A MATRIX DISPLACEMENT METHOD ON PRE- AND POST-BUCKLING ANALYSIS OF LINERS FOR REACTOR VESSELS

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ABSTRACT

A matrix displacement method is suggested for the solution of instability problems for liners of nuclear containment vessels. The method uses the incremental stiffness formulations for the plate and shell discrete elements to predict the pre- and post-buckling behavior by either an iterative or an incremental procedure. The inclusion of initial displacement terms in the element formulations allows the method to account for the possible initial displacement of the liner caused by constructional inaccuracies. Applications are illustrated by using the formulation for a doubly curved shell discrete element.

1. INTRODUCTION

In the field of structural mechanics in reactor technology, engineers often find difficulties in performing structural analysis for the liners. The main difficulties may be briefed as follows:

i. The loading conditions are complex. For instance, in a common prestressed concrete nuclear containment vessel, the loading conditions may include prestress, normal pressure, shrinkage and creep of concrete, thermal gradients, dead load, earthquake, wind, etc. In some special geographic locations, the loads caused by tornado, flood, and vacuum, etc., may also occur. These loadings cause not only bending to the liner but also middle-surface compression which results in serious buckling problem. Subjecting to so many possible loading conditions, one must find and consider the worst possible loading combinations.

ii. The initial displacements, such as the initial inward curvature, due to fabricational and constructional inaccuracies are often present in the liner between anchors. The inclusion of initial displacement terms in plate and shell equations complicates the analysis.

iii. The liner supporting conditions depend on the anchorage arrangement. Since the case of missing or failed anchor is not uncommon, a sound structural analysis must account for such irregular supporting conditions. In other words, the structural analysis must guarantee the stability of the liner under such possible irregular anchorage conditions.

iv. Other factors which may complicate the analysis and design of liners are: (1) variation of anchor spacing; (2) misalignment of liner plate seams; (3) variation of plate thickness; (4) variation of liner plate material yield stress; (5) variation of Poisson's ratio for liner plate material; (6) variation of anchor stiffness, etc.

From the above-cited factors and difficulties, it is seen that an analysis method, which can predict the pre- and post-buckling behavior of imperfect plates and shells with arbitrary shape, loading, and supporting conditions, is sought by the structural engineer in the reactor technological field. One of such methods can be found in the framework of discrete element matrix displacement method.

The development of the discrete element method has been motivated partly by its ability to model accurately the most complex geometries, loading conditions, and boundary conditions. The objective of this paper, therefore, is to extend such capabilities to instability problems of liners for reactor vessels.

This paper suggests a procedure which uses the minimum potential energy principle to derive the geometrically nonlinear stiffness matrices for an arbitrary shell discrete element. The formulations are not only appropriate for predicting the Euler buckling load but also large deflection and post-buckling behaviors for the liner plates and shells. The large deflection and post-buckling behavior can be predicted by either an iterative or an incremental method. Examples are illustrated by the use of a doubly-curved shell discrete element. Examples are performed for two categories: (1) the Euler buckling loads for some typical liner plate and shell problems; and (2) the large deflection and post-buckling behavior of liner plates with initial curvatures.

2. FORMULATION OF POTENTIAL ENERGY FUNCTION

The present formulation of the element stiffness equations is based on the application of the minimum potential energy principle. Therefore it is necessary to construct an expression for the potential energy in terms of displacements and their derivatives. This expression is derived on the basis of the following assumptions: (1) the shell is thin, elastic, and isotropic; (2) the shell middle-surface bisects the thickness; (3) transverse section originally plane remains plane after deformation; (4) direct stress in the normal-to-surface direction and transverse shear strains are neglected; (5) the normal displacement w is a function only of the middle-surface coordinates; (6) the deflections are of the order of magnitude of shell thickness; (7) in the strain-displacement relations the second-order terms of displacement and displacement derivatives are to be retained.

Before formulating the potential energy expression, it is necessary to describe the geometry of a differential shell element. This element is shown in fig. 1. This differential element is of constant thickness h and constant radii of curvature R_1, R_2. The curvilinear coordinate axes α_1 and α_2 lie in the middle-surface of the shell. Lamé parameters A_1, A_2 serve to relate the curvilinear and system (x,y,z) coordinates. So

$$A_1^2 = \left(\frac{\partial x}{\partial \alpha_1}\right)^2 + \left(\frac{\partial y}{\partial \alpha_1}\right)^2 + \left(\frac{\partial z}{\partial \alpha_1}\right)^2$$

$$A_2^2 = \left(\frac{\partial x}{\partial \alpha_2}\right)^2 + \left(\frac{\partial y}{\partial \alpha_2}\right)^2 + \left(\frac{\partial z}{\partial \alpha_2}\right)^2 \tag{1}$$

The strain-displacement relationships are derived on the basis of Novozhilov's procedure [1]. The second-order displacement and displacement derivative terms are added here. The middle-surface strains thus derived [2] are:

$$\epsilon_1 = \frac{\partial u}{\partial s_1} + \frac{w}{R_1} + \frac{1}{2}\left(\frac{\partial v}{\partial s_1}\right)^2 + \frac{1}{2}\left(\frac{u}{R_1} - \frac{\partial w}{\partial s_1}\right)^2$$

$$\epsilon_2 = \frac{\partial v}{\partial s_2} + \frac{w}{R_2} + \frac{1}{2}\left(\frac{\partial u}{\partial s_2}\right)^2 + \frac{1}{2}\left(\frac{v}{R_2} - \frac{\partial w}{\partial s_2}\right)^2 \qquad (2)$$

$$\epsilon_{12} = \frac{\partial u}{\partial s_2} + \frac{\partial v}{\partial s_1} - \frac{\partial u}{\partial s_1}\frac{\partial v}{\partial s_1} - \frac{\partial v}{\partial s_1}\frac{w}{R_1} + \frac{\partial w}{\partial s_1}\frac{\partial w}{\partial s_2} - \frac{v}{R_2}\frac{\partial w}{\partial s_1} - \frac{u}{R_1}\frac{\partial w}{\partial s_2} + \frac{u}{R_1}\frac{v}{R_2}$$
$$- \frac{\partial u}{\partial s_2}\frac{\partial v}{\partial s_2} - \frac{\partial u}{\partial s_2}\frac{w}{R_2}$$

