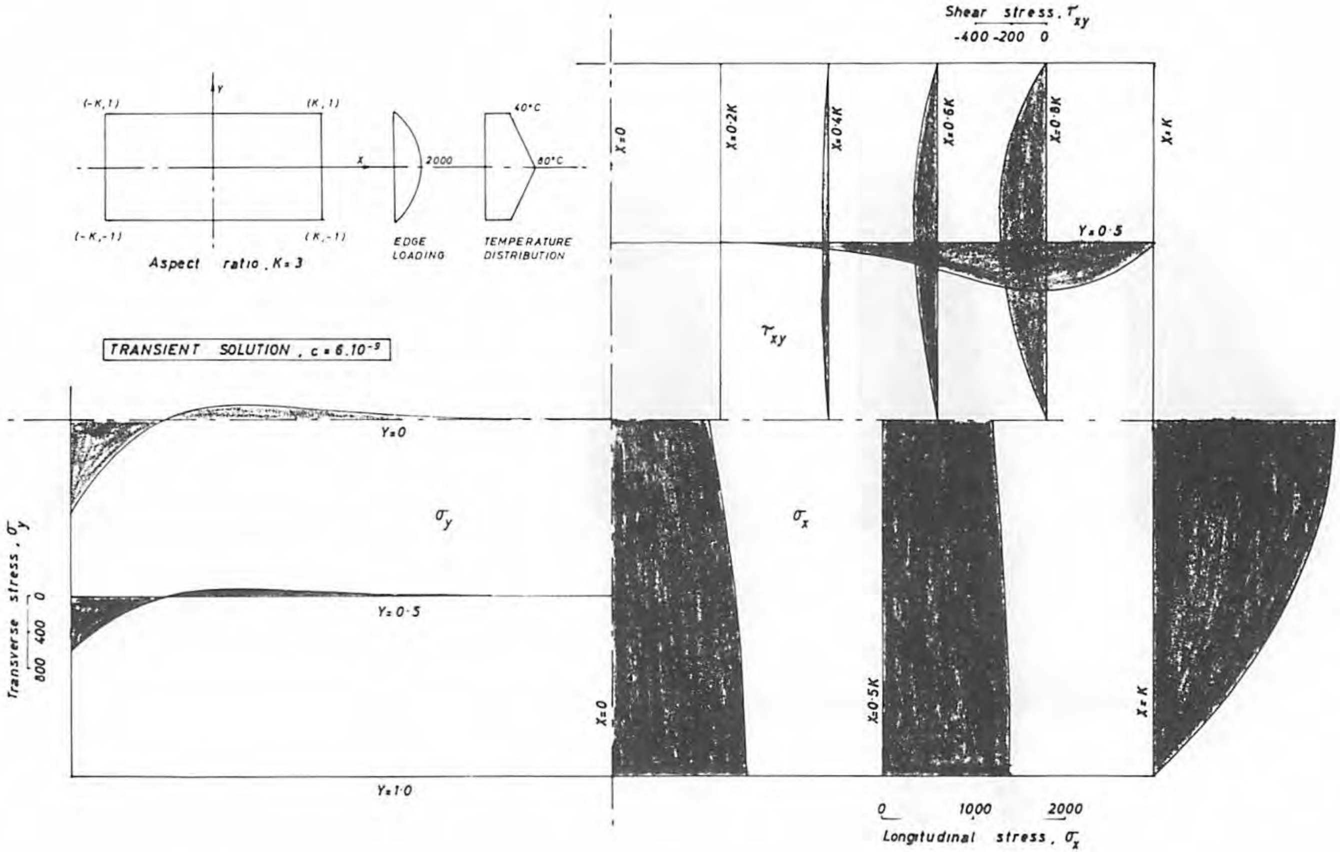
2. Loading and temperature details for flat plate analysis.



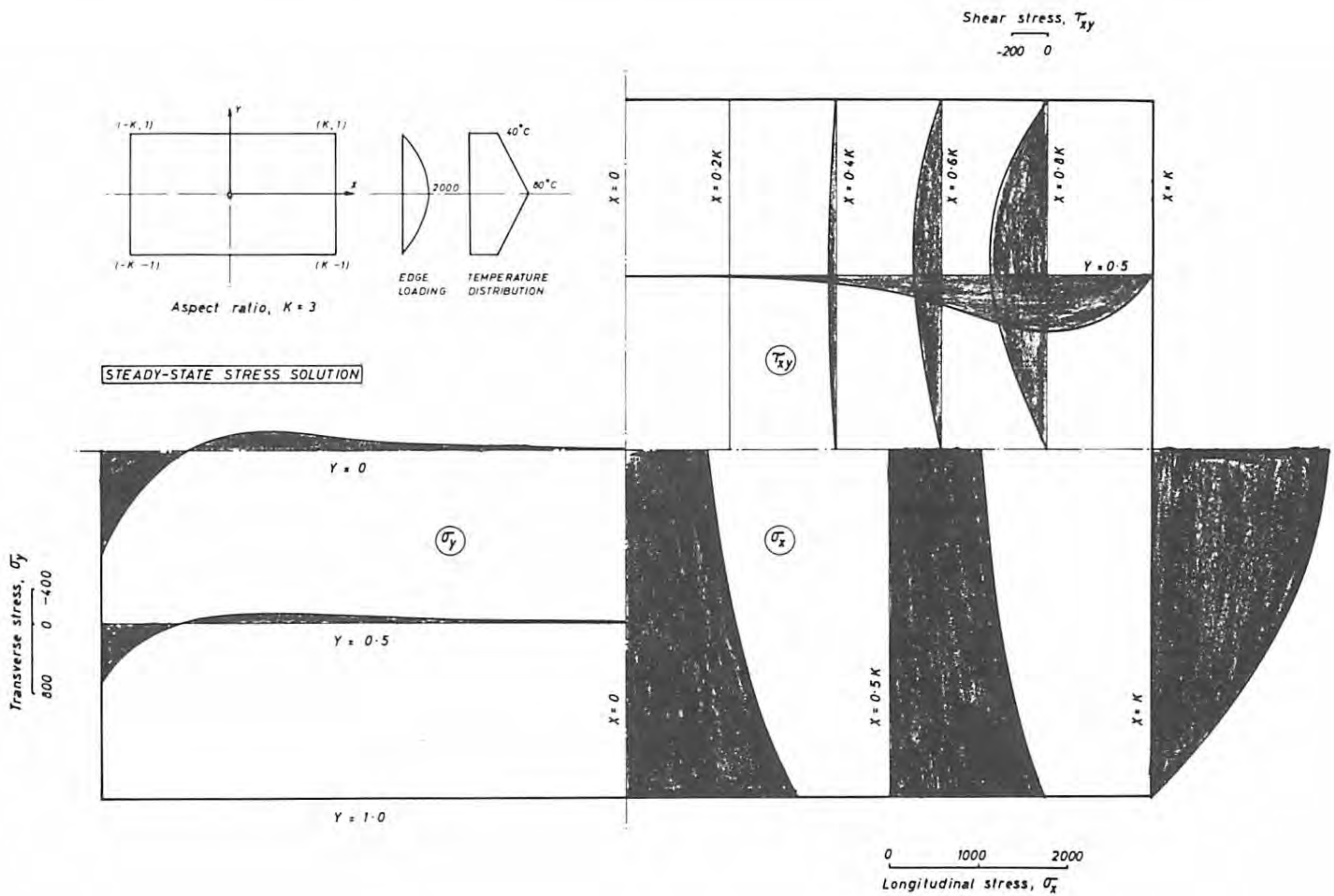
3. Analytical solutions for the stresses in flat plate, thermo-elastic stresses at t = 0.