The strains of a surface parallel to the middle-surface with a distance z are:

$$\epsilon_1^{(z)} = \epsilon_1 + \lambda_1 z$$
$$\epsilon_2^{(z)} = \epsilon_2 + \lambda_2 z \qquad (3)$$
$$\epsilon_{12}^{(z)} = \epsilon_{12} + \lambda_{12} z$$

where

$$\lambda_1 = \frac{\partial^2 w}{\partial s_1^2}\left(-1 + \frac{\partial u}{\partial s_1} + \frac{w}{R_1}\right) - \frac{1}{R_1}\left\{\frac{w}{R_1} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial s_2} + \frac{\partial v}{\partial s_1}\right)^2 - \left(\frac{v}{R_2} - \frac{\partial w}{\partial s_2}\right)^2 + \left(\frac{u}{R_1} - \frac{\partial w}{\partial s_1}\right)^2\right]\right.$$
$$\left. - \left(\frac{\partial w}{\partial s_1}\right)^2 + \left(\frac{\partial u}{\partial s_1}\right)^2 + \left(\frac{\partial v}{\partial s_1}\right)^2 + \left(\frac{\partial u}{\partial s_1}\right)\left(\frac{w}{R_1}\right) + \left(\frac{\partial w}{\partial s_1}\right)\left(\frac{u}{R_1}\right)\right\} \qquad (3a)$$

$$\lambda_2 = \frac{\partial^2 w}{\partial s_2^2}\left(-1 + \frac{\partial v}{\partial s_2} + \frac{w}{R_2}\right) - \frac{1}{R_2}\left\{\frac{w}{R_2} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial s_2} + \frac{\partial v}{\partial s_1}\right)^2 + \left(\frac{u}{R_1} - \frac{\partial w}{\partial s_1}\right)^2 + \left(\frac{v}{R_2} - \frac{\partial w}{\partial s_2}\right)^2\right]\right.$$
$$\left. - \left(\frac{\partial w}{\partial s_2}\right)^2 + \left(\frac{\partial v}{\partial s_2}\right)^2 + \left(\frac{\partial u}{\partial s_2}\right)^2 + \left(\frac{\partial v}{\partial s_2}\right)\left(\frac{w}{R_2}\right) + \left(\frac{\partial w}{\partial s_2}\right)\left(\frac{v}{R_2}\right)\right\} \qquad (3b)$$

$$\lambda_{12} = 2\left\{\frac{\partial^2 w}{\partial s_1 \partial s_2}\left(-1 + \frac{\partial u}{\partial s_1} + \frac{\partial v}{\partial s_2} + \frac{w}{R_1} + \frac{w}{R_2}\right) + \frac{\partial^2 u}{\partial s_1 \partial s_2}\left(\frac{\partial w}{\partial s_1} - \frac{u}{R_1}\right) + \frac{\partial^2 v}{\partial s_1 \partial s_2}\left(\frac{\partial w}{\partial s_2} - \frac{v}{R_2}\right)\right.$$
$$\left. + \frac{1}{2}\left(\frac{\partial u}{\partial s_2} + \frac{\partial v}{\partial s_1}\right)\left(\frac{\partial^2 w}{\partial s_1^2} + \frac{\partial^2 w}{\partial s_2^2}\right)\right\} - \frac{1}{R_1}\left\{\frac{\partial v}{\partial s_1} - \frac{\partial u}{\partial s_2} - \left(\frac{\partial w}{\partial s_1}\right)\left(\frac{\partial w}{\partial s_2}\right) + 2\left(\frac{\partial u}{\partial s_1}\right)\left(\frac{\partial u}{\partial s_2}\right)\right.$$
$$\left. - \left(\frac{\partial u}{\partial s_1}\right)\left(\frac{\partial v}{\partial s_1}\right) + \left(\frac{\partial u}{\partial s_2}\right)\left(\frac{\partial v}{\partial s_2}\right) + \left(\frac{\partial u}{\partial s_2}\right)\left(\frac{w}{R_1}\right) + \left(\frac{\partial u}{\partial s_2}\right)\left(\frac{w}{R_2}\right) + \left(\frac{u}{R_1}\right)\left(\frac{\partial w}{\partial s_2}\right) - 2\left(\frac{\partial v}{\partial s_1}\right)\left(\frac{v}{R_1}\right)\right\}$$

$$- \frac{1}{R_2} \{\frac{\partial u}{\partial s_2} - \frac{\partial v}{\partial s_1} - (\frac{\partial w}{\partial s_1})(\frac{\partial w}{\partial s_2}) + 2(\frac{\partial v}{\partial s_1})(\frac{\partial v}{\partial s_2}) - (\frac{\partial u}{\partial s_2})(\frac{\partial v}{\partial s_2}) + (\frac{\partial u}{\partial s_1})(\frac{\partial v}{\partial s_1})$$

$$+ (\frac{\partial v}{\partial s_1})(\frac{w}{R_2}) + (\frac{\partial v}{\partial s_1})(\frac{w}{R_1}) + (\frac{v}{R_2})(\frac{\partial w}{\partial s_1}) - 2(\frac{\partial u}{\partial s_2})(\frac{w}{R_2})\} \quad (3c)$$

with $s_1 = A_1\alpha_1$ and $s_2 = A_2\alpha_2$.

Having obtained the strain-displacement relationships, it is then possible to obtain the potential energy expression,

$$\Pi = U - W \quad (4)$$

where

$$U = \frac{1}{2}\int_V (\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_{12}\varepsilon_{12})dV \quad (5)$$

The stresses σ_1, σ_2, and σ_{12} are in terms of strains $\varepsilon_1^{(z)}$, $\varepsilon_2^{(z)}$, and $\varepsilon_{12}^{(z)}$ [1],

$$\sigma_1 = \frac{E}{1-\nu^2}(\varepsilon_1^{(z)} + \nu\varepsilon_2^{(z)})$$

$$\sigma_2 = \frac{E}{1-\nu^2}(\varepsilon_2^{(z)} + \nu\varepsilon_1^{(z)}) \quad (6)$$

$$\sigma_{12} = \frac{E}{2(1+\nu)}\varepsilon_{12}^{(z)}$$

The explicit expression for energy U can be obtained by substituting eqs. (3), (3a), (3b) and (3c) into eq. (6) and then substituting the results into eq. (5). It takes the form of

$$U = U_1 + U_2 + U_3 \quad (7)$$

with

$$U_1 = \frac{1}{2}S_1\int_A\{(\frac{\partial v}{\partial s_1})^2 + (\frac{u}{R_1})^2\}ds_1ds_2 + \frac{1}{2}S_2\int_A\{(\frac{\partial u}{\partial s_2})^2 + (\frac{v}{R_2})^2\}ds_1ds_2$$

$$- \frac{1}{2}(\frac{M_1}{R_1} + \frac{M_2}{R_2})\int_A\{(\frac{\partial u}{\partial s_2})^2 + (\frac{\partial v}{\partial s_1})^2 + 2(\frac{\partial u}{\partial s_2})(\frac{\partial v}{\partial s_1}) + (\frac{u}{R_1})^2 + (\frac{v}{R_2})^2\}ds_1ds_2$$

$$- \frac{M_1}{R_1}\int_A\{(\frac{\partial u}{\partial s_1})^2 + (\frac{\partial v}{\partial s_1})^2\}ds_1ds_2 - \frac{M_2}{R_2}\int_A\{(\frac{\partial u}{\partial s_2})^2 + (\frac{\partial v}{\partial s_2})^2\}ds_1ds_2$$

$$- S_{12}\int_A\{(\frac{\partial u}{\partial s_1})(\frac{\partial v}{\partial s_1}) - (\frac{u}{R_1})(\frac{v}{R_2}) + (\frac{\partial u}{\partial s_2})(\frac{\partial v}{\partial s_2})\}ds_1ds_2$$

$$- M_{12}\int_A\{2(\frac{\partial^2 u}{\partial s_1\partial s_2})(\frac{u}{R_1}) + 2(\frac{\partial^2 v}{\partial s_1\partial s_2})(\frac{v}{R_2}) + \frac{2}{R_1}(\frac{\partial u}{\partial s_1})(\frac{\partial u}{\partial s_2}) + \frac{2}{R_2}(\frac{\partial v}{\partial s_1})(\frac{\partial v}{\partial s_2})$$

$$- (\frac{1}{R_1} - \frac{1}{R_2})[(\frac{\partial u}{\partial s_1})(\frac{\partial v}{\partial s_1}) - (\frac{\partial u}{\partial s_2})(\frac{\partial v}{\partial s_2})]\}ds_1ds_2 \quad (7a)$$

$$U_2 = - S_1 \int_A (\frac{u}{R_1})(\frac{\partial w}{\partial s_1})ds_1ds_2 - S_2 \int_A (\frac{v}{R_2})(\frac{\partial w}{\partial s_2})ds_1ds_2$$

$$+ M_1 \int_A \{(\frac{\partial^2 w}{\partial s_1^2})(\frac{\partial u}{\partial s_1}) + \frac{1}{R_1}[(\frac{v}{R_2})(\frac{\partial w}{\partial s_2}) - (\frac{\partial u}{\partial s_1})(\frac{w}{R_1})]\}ds_1ds_2$$

$$+ M_2 \int_A \{(\frac{\partial^2 w}{\partial s_2^2})(\frac{\partial v}{\partial s_2}) + \frac{1}{R_2}[(\frac{u}{R_1})(\frac{\partial w}{\partial s_1}) - (\frac{\partial v}{\partial s_2})(\frac{w}{R_2})]\}ds_1ds_2$$

$$- S_{12} \int_A \{(\frac{\partial u}{\partial s_2})(\frac{w}{R_2}) + (\frac{\partial v}{\partial s_1})(\frac{w}{R_1}) + (\frac{\partial w}{\partial s_1})(\frac{v}{R_2}) + (\frac{\partial w}{\partial s_2})(\frac{u}{R_1})\}ds_1ds_2$$

$$+ M_{12} \int_A \{2(\frac{\partial^2 w}{\partial s_1 \partial s_2})(\frac{\partial u}{\partial s_1} + \frac{\partial v}{\partial s_2}) + 2(\frac{\partial^2 u}{\partial s_1 \partial s_2})(\frac{\partial w}{\partial s_1}) + 2(\frac{\partial^2 v}{\partial s_1 \partial s_2})(\frac{\partial w}{\partial s_2})$$

$$+ (\frac{\partial u}{\partial s_2} + \frac{\partial v}{\partial s_1})(\frac{\partial^2 w}{\partial s_1^2} + \frac{\partial^2 w}{\partial s_2^2}) + \frac{1}{R_1}[2(\frac{\partial v}{\partial s_1})(\frac{w}{R_1}) - (\frac{\partial u}{\partial s_2})(\frac{w}{R_1})$$

$$- (\frac{\partial u}{\partial s_2})(\frac{w}{R_2}) - (\frac{u}{R_1})(\frac{\partial w}{\partial s_2})] + \frac{1}{R_2}[2(\frac{\partial u}{\partial s_2})(\frac{w}{R_2}) - (\frac{\partial v}{\partial s_1})(\frac{w}{R_2})$$

$$- (\frac{\partial v}{\partial s_1})(\frac{w}{R_1}) - (\frac{v}{R_2})(\frac{\partial w}{\partial s_1})]\}ds_1ds_2 \quad (7b)$$

and

$$U_3 = \frac{1}{2} S_1 \int_A (\frac{\partial w}{\partial s_1})^2 ds_1ds_2 + \frac{1}{2} S_2 \int_A (\frac{\partial w}{\partial s_2})^2 ds_1ds_2$$

$$+ \frac{M_1}{R_1} \int_A \{(\frac{\partial^2 w}{\partial s_1^2})(w) + \frac{1}{2}[(\frac{\partial w}{\partial s_1})^2 - (\frac{\partial w}{\partial s_2})^2]\}ds_1ds_2$$

$$+ \frac{M_2}{R_2} \int_A \{(\frac{\partial^2 w}{\partial s_2^2})(w) + \frac{1}{2}[(\frac{\partial w}{\partial s_2})^2 - (\frac{\partial w}{\partial s_1})^2]\}ds_1ds_2$$

$$+ S_{12} \int_A (\frac{\partial w}{\partial s_1})(\frac{\partial w}{\partial s_2})ds_1ds_2$$

$$+ M_{12} (\frac{1}{R_1} + \frac{1}{R_2}) \int_A \{2(\frac{\partial^2 w}{\partial s_1 \partial s_2})(w) + (\frac{\partial w}{\partial s_1})(\frac{\partial w}{\partial s_2})\}ds_1ds_2 \quad (7c)$$

where U_1 is the energy caused by membrane action only; U_2 is the energy caused by the coupling of membrane and flexure actions; U_3 is the energy caused by flexure action only; S_1, S_2, and S_{12} are the resultant membrane forces caused by σ_1, σ_2, and σ_{12}, respectively; and M_1, M_2, and M_{12} are the resultant bending and twisting moments caused by σ_1, σ_2, and σ_{12}, respectively. These resultant forces are introduced for convenience of predicting buckling load. They may, however, be written in terms of displacements and displacement derivatives.

3. ELEMENT FORMULATION AND ASSEMBLAGE FOR AN ARBITRARY SHELL DISCRETE ELEMENT

Once the element model has been appropriately chosen and its corresponding displacement functions have been decided, the potential energy expression for this element can be obtained by substituting the displacement functions into eq. (4). The results can be written in a matrix form

$$\Pi = U - W = U_k + U_1 + U_2 - W \tag{8}$$

where

$$U_k = \frac{1}{2} \lfloor q \rfloor \; [k]\{q\} \tag{8a}$$

$$U_1 = \frac{1}{6} \lfloor q \rfloor \; [n_1]\{q\} \tag{8b}$$

$$U_2 = \frac{1}{12} \lfloor q \rfloor \; [n_2]\{q\} \tag{8c}$$

and

$$W = \lfloor q \rfloor \; \{p\} \tag{8d}$$

in which the energy U_k is the well-known strain energy obtained from small deflection theory; the energy U_1 consists of third-order displacement and displacement derivative terms, which serves to couple membrane and flexure action; the energy U_2 consists of fourth-order displacement and displacement derivative terms, which affects flexural action only; the work W designates the potential energy of external loads.