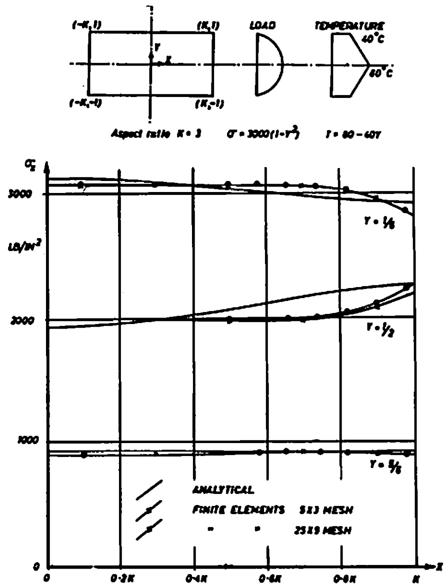
4. Analytical solutions for the stresses in flat plate, transient creep solution for creep parameter, c = 3.10^{-9}.



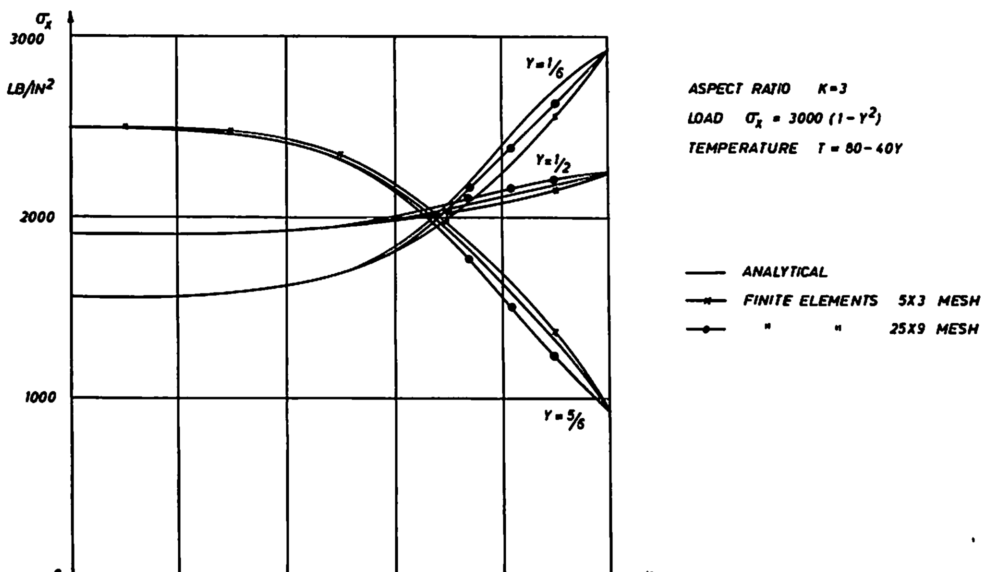
5. Analytical solutions for the stresses in flat plate, transient creep solution for creep parameter, $c = 6.10^{-9}$.



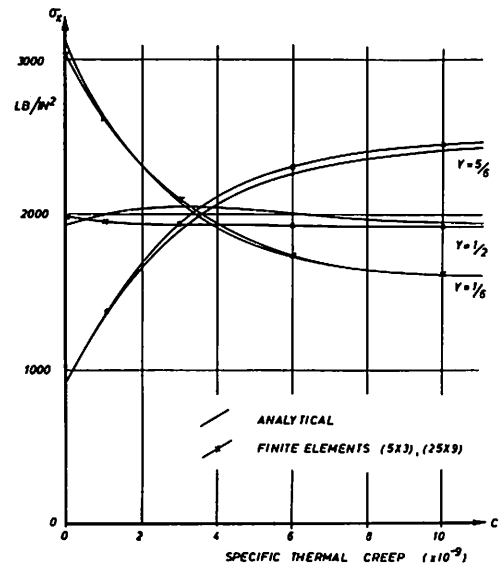
6. Analytical solutions for the stresses in flat plate, long-term steady-state stresses.