According to the principle of minimum potential energy, the state of equilibrium of a deformed shell element is obtained by that for which the first variation of the total potential energy vanishes,

$$\delta\Pi = \delta U - \delta W = 0 \tag{9}$$

Substituting eqs. (8a) to (8d) into eq. (9), the matrix stiffness equation of equilibrium for a shell element is obtained in the form

$$\{p\} = ([k] + \frac{1}{2}[n_1] + \frac{1}{3}[n_2])\{q\} \tag{10}$$

The coefficients in these three matrices are obtained by performing the second partial differentiation of the strain energy with respect to proper nodal displacements. This procedure for obtaining the incremental stiffness matrices $[n_1]$ and $[n_2]$ was introduced in ref. [3]. Eq. (10) is the element nonlinear stiffness formulation which is appropriate for a direct iterative analysis. The iterative method can be based on a variety of formalized as well as informal procedures. The most formalized method is the Newton-Raphson method.

One of the alternative numerical methods for solving geometrically nonlinear problems lies in the incremental procedure. The incremental stiffness equation can be obtained by applying an incremental operator Δ to eq. (10). The incremental operator is given definition by Taylor expansion about a known equilibrium state $(\{\bar{q}\}, \{\bar{p}\})$. Assuming f is a dummy function of {q}, its second-order Taylor expansion takes the form

$$\Delta f_i = f_i(\{\bar{q}\} + \{\Delta q\}) - f_i(\{\bar{q}\})$$

$$= \frac{\partial f_i(\{q\})}{\partial q_j}\bigg|_{\{\bar{q}\}} \Delta q_j + \frac{1}{2}\frac{\partial^2 f_i(\{q\})}{\partial q_j \partial q_k}\bigg|_{\{\bar{q}\}} \Delta q_j \Delta q_k \qquad (11)$$

Applying the incremental operator Δ, defined by eq. (11), to eq. (10), the nonlinear incremental equation for a shell element is obtained. If the last term in eq. (11) is neglected, the incremental equation obtained is linearized. The present study is focused on the linearized incremental procedure. The element formulation is of the form

$$\{\Delta p\} = ([k] + [n_1] + [n_2])\{\Delta q\} \qquad (12)$$

Upon assemblage of individual element formulations, such as eq. (10) or (12), the overall structural stiffness equations are obtained. In this representation, the total system stiffness equations are written by capital letters. Thus for the iterative method, one has the equation

$$\{P\} = ([K] + \frac{1}{2}[N_1] + \frac{1}{3}[N_2])\{Q\} \qquad (13)$$

For the piecewise linear incremental method, one has the equation for the total system

$$\{\Delta P\} = ([K] + [N_1] + [N_2])\{\Delta Q\} \qquad (14)$$

Since the incremental displacement distribution {ΔQ} cannot be assumed, eq. (14) is not directly applicable. However, the incremental load distribution can always be assumed. Eq. (14) must be rewritten in an inversed form

$$\{\Delta Q\}_{i+1} = ([K] + [N_1] + [N_2])_i^{-1}\{\Delta P\}_{i+1} \qquad (15)$$

where i denotes the incremental step number. It is seen from eq. (15) that the sum of stiffness matrices must be inversed in every step. Since the stiffness matrices are given in terms of the geometry of the undeformed elements, they require no coordinate transformation. The accuracy of this piecewise linear incremental approximation can be increased to any desired level by increasing appropriately the number of increments.

One of the most significant and interesting problems in the geometrically nonlinear analysis of liners is to predict the Euler buckling load. The Euler buckling load is defined as that for which the slope of load-deflection curve vanishes. This definition is identical to that for the "neutral equilibrium" state given in the minimum potential energy principle. From this definition, it is readily seen that the linear incremental procedure is very appropriate for finding the Euler buckling load during the process of predicting nonlinear behavior. During the incremental process, the buckling load is defined as the loading level for which the slope of the load-deflection curve vanishes, or

$$\det.\left|[K] + [N_1] + [N_2]\right| = 0 \tag{16}$$

In some cases, the engineers may be only interested in obtaining the linear Euler buckling load rather than the nonlinear behavior. If that is the case, the second-order incremental stiffness matrix $[N_2]$ can be neglected. Furthermore, a dummy parameter λ may be placed in front of $[N_1]$ in order to have an eigen solution.

$$\det.\left|[K] + \lambda[N_1]\right| = 0 \tag{17}$$

or

$$\frac{1}{\lambda} = - [K]^{-1} [N_1] \tag{18}$$

This equation is solved by an extrapolation technique. For an assumed loading level, there exists a value of λ. If the trial loading level reaches the Euler buckling load, the parameter λ equals to unity. Usually three trials will be sufficient for extropolating the buckling load for which the corresponding λ equals to unity. This approximation procedure is illustrated in fig. 2.

4. ELEMENT FORMULATION AND ASSEMBLAGE FOR PLATE ELEMENT WITH INITIAL DEFLECTION

In this study, the initial deflection terms are included for a special form of shell element - flat plate element.

For the flat plate with initial deflections, the strain-displacement relations for a surface at a distance z from the middle-surface are [4]

$$\epsilon_1^{(z)} = \frac{\partial u}{\partial x} - z(\frac{\partial^2 w}{\partial x^2}) + \frac{1}{2}[\frac{\partial(w + w_o)}{\partial x}]^2 - \frac{1}{2}(\frac{\partial w_o}{\partial x})^2 \tag{19a}$$

$$\epsilon_2^{(z)} = \frac{\partial v}{\partial y} - z(\frac{\partial^2 w}{\partial y^2}) + \frac{1}{2}[\frac{\partial(w + w_o)}{\partial y}]^2 - \frac{1}{2}(\frac{\partial w_o}{\partial y})^2 \tag{19b}$$

$$\epsilon_{12}^{(z)} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\,\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial(w + w_o)}{\partial x}\,\frac{\partial(w + w_o)}{\partial y} - (\frac{\partial w_o}{\partial x})(\frac{\partial w_o}{\partial y}) \tag{19c}$$