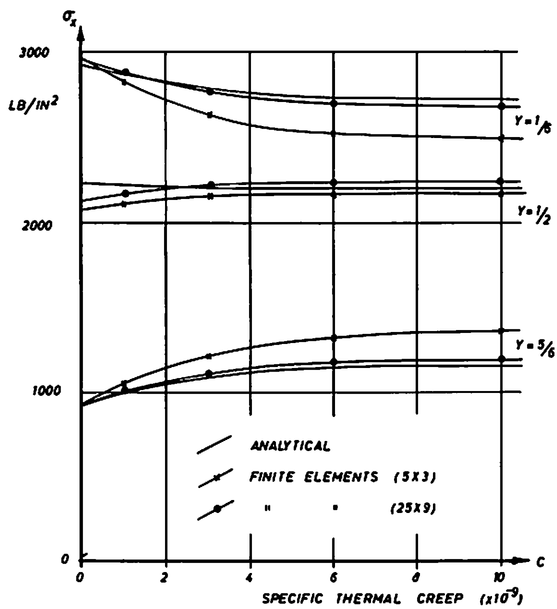
7. Comparison of analytical and finite elements elastic solution for flat plate. Variation of longitudinal stress, σ_x, along plate.



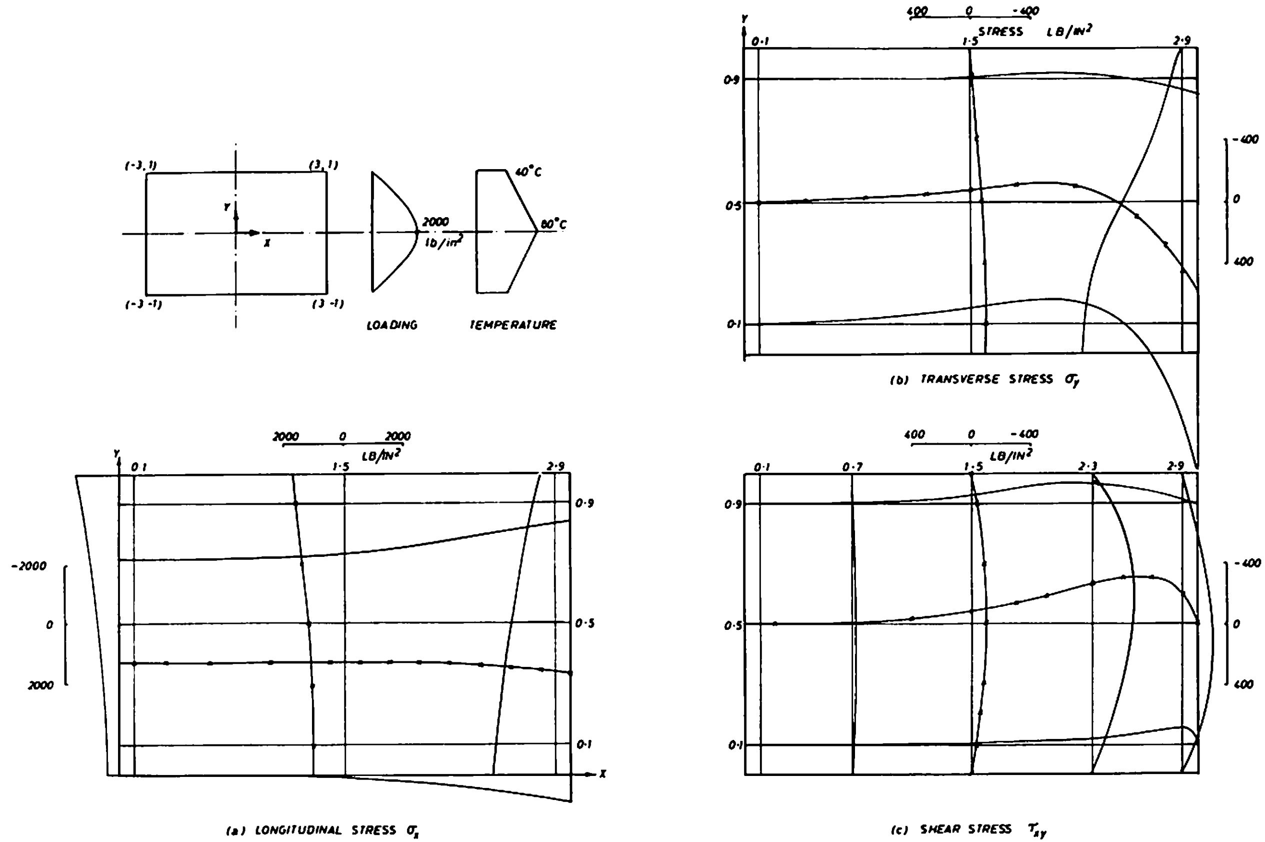
8. Comparison of analytical and finite elements steady-state stess solutions for flat plate. Variation of longitudinal stress, σ_x, along plate.