Upon substitution of eqs. (19a) to (19c) into eq. (6) and then into eq. (5), the strain energy expression is obtained,

$$U = U_k + U_o + U_1 + U_2 \tag{20}$$

where

$$U_k = \frac{D}{2}\iint\{(\nabla^2 w)^2 + \frac{12}{h^2}\,(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})^2 - 2(1-\nu)[\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - (\frac{\partial^2 w}{\partial x \partial y})^2 + \frac{12}{h^2}\,\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial y}$$

$$- \frac{3}{h^2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] \Big\} dxdy \tag{20a}$$

$$U_o = - \frac{3D}{h^2} \iint \Big\{ [(\frac{\partial w_o}{\partial x})^2 + \nu(\frac{\partial w_o}{\partial y})^2][\frac{\partial(w + w_o)}{\partial x}]^2 + [(\frac{\partial w_o}{\partial y})^2 + \nu(\frac{\partial w_o}{\partial x})^2]$$

$$[\frac{\partial(w + w_o)}{\partial y}]^2 + 2(1-\nu) \frac{\partial w_o}{\partial x} \frac{\partial w_o}{\partial y} \frac{\partial(w + w_o)}{\partial x} \frac{\partial(w + w_o)}{\partial y} \Big\} dxdy \tag{20b}$$

$$U_1 = \frac{6D}{h^2} \iint \Big\{ (\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y})[\frac{\partial(w + w_o)}{\partial x}]^2 + (\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x})[\frac{\partial(w + w_o)}{\partial y}]^2$$

$$+ (1-\nu)(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \frac{\partial(w + w_o)}{\partial x} \frac{\partial(w + w_o)}{\partial y} \Big\} dxdy \tag{20c}$$

$$U_2 = \frac{3D}{2h^2} \iint \Big\{ [\frac{\partial(w + w_o)}{\partial x}]^2 + [\frac{\partial(w + w_o)}{\partial y}]^2 \Big\}^2 dxdy \tag{20d}$$

The zero and first order displacement derivative terms, such as $\frac{1}{4}(\frac{\partial w_o}{\partial x})^4$, $-\frac{\partial u}{\partial x}(\frac{\partial w_o}{\partial x})^2$.....etc., do not contribute to the stiffness matrices. They are thus neglected from the above energy expressions.

Using this energy expression, following the procedure described in the previous section, the element formulations for iterative and linear incremental analysis are obtained, respectively, as follows,

$$\{p\} = [k]\{q\} + ([n_o] + \frac{1}{2}[n_1] + \frac{1}{3}[n_2])\{q + q_o\} \tag{21}$$

and

$$\{\Delta p\} = ([k] + [n_o] + [n_1] + [n_2])\{\Delta q\} \tag{22}$$

A comparison of eqs. (21) and (22) with eqs. (10) and (14), respectively, shows that an additional zero-order incremental stiffness matrix $[n_o]$ must be added in order to account for the initial deflections. Furthermore, the matrices $[n_1]$ and $[n_2]$ in eqs. (21) and (22) are in terms of first- and second-order gross displacement vectors $\{q + q_o\}$, rather than net displacement vectors $\{q\}$.

The assemblage and solution procedure for the plate with initial deflections are identical to those described in previous sections for shell discrete element.

5. TREATING BOUNDARY CONDITIONS AND DISTRIBUTING MEMBRANE STRESSES

In the analysis of large deflection and post-buckling behavior of plates and shells, the edge movements in the middle-surface direction constitute, in addition to the well-known flexural boundary conditions, the so-called membrane boundary conditions. The treatment of flexural boundary conditions is a standard procedure which is well explained in the common texts [5]. The treatment of membrane boundary conditions for the large deflection and post-buckling behavior seems seldom discussed. It is therefore to be elaborated on here.

In order to treat the membrane edge conditions, the distribution of membrane stress and membrane displacement, particularly at the edges, must be determined. The equations for membrane stresses written in terms of membrane and flexure displacements are given by the combination of eqs. (2) and (6). By statics, the equations can be written in a matrix form for a discrete element

$$\{p_m\} = [k_m]\{q_m + q_{o_m}\} + \lfloor q_f + q_{o_f} \rfloor \; [g]\{q_f + q_{o_f}\} \tag{23}$$

in which $\{p_m\}$ is the element nodal membrane force vector; $\{q_m\}$ and $\{q_f\}$ are the vectors of element nodal membrane and flexure displacements, respectively; $[k_m]$ is the membrane portion of the linear stiffness matrix; matrix $[g]$ associates $\{p_m\}$ with $\{q_f + q_{o_f}\}$, and subscript "o" denotes initial quantities.

Upon assemblage of individual elements, the total system membrane force vs. displacement equation is obtained,

$$\{P_m\} = [K_m]\{Q_m + Q_{o_m}\} + \lfloor Q_f + Q_{o_f} \rfloor \; [G]\{Q_f + Q_{o_f}\} \tag{24}$$

All the terms in the square matrices of above equation are composed of only constant terms. The implication is that no coordinate transformation is required in the process of nonlinear behavior prediction.

At a certain equilibrium state, the flexure displacements are first obtained by either an iterative solution for eq. (13) or an incremental solution for eq. (15). Substituting such displacements into eq. (24) and imposing the membrane boundary condition, the distribution of membrane displacements in the undeformed middle-surface is obtained,

$$\{Q_m + Q_{o_f}\} = [K_m]^{-1}(\{P_m\} - \lfloor Q_f + Q_{o_f} \rfloor \; [G]\{Q_f + Q_{o_f}\}) \tag{25}$$

From these flexure and membrane displacements, the distribution of membrane stress can be obtained by using eq. (24). If the incremental procedure is used, these flexure and membrane displacements also serve to form the new incremental stiffness matrix for the next incremental step.

6. APPLICATIONS

a. The Formulations for a Doubly Curved Shell Element.

The formulations and procedure presented in this paper are suitable for an arbitrary shell element, i.e., arbitrary shape and displacement functions. The examples of application here are, however, obtained by the use of a doubly-curved shell discrete element [6]. The geometry of the element is shown in fig. 1. The element is of constant thickness and constant radii of curvature R_1 and R_2. Each node has six degrees of freedom: two membrane displacements u and v; a normal-to-surface deflection w; two rotations θ_1 and θ_2; and a generalized twist degree of freedom θ_{12}.

The u and v functions chosen are the so-called bilinear or first-order Lagrangian interpolation formulas which provide linear representation in the x and y directions. The w function is obtained from the fourth-order Hermitian polynomial approach. The nodal twist derivatives are adopted to assure inclusion of the strain due to simple twist. These functions are assumed as follows [6]:

$$u = \frac{1}{ab}\left[(\alpha_1-a)(\alpha_2-b)u_1 - \alpha_1(\alpha_2-b)u_2 + \alpha_1\alpha_2 u_3 - (\alpha_1-a)\alpha_2 u_4\right]$$

$$v = \frac{1}{ab}\left[(\alpha_1-a)(\alpha_2-b)v_1 - \alpha_1(\alpha_2-b)v_2 + \alpha_1\alpha_2 v_3 - (\alpha_1-a)\alpha_2 v_4\right]$$

$$\begin{aligned}
a^3b^3w = \; & (a^3 + 2\alpha_1^3 - 3a\alpha_1^2)(b^3 + 2\alpha_2^3 - 3b\alpha_2^2)w_1 + (3a\alpha_1^2 - 2\alpha_1^3)(b^3 + 2\alpha_2^3 - 3b\alpha_2^2)w_2 \\
& + (3a\alpha_1^2 - 2\alpha_1^3)(3b\alpha_2^2 - 2\alpha_2^3)w_3 + (a^3 + 2\alpha_1^3 - 3a\alpha_1^2)(3b\alpha_2^2 - 2\alpha_2^3)w_4 \\
& + a\alpha_1(\alpha_1 - a)^2(b^3 + 2\alpha_2^3 - 3b\alpha_2^2)\theta_{1_1} + a(\alpha_1^3 - a\alpha_1^2)(b^3 + 2\alpha_2^3 - 3b\alpha_2^2)\theta_{1_2} \\
& + a(\alpha_1^3 - a\alpha_1^2)(3b\alpha_2^2 - 2\alpha_2^3)\theta_{1_3} + a\alpha_1(\alpha_1 - a)^2(3b\alpha_2^2 - 2\alpha_2^3)\theta_{1_4} \\
& + b(a^3 + 2\alpha_1^3 - 3a\alpha_1^2)\alpha_2(\alpha_2 - b)^2\theta_{2_1} + b(3a\alpha_1^2 - 2\alpha_1^3)\alpha_2(\alpha_2 - b)^2\theta_{2_2} \\
& + b(3a\alpha_1^2 - 2\alpha_1^3)(\alpha_2^3 - b\alpha_2^2)\theta_{2_3} + b(a^3 + 2\alpha_1^3 - 3a\alpha_1^2)(\alpha_2^3 - b\alpha_2^2)\theta_{2_4} \\
& + ab\alpha_1\alpha_2(\alpha_1 - a)^2(\alpha_2 - b)^2\theta_{12_1} + ab\alpha_1\alpha_2(\alpha_1^2 - a\alpha_1)(\alpha_2 - b)^2\theta_{12_2} \\
& + ab\alpha_1\alpha_2(\alpha_1^2 - a\alpha_1)(\alpha_2^2 - b\alpha_2)\theta_{12_3} + ab\alpha_1\alpha_2(\alpha_1 - a)^2(\alpha_2^2 - b\alpha_2)\theta_{12_4}
\end{aligned} \tag{26}$$

The element formulations for the basic stiffness matrix [k] and the incremental stiffness matrix [n] have been presented in ref. [2]. For brevity of presentation, they are not shown here. If the curvilinear coordinates α_1 and α_2 in eq. (26) are replaced by Cartesian coordinates x and y, a rectangular plate element is obtained. The element stiffness matrix [k] and incremental stiffness matrices $[n_1]$ and $[n_2]$ have been presented in ref. [7]. For the formulations for rectangular plate element with initial deflections, it is referred to ref. [8].

b. Euler Buckling Load Predictions.

One of the most common problems in designing the liner is to predict its Euler buckling load. The liners are commonly anchored to the concrete containment vessels. Due to the arrangement of anchorage system, the liner components to be considered for buckling analysis may have uncommon shape and complex boundary conditions. It is proposed to employ the present discrete element approach to deal with such complex situation. In order to evaluate the present formulations and procedure, the examples with known alternative analytical solutions are chosen. The first example considered is a liner supported by equally spaced rows and columns of angle-shaped anchors as shown in fig. 3. The liner is subjected to high thermal expanding effect which causes circumferential compression. The anchorage system provides with clamped edge conditions. The case for a clamped rectangular plate has been solved in ref. [9] by an energy approach. The buckling stress solution takes the form

$$\sigma_{cr} = k \frac{\pi^2 D}{L_2^2 h} \tag{27}$$

The k values are shown in Fig. 3 for different aspect ratios. The k values obtained by using 16 elements to idealize one quadrant of a typical component of liner is also shown for comparison. Good agreement is indicated.

The writer considers next a steel liner segment where the curvature is not negligible (fig. 4). This segment of liner is assumed to be subjected to compression along the generator direction. All edges are assumed as simply supported, i.e., the displacement components u, v, and w are all zeros and the bending moments about the edge lines are also zero. The prebuckling middle-surface stress distribution are obtained by the membrane theory. The results prove to be in close agreement with the classical solution

$$\sigma_{cr} = \frac{Eh}{R\sqrt{3(1-\nu^2)}} \tag{28}$$

The present formulations have also been proven to be accurate in the prediction of Euler buckling load for cylindrical shell with other edge conditions and translational shell subjected to middle-surface or normal-to-surface loads. For detail, it is referred to ref. [2].

One of the advantages of discrete element approach is its versatility. The boundary conditions are irrelevant to the analysis procedure. Fig. 5 shows a common arrangement of anchorage system which constitutes a complicated boundary condition if classical analytical method is employed [9]. Such edge supports, however, make no difference from the regular boundary conditions (such as with all edges simply supported) if the present discrete element approach is used.

c. Postbuckling Behavior of Liner Plates.

The present development has extended the matrix displacement method from conventional linear elastic instability predictions to the analysis of geometrically nonlinear post-buckling responses for liner plates. The illustrated examples are performed here for some typical plate problems for which the alternative analytic solutions are available for comparison.

The first example chosen is a square plate under a middle-surface compression on two opposite edges. Two different edge conditons are considered: (1) The loaded edges are maintained straight with zero shearing stress. The unloaded edges are maintained straight by a distribution of normal membrane stress, the resultant of which is zero. The shearing stress is also zero; (2) The loaded edges are maintained straight with zero shearing stress. The unloaded edges are free to move and the normal membrane stress and shearing stress are zero.

Because of the symmetrical nature of the problem, only a quadrant needs to be analyzed. 16 elements are used.

Based on eq. (17), the Euler buckling load is first predicted. A very small lateral disturbing pressure is then applied in order to produce a curvature to the plate. The disturbing pressure is maintained constant from then on. The plate is then subjected to an increment of middle-surface compression which is equivalent to a lateral pressure,

$$p_z = p_x \frac{\partial^2 w}{\partial x^2} + p_y \frac{\partial^2 w}{\partial y^2} + 2p_{xy} \frac{\partial^2 w}{\partial x \partial y} \tag{29}$$

where p denotes the distribution of membrane stress induced by the increment of middle-surface compression. The postbuckling behavior is then predicted by a step-by-step linear incremental procedure (eq. (22)). The load vs. center deflection results for the postbuckling response are shown in fig. 6 for both boundary conditions. Alternative analytical solutions are available in refs. [10, 11] and are also shown in fig. 6 for comparison. Acceptable agreement is indicated.