9. Comparison of analytical and finite elements solutions at X = 0.1K, for time-varying longitudinal stresses, σ_x, in flat plate.



lo. Comparison oı analytical and finite elements solutions, at X = 0.9K, for time-varying stresses, σ_x, in flat plate.



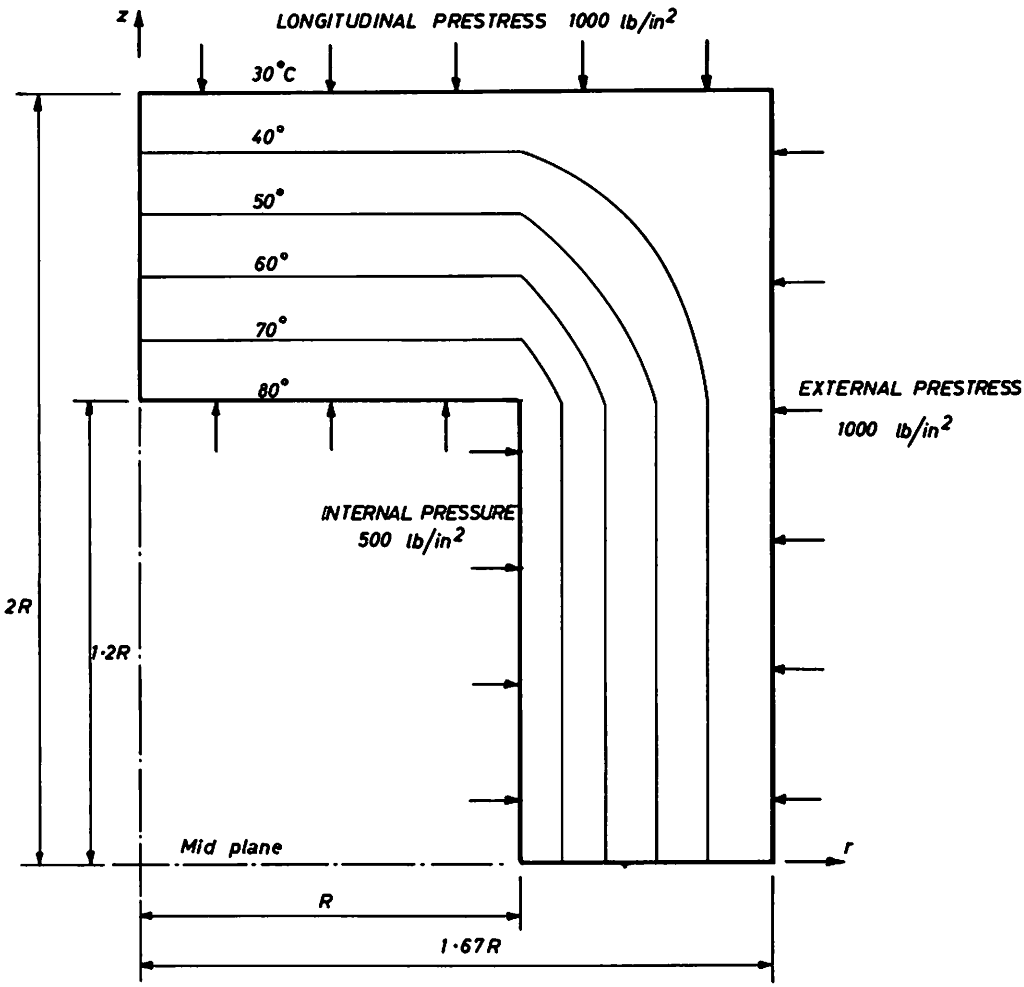
x RESULTS FROM VARIATIONAL PRINCIPLE

11. Comparison of finite elements variational principle solution with