The writer considers next the liner plates with initial deflections. Again the plates are square in shape and subjected to a compression acted on two opposite edges. The initial deflections are defined by the function

$$\bar{w}_o = W_o \sin \frac{\pi x}{A} \sin \frac{\pi y}{B} \tag{30}$$

where x and y are the Cartesian coordinates with origin lying at left lower corner of the plate; A and B are the length and width of the plate, respectively. Again, 16 elements are used to idealize a quadrant of the plate.

Boundary condition (1) is first considered. The load vs. center deflection behaviors for different degrees of initial curvature are obtained and shown in fig. 7. The alternative classical solution given by ref. [12] is also shown for comparison. Agreement is acceptable. The same problem with boundary condition (2) is also examined. The results for load vs. center deflection for square plate with a small initial curvature (W_o = 0.1h) is shown in fig. 8. An alternative solution given by ref. [11] is also shown for comparison. The agreement is acceptable for engineering purpose.

7. CONCLUSIONS

A matrix displacement discrete element method has been presented for the prediction of pre- and post-buckling behavior of liners for the reactor vessels. The formulations and procedure has been evaluated by the use of a doubly-curved discrete element. The results have shown that the method is accurate in the prediction of Euler buckling loads for liner shells. Besides, it is also accurate in the prediction of pre- and post-buckling behavior of liner plates.

Due to the versatile nature of the discrete element displacement method, the liner components to be considered may be of arbitrary shape, complicated anchorage system, and complex loading conditions.

The formulations and the method presented here is applicable to any discrete element, i.e., arbitrary shape and displacement patterns. Obviously, one of the main objectives of this paper is to present a general basis for the extension of the discrete element displacement approach for the pre- and post-buckling analysis of liners.

The future extension of this research will be to include the initial displacement terms in the shell equations. Besides, in the buckling process the liner should be considered to be deflected in the direction away from the concrete wall.

8. NOTATIONS

a,b	= element dimensions in the α_1 and α_2 directions, respectively
D	= flexural rigidity, $Eh^3/12(1-\nu^2)$
E	= modulus of elasticity
h	= thickness of element
[k]	= element basic stiffness matrix
$[n_o]$, $[n_1]$, $[n_2]$	= zero, first, and second order incremental stiffness matrices, respectively
R_1, R_2	= radii of curvature shown in fig. 1
s_1, s_2	= lengths $A_1d\alpha_1$ and $A_2d\alpha_2$ along α_1 and α_2 directions, respectively
u, v, w	= displacement components in α_1, α_2, and z directions, respectively
W_o	= maximum initial deflection of liner plate
x, y, z	= Cartesian coordinates
α_1, α_2	= curvilinear coordinates
Δ	= incremental operator
σ_1, σ_2, σ_{12}	= stresses in α_1, α_2 directions and shearing stresses, respectively
ε_1, ε_2, ε_{12}	= strains in α_1, α_2 directions and shearing strain in the middle-surface, respectively
$\varepsilon_1^{(z)}$, $\varepsilon_2^{(z)}$, $\varepsilon_{12}^{(z)}$	= strains in a surface parallel to middle-surface with a distance z
{ }, ⌊ ⌋	= column and row matrices, respectively
[ ]	= rectangular matrix.

9. ACKNOWLEDGEMENT

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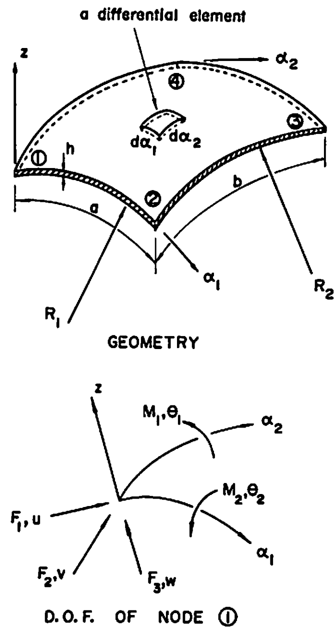
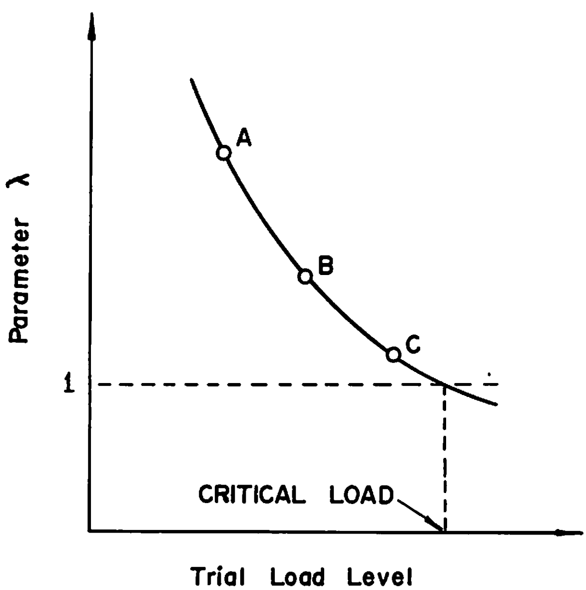
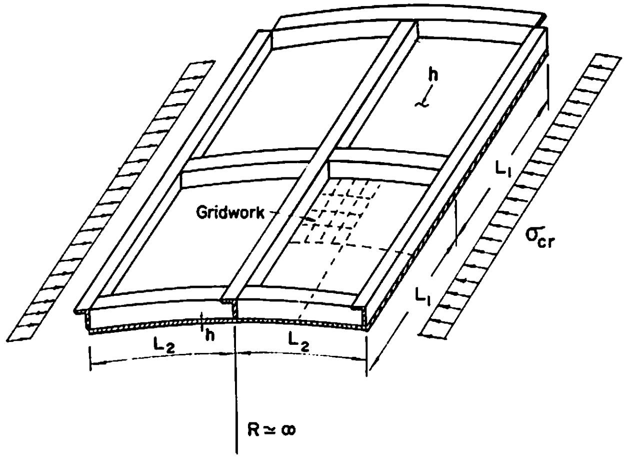


1. Geometry of a Differential Shell Element as well as a Doubly-Curved Shell Discrete Element.

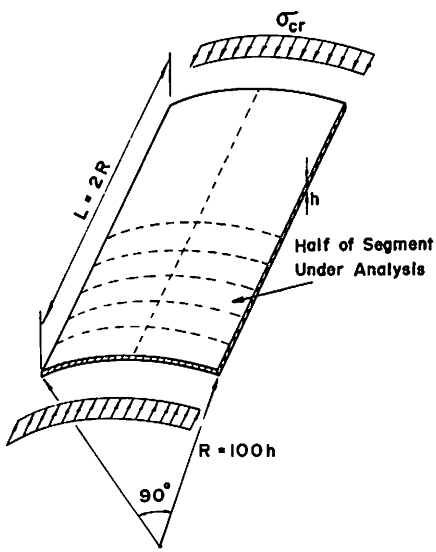


2. Euler Buckling Load Extrapolation.



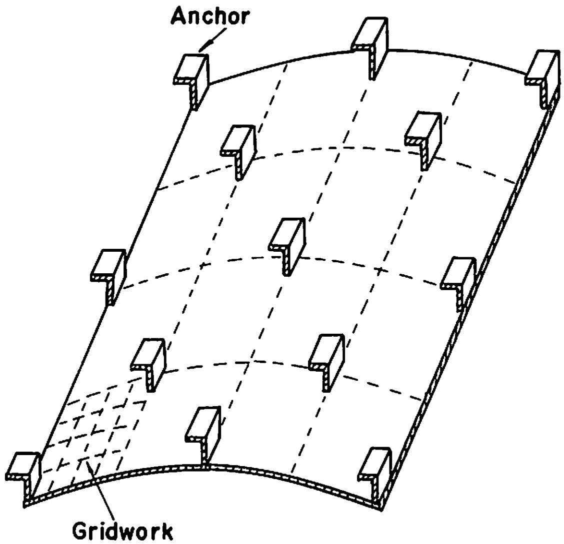
$L_2 : L_1$	0 75	1	2	3	4
k (ref 9)	11.69	10.07	78.80	73.70	72.30
k (16 Elements)	11.77	10.12	79.50	74.56	73.28

3. Euler Buckling Loads for an Anchored Liner Plate.

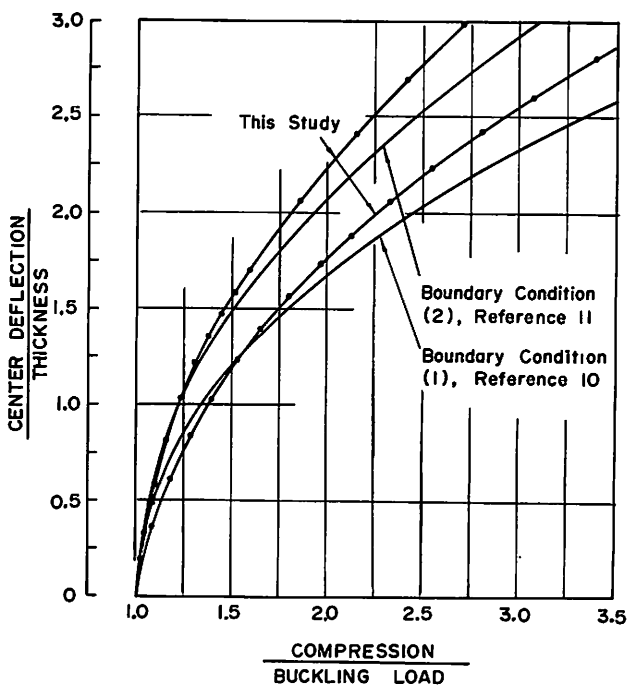


No of Elements	4	10	18
σ_{cr}/σ_{cr} (ref 9)	1.175	1.028	0.994

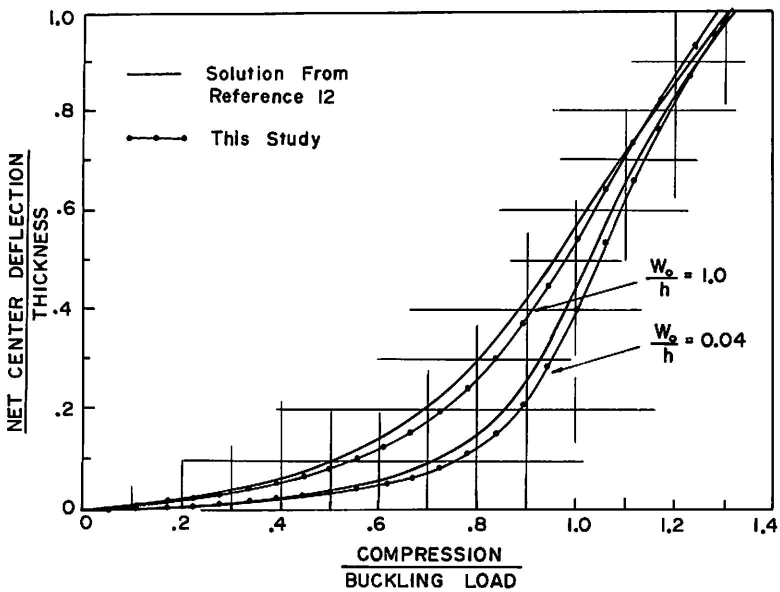
4. Euler Buckling Load for a Segment of Liner.



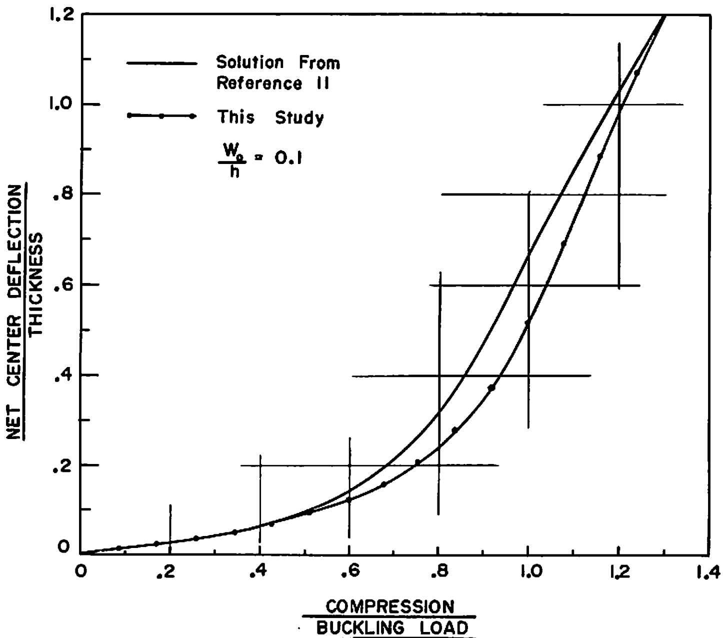
5. An Example of Liner Anchorage System.



6. Comparison of Post-Buckling Center Deflection of a Square Plate.



7. Comparison of Pre- and Post-Buckling Net Center Deflection for Square Plate with Edge Condition (1).



8. Comparison of Pre- and Post-Buckling Net Center Deflection for Square Plate with Edge Condition (2).