finite element step-by-step solution for steady-state stresses in flat plate.

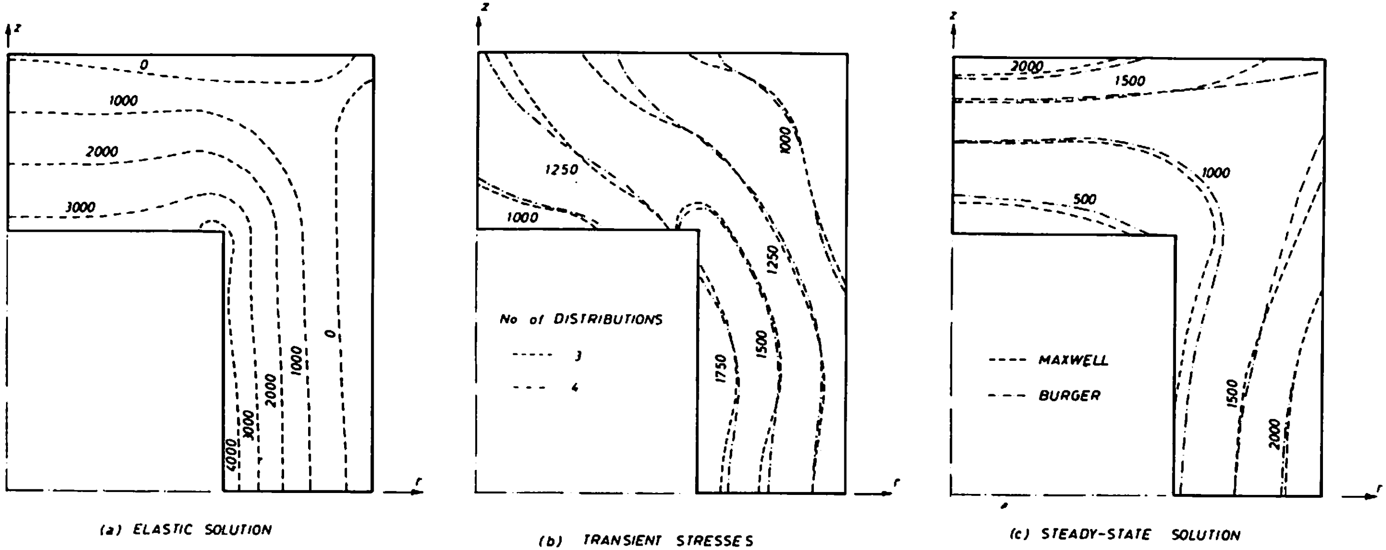
(a) longitudinal stress, σ_x.

(b) transverse stress, σ_y.

(c) shear stress, τ_{xy}.



12. Loading and temperature details for analysis of short cylinder vessel with end caps.



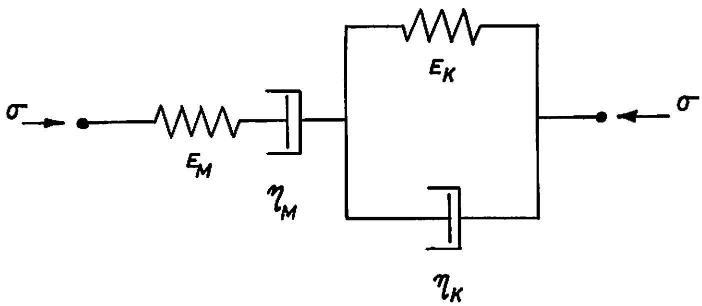
13. Contours of circumferential stress in hollow cylinder.

(a) thermo-elastic.

(b) creep solution at approximately 70 days after heating.

(c) steady-state stress condition.

Comparison of predicted stresses in the steady-state condition by Maxwell and Burger's body material representation.



14. The Burger's body four element model representation for material which exhibits creep and creep recovery.

DISCUSSION

Q P. J. E. SULLIVAN, U. K.

I would like to congratulate Dr. England and Dr. Allen on their excellent paper on the approximate method for calculating time dependent creep stresses as applied to pressure vessels.

I have one point to raise regarding the application of the Burger's body to his technique to calculate creep strains by the power variational technique. For his technique to work Dr. England stipulates that the differential equation of the Burger body should be reduced to the following form :

$$0. = q_o \sigma + q_1 \ddot{\sigma} + q_2 \ddot{\sigma}$$

The solution of the above equation only gives the particular integral and not the complete solution of the Burger model.

Could Dr. England explain the physical significance of omitting the complementary function solution if he uses a Burger's model for his material representation ?

A G. L. ENGLAND, U. K.

In reply to Mr. Sullivan's question relating to equations (29) and (30) it should be emphasized that when the integration is taken over the entire volume of the structure, the right hand side of equation (29) becomes,

$$\int (q_o \sigma + q_1 \dot{\sigma} + q_2 \ddot{\sigma}) \delta\sigma \, dV$$

which is equal to zero by consequence of the zero which results from the volume integral of the left hand side of equation (29) when the ratio $\frac{p_2}{p_1}$ conforms to the restriction mentioned. A complete solution to the problem is thus obtained from the set of equations (33), which are 'n' in number, corresponding to the 'n' independent self-equilibrating distributions of stress, $\delta\sigma$.

$$\int (q_o \sigma + q_1 \dot{\sigma} + q_2 \ddot{\sigma}) \delta\sigma \, dV = 0 \qquad (33)$$

Q Z. P. BAŽANT, U. S. A.

I wish to emphasize the well-known fact that the rate-of-creep stress-strain law is a rather inaccurate representation of the actual behaviour of concrete. Especially, it strongly underestimates the creep response of an aged concrete and neglects recovery. Perhaps it would be worthwhile to extend your very valuable analysis to a more accurate creep law. With the use of a computer it should not pose unsurmountable difficulties.

Second, I should like to mention that your basic stress distributions across the beam depth are in fact similar to the expansion in Lengendre polynomials used in the refined theories of plate bending.

Finally, let me note that a variational principle for the rate-of-creep stress-strain law has also been presented in the Proceedings of the 7th Congress of International Association for Bridge and Structural Engineering, pp. 887-897 (1964).

A G. L. ENGLAND, U. K.

Although I agree with Dr. Bažant that the rate of creep strain law for concrete does not represent the true material behaviour, there is experimental evidence to suggest that its use can give good predictions of structural behaviour for prestressed concrete structures subjected to non-uniform heating. With the additional feature of including delayed elastic and recovery strains, which are independent of temperature, into the material characteristics the predictions of structural behaviour are likely to be better than obtained from the simpler Maxwellian material approach and I would therefore expect worthwhile results to emerge.

In reply to the second question, I do not claim any originality for the stress distributions shown for the analysis of the restrained beam of my introduction. They are simply self-equilibrating distributions of stress. For a complex structure such as a PCRV self-equilibrating distributions that are required for the analysis of the creep phase of the problem, they may be obtained in a number of ways, e. g. they may be deduced from the difference of two non-homogeneous elastic solutions for the same boundary loading, or they may be obtained directly by solving a non-uniform temperature problem without boundary contraints. Such solutions would normally be obtained with the help of finite element techniques. They form the basic information for the creep analysis which follows. The author is not aware of other creep analyses of the form reported in the paper and thus any originality that might exist is attributed to the time-dependent solution techniques employed.